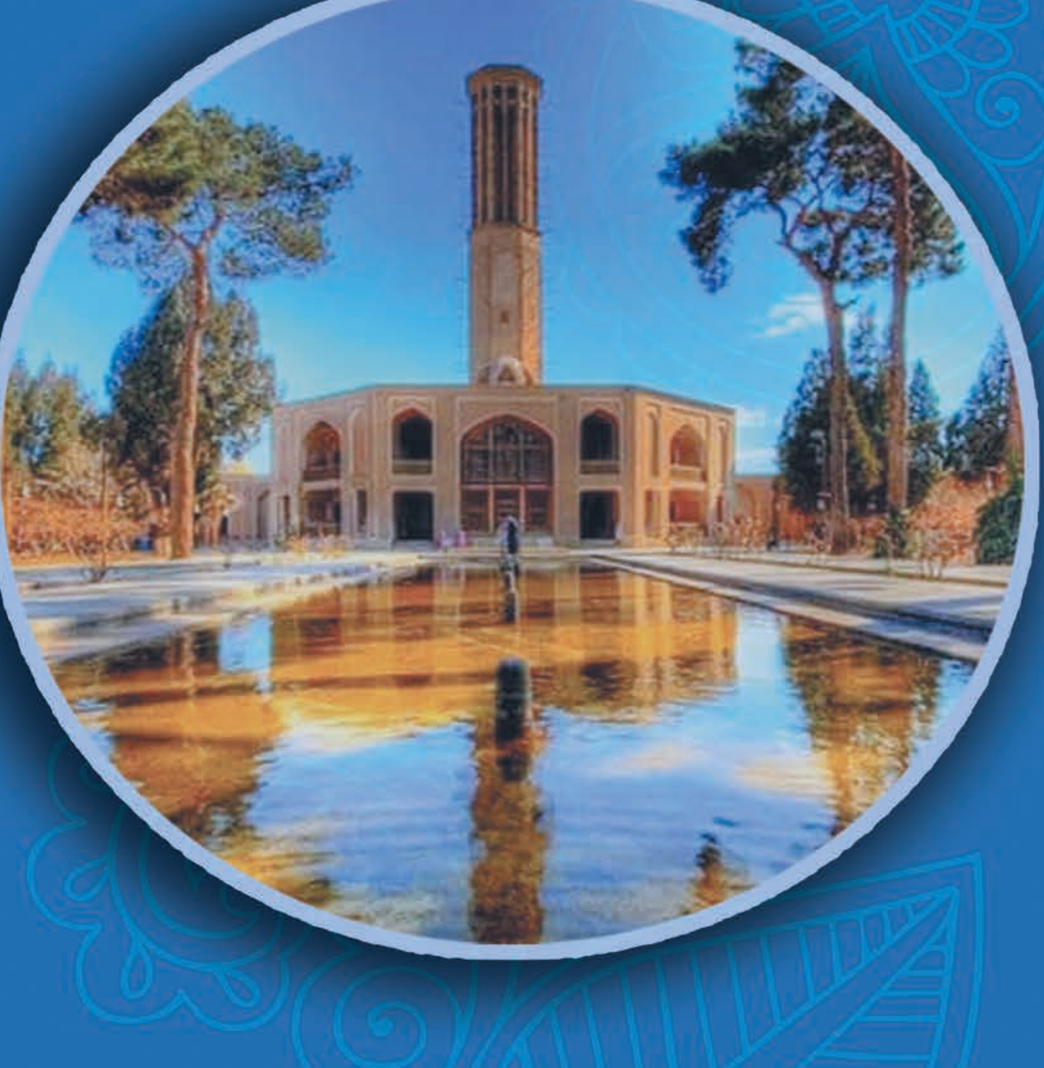
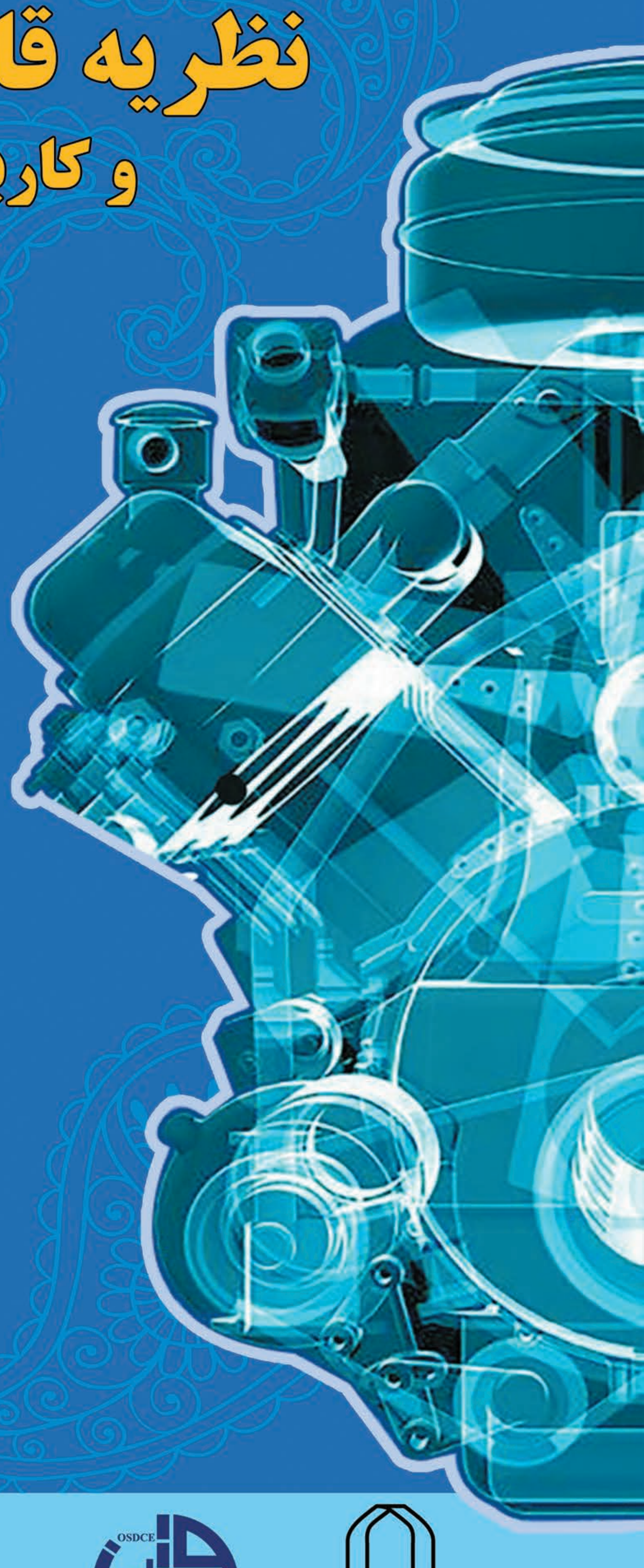
کتابچه راهنما و خلاصه مقالات

پنجمین سمینار تخصصی

نظریه قابلیت اعتماد و کاربردهای آن





۲۸ و ۲۹ فروردین ۱۳۹۸

دانشکده علوم ریاضی

دانشگاه یزد

Proceeding of

The 5th Seminar on Reliability Theory and its Applications

Department of Statistics
Yazd University,
Yazd, Iran

17-18 April, 2019

Preface

Following the series of workshops on “Reliability Theory and its Applications” in Ferdowsi University of Mashhad and three seminars in University of Isfahan (2015), University of Tehran (2016) and Ferdowsi University of Mashhad (2017) we are pleased to organize the 5th Seminar on **“Reliability Theory and its Applications”** during 17-18 April, 2019 at the Department of Statistics, Yazd University. On behalf of the organizing and scientic committees, we would like to extend a very warm welcome to all participants, hoping that their stay in Yazd will be happy and fruitful. Hope that this seminar provides an environment of useful discussions and would also exchange scientic ideas through opinions. We wish to express our gratitude to the numerous individuals that have contributed to the success of this seminar, in which around 70 colleagues, researchers, and postgraduate students from universities and organizations have participated.

Finally, we would like to extend our sincere gratitude to the Research Council of Yazd University, the administration of College of Sciences, the Ordered and Spatial Data Center of Excellence, the Islamic World Science Citation Center, the Fars Science and Technology Park, the Iranian Statistical Society, the Scientic Committee, the Organizing Committee, the referees, and the students and staff of the Department of Statistics at Yazd University for their kind cooperation.

Eisa Mahmoudi (Chair)

April, 2019

Topics

The aim of the seminar is to provide a forum for presentation and discussion of scientic works covering theories and methods in the field of reliability and its application in a wide range of areas:

- Accelerated life testing
- Bayesian methods in reliability systems
- Case studies in reliability analysis
- Computational algorithms in reliability
- Data mining in reliability
- Degradation models
- Lifetime data analysis
- Lifetime distributions theory
- Maintenance modeling and analysis
- Networks reliability
- Optimization methods in reliability
- Reliability of coherent
- Safety and risk assessment
- Software reliability
- Stochastic aging
- Stochastic dependence in reliability
- Stochastic orderings in reliability
- Stochastic processes in reliability
- Stress-strength modeling
- Survival analysis

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Estimation of $P(Y < X)$ for Two-Parameter Lindley Logarithmic Distribution

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Abstract: In this paper we study the stress-strength parameter $R = P(X < Y)$, when X and Y are independent and both have two-parameter Lindley Logarithmic (LL) distributions. We consider the computation of R in closed form, as well as its maximum likelihood estimator. Furthermore via simulation study the root mean square error (RMSE), the percentage relative bias (RB) of the estimator and also two confidence intervals has presented.

Keywords Lindley Logarithmic Distribution, Stress-Strength model, Maximum Likelihood Estimator.
Mathematics Subject Classification (2010) : 47A55, 39B52, 34K20, 39B82.

1 Introduction

Reliability is defined as the ability of a system or component to perform its required functions under stated conditions for a specified period of time. The stress-strength interference model is one that is used to compute reliability. It is found to be useful in situations where the reliability of a component or system is defined by the probability that a random variable X (representing strength) is greater than another random variable Y (representing stress). Once the distribution and parameters of X and Y are determined, the reliability can be calculated by computing the $R = P(Y < X)$.

It may be mentioned that R is of greater interest than just reliability since it provides a general measure of the difference between two populations and has applications in many area. For example, if X is the response for a control group, and Y refers to a treatment group, R is a measure of the effect of the treatment. In addition, it may be mentioned that R equals the area under the receiver operating characteristic (ROC) curve for diagnostic test or biomarkers with continuous outcome (Bamber, (4)). The ROC curve is widely used, in biological, medical and health service research, to evaluate the ability of diagnostic tests or biomarkers to distinguish

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between two groups of subjects, usually non-diseased and diseased subjects. For more details, one can be advised to Kotz et. al. (10).

Many authors have studied the stress-strength parameter R. Gogoi and Borah (9) deals with the stress vs. strength problem incorporating multi-component for systems viz. standby redundancy in the case of Exponential, Gamma and Lindley distributions. Singh et. al.(13) have developed a re-modeling of stress-strength system reliability where they have defined the probability that the system is capable to withstand the maximum operated stress at its minimum strength when both stress and strength variables are Weibull distributed. Barbiero (5) studied statistical inference for the reliability of stress-strength models when stress and strength are independent Poisson random variables, whereas, Ali et. al. (1) have investigated the estimation of $Pr(X < Y)$, when X and Y belong to different distribution families. One can refer to recently works by Kzlaslan (11), Chaudhary and Tomer (7), Bai et al. (2; 3), Eryilmaz (8), Wang et al. (14), Yadav and Singh (15) and Cetinkaya and Gen (6).

Mahmoudi and Abolhosseini (12) introduced a new distribution with increasing and bathtub shaped failure rate, called as the Lindley logarithmic (LL) distribution. The main reasons for introducing the LL distribution are:

1. This distribution is generalized of Lindley distribution. It is more flexible than the Lindley distribution because of hazard rate function.
2. It can be used in several areas such as public health, actuarial science, biomedical studies, demography and industrial reliability.

Suppose $X_1, \cdots, X_N$ be independent and identify distributed random variables from Lindley distribution and N has the Logarithmic distribution. Let $Y = X_{1:n} = \min_{1 \le i \le N} X_i$, then cdf of $Y|N = n$ is given by

$$F_{Y|N=n}(y;\gamma) = 1 - \left[(1 + \frac{\gamma y}{\gamma+1})e^{-\gamma y}\right]^n.$$

The cdf and pdf of Lindley Logarithmic (LL) distribution are given, respectively, by

$$F(y;\theta,\gamma) = 1 - \frac{\log(1 - \theta(1 + \frac{\gamma y}{\gamma+1})e^{-\gamma y})}{\log(1-\theta)}, \tag{1.1}$$

$$f(y;\theta,\gamma) = \frac{\theta \frac{\gamma^2}{\gamma+1} e^{-\gamma y}(1+y)}{(\theta(1 + \frac{\gamma y}{\gamma+1})e^{-\gamma y} - 1)\log(1-\theta)}, \tag{1.2}$$

where $0 < \theta < 1, \gamma > 0$. The survival and hazard rate functions of LL distribution are given, respectively, by

$$S(y;\theta,\gamma) = \frac{\log(1 - \theta(1 + \frac{\gamma y}{\gamma+1})e^{-\gamma y})}{\log(1-\theta)}, \tag{1.3}$$

and

$$h(y;\theta,\gamma) = \frac{\theta\frac{\gamma^2}{\gamma+1}e^{-\gamma y}(1+y)}{(\theta(1+\frac{\gamma y}{\gamma+1})e^{-\gamma y}-1)\log(1-\theta(1+\frac{\gamma y}{\gamma+1})e^{-\gamma y})}. \quad (1.4)$$

Figures 1 and 2, respectively, represent the graphs of the distribution functions, density, and survival for different values of the parameter.

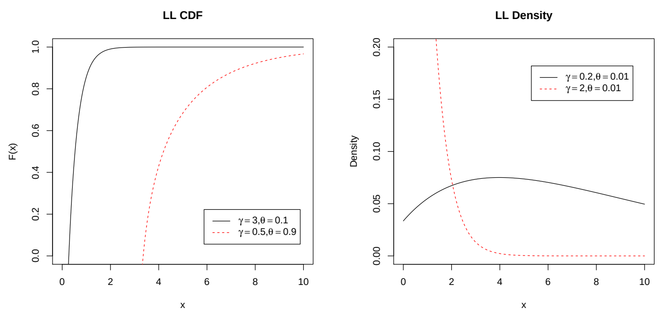

Figure 1: Plots of pdf and cdf of LL distribution

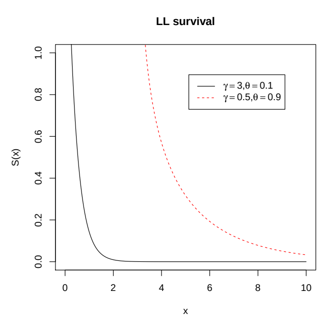

Figure 2: Plot of of the survival function of LL ditribution

The ξth quantile of the LL distribution, which is used for data generation from the LL distribution, is given by

$$(1-\xi)\log(1-\theta) = \sum_{j=1}^{\infty} \frac{(-\theta(1+\frac{\gamma x_\xi}{\gamma+1})e^{-\gamma x_\xi})^j}{j}.$$

For a random variable Y with LL distribution the moment generating function and kth order moment are given, respectively, by The random variable Y has mean and variance given, respectively, by

$$E[Y] = \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{-\theta^n\gamma^{i+2}}{(\gamma+1)^{i+1}\log(1-\theta)}\left[\frac{\Gamma(i+3)}{(n\gamma)^{i+3}}+\frac{\Gamma(i+2)}{(n\gamma)^{i+2}}\right],$$

and

$$Var[Y] = \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{-\theta^n\gamma^{i+2}}{(\gamma+1)^{i+1}\log(1-\theta)}\left[\frac{\Gamma(i+4)}{(n\gamma)^{i+4}}+\frac{\Gamma(i+3)}{(n\gamma)^{i+3}}\right]-E^2(Y).$$

The paper is organized as follows. In section 2, an approximation of the stress-strength parameter $R = P(X < Y)$ of Lindley Logarithmic (LL) distribution is obtained. Maximum likelihood estimator of R has studied in Section 3. Furthermore a simulation study has presented in Section 4.

2 Stress-Strength Parameter

For constructing the stress strength parameter consider two cases:

Case I. Suppose X (stress) and Y (strength) are two independent random variables, following $LL(\theta_1,\gamma)$ and $LL(\theta_2,\gamma)$ respectively. After some algebric calculations, the reliability of the stress-strength model is given by,

$$\begin{aligned}
R &= P(X<Y) = \sum_{j=0}^{\infty}\sum_{i=0}^{j}\sum_{k=1}^{\infty}\sum_{m=0}^{k}\binom{j}{i}\binom{k}{m}\frac{(-1)^{2k}}{\log(1-\theta_1)\log(1-\theta_2)} \\
&\times \frac{(\theta_1)^{j+1}(\theta_2)^k(\gamma)^{m+i+2}}{(1+\gamma)^{m+i+1}}\left[\frac{\Gamma(m+i+1)}{(\gamma(j+k+1))^{m+i+1}}+\frac{\Gamma(m+i+2)}{(\gamma(j+k+1))^{m+i+2}}\right]. \\
&= \lim_{z\to\infty}\sum_{j=0}^{z}\sum_{i=0}^{j}\sum_{k=1}^{z}\sum_{m=0}^{k}\binom{j}{i}\binom{k}{m}\frac{(-1)^{2k}}{\log(1-\theta_1)\log(1-\theta_2)} \\
&\times \frac{(\theta_1)^{j+1}(\theta_2)^k(\gamma)^{m+i+2}}{(1+\gamma)^{m+i+1}}\left[\frac{\Gamma(m+i+1)}{(\gamma(j+k+1))^{m+i+1}}+\frac{\Gamma(m+i+2)}{(\gamma(j+k+1))^{m+i+2}}\right].
\end{aligned} \tag{2.1}$$

Case II. By supposing $X \sim LL(\theta,\gamma_1)$ and $Y \sim LL(\theta,\gamma_2)$, we have

$$\begin{aligned}
R &= P(X<Y) = \sum_{j=1}^{\infty}\sum_{i=0}^{j}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{j}{i}\binom{k}{l}\frac{(-1)^{2j+1}}{(\log(1-\theta))^2}\times\frac{(\theta)^{k+j+1}}{j} \\
&\times \frac{(\gamma_1)^{l+2}}{(\gamma_1+1)^{l+2}}(\frac{\gamma_2}{\gamma_2+1})^i\left[\frac{\Gamma(l+i+1)}{(\gamma_1(1+k)+\gamma_2 j)^{l+i+1}}+\frac{\Gamma(l+i+2)}{(\gamma_1(1+k)+\gamma_2 j)^{l+i+2}}\right]. \\
&= \lim_{z\to\infty}\sum_{j=1}^{z}\sum_{i=0}^{j}\sum_{k=0}^{z}\sum_{l=0}^{k}\binom{j}{i}\binom{k}{l}\frac{(-1)^{2j+1}}{(\log(1-\theta))^2}\times\frac{(\theta)^{k+j+1}}{j} \\
&\times \frac{(\gamma_1)^{l+2}}{(\gamma_1+1)^{l+2}}(\frac{\gamma_2}{\gamma_2+1})^i\left[\frac{\Gamma(l+i+1)}{(\gamma_1(1+k)+\gamma_2 j)^{l+i+1}}+\frac{\Gamma(l+i+2)}{(\gamma_1(1+k)+\gamma_2 j)^{l+i+2}}\right].
\end{aligned} \tag{2.2}$$

It should be noted that the series of (2.1) and (2.2) are rapidly converge and the reliability can be actually computed taking into account only its first terms. As an example, we compute the reliability R for $\gamma = 1$ and different values of θ_1 and θ_2. The partial sums are reported in Table 1. As it is seen the values of R are already stable at the 4th decimal digit when $z = 20$.

Table 1: Partial sums for the computation of R for a Lindley Logarithmic stress strength model ($\gamma = 1$).

$z = 50$	$z = 20$	$z = 10$	$z = 5$	$z = 4$	$z = 3$	$z = 2$	$z = 1$	(θ_1, θ_2)
0.6446	0.6446	0.6272	0.5589	0.5237	0.4710	0.3887	0.2516	(0.1,0.8)
0.5180	0.5180	0.5180	0.5180	0.5180	0.5178	0.5151	0.4804	(0.1,0.1)
0.5694	0.5694	0.5693	0.5634	0.5556	0.5369	0.4902	0.365	(0.5,0.1)
0.7234	0.7234	0.7234	0.6349	0.5869	0.5161	0.4072	0.2390	(0.8,0.5)
0.4838	0.4838	0.4838	0.4838	0.4838	0.3739	0.2522	0.1222	(0.9,0.9)

3 Maximum Likelihood Estimator of R

Let $X_1, \cdots, X_N$ and $Y_1, \cdots, Y_M$ are independent random samples from $LL(\theta_1, \gamma)$ and $LL(\theta_2, \gamma)$, respectively. Then the log-likelihood function is

$$\begin{aligned} l_n \equiv l_n(\underline{x}, \underline{y}; \theta_1, \theta_2, \gamma) \quad = \quad & n \log \theta_1 + m \log \theta_2 + (m+n) \log(\frac{\gamma^2}{\gamma+1}) - \gamma(\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j) \\ & + \sum_{i=1}^{n} \log(1+x_i) + \sum_{j=1}^{m} \log(1+y_j) \\ & - \sum_{i=1}^{n} \log(\theta_1(1+\frac{\gamma x_i}{\gamma+1})e^{-\gamma x_i} - 1) - n \log(\log(1-\theta_1)) \\ & - \sum_{j=1}^{m} \log(\theta_2(1+\frac{\gamma y_i}{\gamma+1})e^{-\gamma y_i} - 1) - m \log(\log(1-\theta_2)). \end{aligned}$$

The components of score function are as follows

$$\begin{aligned} \frac{\partial l_n}{\partial \theta_1} \quad &= \frac{n}{\theta_1} - \sum_{i=1}^{n} \frac{(1+\frac{\gamma x_i}{\gamma+1})e^{-\gamma x_i}}{(\theta_1(1+\frac{\gamma y_i}{\gamma+1})e^{-\gamma y_i}-1)} + \frac{n}{(1-\theta_1)(\log(1-\theta_1))}, \\ \frac{\partial l}{\partial \theta_2} \quad &= \frac{m}{\theta_2} - \sum_{j=1}^{m} \frac{(1+\frac{\gamma y_j}{\gamma+1})e^{-\gamma y_j}}{(\theta_2(1+\frac{\gamma y_j}{\gamma+1})e^{-\gamma y_j}-1)} + \frac{m}{(1-\theta_2)(\log(1-\theta_2))}, \\ \frac{\partial l}{\partial \gamma} \quad &= (m+n)\frac{(\gamma+2)}{\gamma(\gamma+1)} - (\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j) - \sum_{i=1}^{n} \frac{\gamma\theta_1 x_i(\gamma+\gamma x_i+x_i+2)}{(\gamma+1)(\theta_1+\gamma\theta_1(x_i+1)+(\gamma+1)(-e^{\gamma x_i}))} \\ & - \sum_{j=1}^{m} \frac{\gamma\theta_2 y_j(\gamma+\gamma y_j+y_j+2)}{(\gamma+1)(\theta_2+\gamma\theta_2(y_j+1)+(\gamma+1)(-e^{\gamma y_j}))}. \end{aligned}$$

The MLEs of parameters θ_1, θ_2 and γ can be obtained through solving system of nonlinear equations via EM algorithm. This system of nonlinear equations does not have closed form. The

MLE of R is obtained by replacement estimation of parameters beacuse of invariance property of MLEs.

4 Simulation Study

In this section based on Bootstrap method, different samples with various pamateres and sample sizes are drawn from $LL(\theta_1, 1)$ and $LL(\theta_2, 1)$ independently. Different and unequal sample sizes are here considered. The MLE estimators of R are computed on each sample and their approximate variances are calculated. In more detail, the root mean square error (RMSE) and the percentage relative bias (RB) of the estimators are provided by,

$$RMSE(\tilde{R}) = \sqrt{\frac{1}{B}\sum_{s=1}^{B}(\tilde{R}(s) - R)^2},$$

$$RB(\tilde{R}) = \frac{\Big((1/B)\sum_{s=1}^{B}\tilde{R}(s) - R\Big)}{R} \cdot 100,$$

where $\tilde{R}(s)$ denotes the value of $\tilde{R}$ for the sth sample and B is the replication of Bootstrap which is equal to 1000 in this study. Table 4 shows the approxiamtion of R, $RMSE(\tilde{R})$ and $RB(\tilde{R})$ for different values of parameters and sample sizes. It is seen that, as the sample sizes are increased the RMSE is decreased.

5 Confidence Intervals For R

In this section based on bootstarp method we present two confidence intervals for R as follow,

Normal Interval.

$$(\tilde{R} - Z_{\alpha/2}\hat{s.e}_{boot}, \tilde{R} + Z_{\alpha/2}\hat{s.e}_{boot})$$

where $\hat{s.e}_{boot}$ is the bootstrap estimate of the standard error.

Percentile Intervals.

$$(R^*_{\frac{\alpha}{2}}, R^*_{1-\frac{\alpha}{2}}),$$

where $R^*_{\frac{\alpha}{2}}$ and $R^*_{1-\frac{\alpha}{2}}$ are the $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ quantiles of the bootstrap sample respectively.

According to the Table 2.5, it seems that the accuracy of the two confidence intervals methods are almost the same.

References

[1] Ali, M., Pal, M. and Woo, J. (2010), Estimation of $Pr(Y < X)$ when X and Y belong to different distribution families, *Journal of Probability and Statistical Science*, **8**, 35-48.

[2] Bai, X., Shi, Y., Liu, Y. and Liu, B. (2018), Reliability estimation of multicomponent stress-strength model based on copula function under progressively hybrid censoring, *Journal of Computational and Applied Mathematics*, **344**, 100-114.

[3] Bai, X., Shi, Y., Liu, Y. and Liu, B. (2019), Reliability inference of stressstrength model for the truncated proportional hazard rate distribution under progressively Type-II censored samples, *Applied Mathematical Modelling*, **65**, 377-389.

[4] Bamber, D. (1975), The area above the ordinal dominance graph and the area below the receiver operating graph, *Journal of Mathematical Psychology*, **12**, 387-415.

[5] Barbiero A. (2013), Inference on Reliability of Stress-Strength Models for Poisson Data, *Journal of Quality and Reliability Engineering*, Article ID 530530, 8 pages.

[6] Cetinkaya, C. and Gen, A.I. (2019). Stress-strength reliability estimation under the standard two-sided power distribution, *Applied Mathematical Modelling*, **65**, 72-88.

[7] Chaudhary, S. and Tomer, S.K. (2018), Estimation of stress-strength reliability for Maxwell distribution under progressive Type-II censoring scheme, *International Journal of System Assurance Engineering and Management*, **11**, 1-13.

[8] Eryilmaz, S. (2018), Phase type stress-strength models with reliability applications, *Communications in Statistics-Simulation and Computation*, **47**(4), 954-963.

[9] Gogoi J. and Borah M. (2012), Estimation of Reliability for Multicomponent Systems Using Exponential, Gamma and Lindley Stress-Strength Distributions, *Journal of Reliability and Statistical Studies*, **5**(1), 33-41

[10] Kotz, S., Lumelskii, Y. and Pensky, M. (2003), *The Stress-Strength Model and its Generalizations: Theory and Applications*, World Scientific, New York.

[11] Kzlaslan, F. (2017), Classical and Bayesian estimation of reliability in a multicomponent stress-strength model based on the proportional reversed hazard rate mode, *Mathematics and Computers in Simulation*, **136**, 36-62.

[12] Mahmoudi, E. and Abolhosseini, S. (2016), Lindley-Logarithmic Distribution: Model and Properties, *Journal of Statistical sciences*, **10**(1), 139-158.

[13] Singh B., Rathi S. and Singh G. (2011), Inferential analysis of the re-modeled Stress-strength system reliability with Application to the real data, *Journal of Reliability and Statistical Studies*, **4**(2), 1-23.

[14] Wang, B., Geng, Y. and Zhou, J. (2018), Inference for the generalized exponential stress-strength model, *Applied Mathematical Modelling*, **53**, 267-275.

[15] Yadav, A.S. and Singh, S.K.S.U. (2018), Estimation of stress-strength reliability for inverse Weibull distribution under progressive Type-II censoring scheme, *Journal of Industrial and Production Engineering*, **35**(1), 48-55 .

Table 2: Approximation, root mean square error (RMSE) and relative bias (RB) of the parameter R via bootstrap with 1000 replication

	$(n_1, n_2) = (20, 10)$	
	$(\theta_1, \theta_2) = (0.3, 0.5)$	$(\theta_1, \theta_2) = (0.7, 0.6)$
$\tilde{R}$	0.62065	0.67535
$RMSE(\tilde{R})$	0.09879	0.10926
$RB(\tilde{R})$	-7.58e-15	-5.09e-15
	$(n_1, n_2) = (20, 20)$	
	$(\theta_1, \theta_2) = (0.3, 0.5)$	$(\theta_1, \theta_2) = (0.7, 0.6)$
$\tilde{R}$	0.60887	0.69114
$RMSE(\tilde{R})$	0.08398	0.10814
$RB(\tilde{R})$	9.37e-15	3.88e-15
	$(n_1, n_2) = (50, 50)$	
	$(\theta_1, \theta_2) = (0.3, 0.5)$	$(\theta_1, \theta_2) = (0.7, 0.6)$
$\tilde{R}$	0.60049	0.70830
$RMSE(\tilde{R})$	0.07497	0.10024
$RB(\tilde{R})$	-5.01e-15	-7.96e-15

Table 3: Two bootstrap 95% confidence intervals for R with 1000 replication

(θ_1, θ_2)	(n_1, n_2)	*Normal Int.*	*Percentile Int.*
	(10,20)	(0.20, 0.81)	(0.10, 0.80)
(0.3,0.5)	(20,20)	(0.27, 0.69)	(0.30, 0.65)
	(50,50)	(0.32, 0.60)	(0.32, 0.58)
	(10,20)	(0.14, 0.71)	(0.10, 0.70)
(0.7,0.9)	(20, 20)	(0.20, 0.63)	(0.20, 0.60)
	(50,50)	(0.32, 0.60)	(0.28, 0.57)

پنجمین سمینار تخصصی نظریه قابلیت اعتماد و کاربردهای آن
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5th Seminar on Reliability Theory and its Applications

A New Bathtub Shaped Extension of the Weibull Distribution with Analysis to Reliability Data

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Abstract: In this article, a new flexible extension of the Weibull distribution is proposed which is capable of modeling lifetime data with bathtub-shaped hazard rate function. The new model is introduced by considering a system of two logarithms of cumulative hazard rate functions. The proposed distribution will be named as a new flexible extended Weibull distribution. Some mathematical properties and characterizations along with the estimation of the model parameters through maximum likelihood method are discussed. Finally, to illustrate the importance of the proposed distribution, two real life applications with bathtub-shaped hazard functions are analyzed demonstrating that the new model provides adequate fits in comparison with the other modified forms of the Weibull model including the exponentiated Weibull, Marshall-Olkin Weibull, Additive Weibull, new modified Weibull and additive Perks-Weibull distributions.

Keywords Bathtub-shaped hazard rate function, Modeling reliability data, Characterizations, Maximum likelihood estimates, Weibull distribution.

Mathematics Subject Classification (2010): 60E05, 62F10.

1 Introduction

The hazard rate function (also known as a failure rate function) is one of the most important reliability characteristics that describes the failure mechanism of the system during its lifetime period. It deals with an immediate risk of failure of the system at the time, say t, given that the system has not failed up to that time. Among the hazard rate functions, the bathtub hazard rate curve is a well-known concept in reliability engineering. It represents the failure behavior of various engineering systems having initially a decreasing failure rate during the very first phase, a constant failure rate in the middle part of the life (usually called useful life period)

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and finally an increasing failure rate in the last phase. In the context of reliability theory, these three phases are, respectively, known as burning, random and wear-out failure regions.

In the last two decades, many new life distributions capable of modeling data with the bathtub hazard rate function have been introduced in the literature. Most of them are the modifications and extensions of the two-parameter Weibull distribution, including a four-parameter Additive Weibull (AW) with bathtub hazard rate function consisting of two Weibull hazard functions proposed by Xie and Lee [7], a five-parameter new modified Weibull (NMW) of Almalki and Yuan [2], which has a bathtub-shaped hazard function consisting of modified Weibull and Weibull hazards, an Additive PerksWeibull (APW) of Singh [6] by combining the sum of the hazard rates of the Perks and Weibull distributions has the bathtub shaped failure rate function, among others.

The key goal of the modification and extension forms of the Weibull model is to describe and fit the data sets with non-monotonic hazard rate, such as the bathtub, unimodal and modified unimodal hazard rate. Many extensions of the Weibull distribution have achieved the above purpose. However, the number of parameters has increased up to 5 or more, the forms of the survival and hazard functions have been complicated and estimation problems have risen. On the other hand, unfortunately, some of the modifications do not have a closed form for their cumulative distribution functions (CDFs). Furthermore, as we have seen, the bathtub and the modified unimodal shapes have three phases: initially decreasing phase, relatively constant phase and then an increasing phase for the bathtub shape and the phases of the modified unimodal shape are initially increasing, then decreasing, then increasing again. The main weakness of some modified Weibull distributions is that they are unable to fit the last phase of the bathtub shapes, which is an essential part, as well as the first and middle phases. In terms of cumulative hazard rate function (CHRF), the CDF can be expressed as

$$G\left(x\right) = \ 1 - \ e^{-H(x)}, \tag{1.1}$$

where, the CHRF denoted by $H(x)$ satisfies the following properties

i. $H(x)$ is differentiable non-negative and increasing function of x,

ii. $\lim_{x\to 0}\ H(x) = 0$ and $\lim_{x\to\infty} H(x) = \infty$

It may be a very useful approach to combine two or cumulative hazard functions and generate a new function as

$$H(x) = \beta H_1\left(x\right) + \sigma H_2\left(x\right). \tag{1.2}$$

The expression (1.2) is bounded. However, in this article, a new function *log H(x)* instead of *H(x)* is used to relax the boundary conditions. Hence, one can write (1.2) as

$$\log H(x) = \beta \log H_1(x) + \sigma \log H_2(x). \tag{1.3}$$

Here, a mixture of the two logarithm of cumulative hazard functions, taken as x^{α} and $\left\{-\left({}^{1}/_{x^{\lambda}}\right)\right\}$ is used to introduce a new flexible lifetime distribution. So, the expression (1.3) can be written as

$$H(x) = e^{\beta x^{\alpha} - \frac{\sigma}{x^{\lambda}}}. \tag{1.4}$$

Using (1.4) in (1.1), one may easily arrive at the CDF of the new flexible extended Weibull (NFEW) distribution. The rest of the paper is designed as follows: Section 2 provides the definition and visual sketching of the proposed distribution. Two real-life applications are provided in Section 3. Finally, some concluding remarks are provided Section 4.

2 New flexible extended Weibull distribution

The CDF of the NFEW distribution is given by

$$G(x) = 1 - \exp\left\{-e^{\beta x^{\alpha} - \frac{\sigma}{x^{\lambda}}}\right\}, \qquad x \geq 0,\ \alpha, \beta, \sigma, \lambda > 0. \tag{2.1}$$

The probability density function (PDF) corresponding to (2.1) is given by

$$g(x) = \left(\alpha\beta x^{\alpha-1} + \frac{\lambda\sigma}{x^{\lambda+1}}\right) e^{\left(\beta x^{\alpha} - \frac{\sigma}{x^{\lambda}}\right)} \exp\left\{-e^{\beta x^{\alpha} - \frac{\sigma}{x^{\lambda}}}\right\}, \qquad x \geq 0. \tag{2.2}$$

The survival function (SF) and hazard rate function (HRF) of the proposed model are given, respectively, by

$$S(x) = \exp\left\{-e^{\beta x^{\alpha} - \frac{\sigma}{x^{\lambda}}}\right\}, \qquad x \geq 0, \tag{2.3}$$

and

$$h(x) = \left(\alpha\beta x^{\alpha-1} + \frac{\lambda\sigma}{x^{\lambda+1}}\right) e^{\left(\beta x^{\alpha} - \frac{\sigma}{x^{\lambda}}\right)}, \qquad x \geq 0. \tag{2.4}$$

Some possible shapes for the hazard rate function (HRF) of the proposed model are sketched in Figure 1 .

Motivations

The key motivations for using the proposed model in practice are as follow:

1. The distribution function as well as the survival function of the proposed model have the closed form.

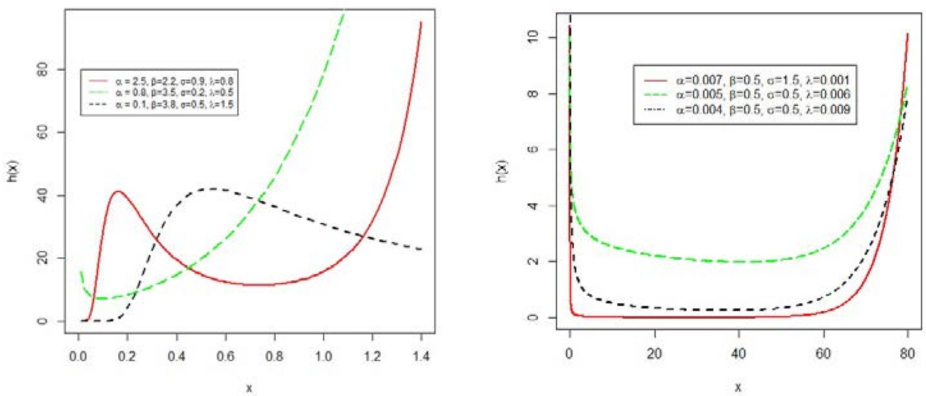

Figure 1: Plots of HRF of the NFEW distribution for selected values of parameters.

2. The proposed model is capable of modeling data with monotonic and non-monotonic failure rates.

3. The proposed model is capable of modeling the last phase of the modified unimodal shaped failure rate function closely (see Figure 1).

4. The proposed model has a long constant failure rate period (as shown in Figure 1) which is capable to model the second phase of the bathtub shaped failure rate function.

5. The proposed model is capable of modeling the last phase of the bathtub shaped failure rate function closely (as described in Figure 1).

6. The proposed model provide a best fit to the reliability data having a bathtub shaped failure rate function than the other well-known bathtub shaped extensions of the Weibull distribution having the same and higher number of parameters.

3 Applications

For the practical illustration, the fitting results of the NFEW distribution to two well-known data sets having bathtub shaped failure rates are compared to the goodness-of-fit with the other modified forms of the Weibull distribution. The analytical measures for model comparison such as Akaike information criterion (AIC), Kolmogorov-Smirnov (KS) statistic and the corresponding p-value are considered. Using these statistical measures, it is showed that the

NFEW distribution provides a better fit than the new modified Weibull of Almalki and Yuan [2], Marshall-Olkin Weibull (MOW) of Marshall and Olkin [3], exponentiated Weibull (EW) of Mudholkar and Srivastava [5], additive Perks-Weibull of Singh [6] and Additive Weibull of Xie and Lai [7].

3.1 Arset data

The first data set having the bathtub shaped representing the lifetimes of 50 devices taken from Arset [1]. This data set is known to have a bathtub-shaped hazard function. Table 1 provides goodness of fit measures and maximum likelihood estimates (MLEs) of parameters of the NFEW and other competing distributions along with standard errors in brackets. From Figure 2, it is clear that the cdf of NFEW fits the data well and its survival function follows the cdf and Kaplan–Meier estimate closely.

Table 1: MLEs with their standard errors in brackets for Arset data.

Dist.	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\sigma}$	AIC	KS	P-value
NFEW	2.0110-5	2.1310-9			0.186	0.005	430.70	0.084	0.806
EW	1.373	0.002	0.495				485.97	0.201	0.036
MOW	0.707	0.131		3.620			488.88	0.178	0.076
NMW	7.0110-8	0.071	0.016	0.595	0.197		435.86	0.088	0.803
AW	0.086	1.1310-8	0.102	4.214			451.09	0.127	0.365
APW		7.1510-17	0.443	0.053	0.688		433.75	0.091	0.804

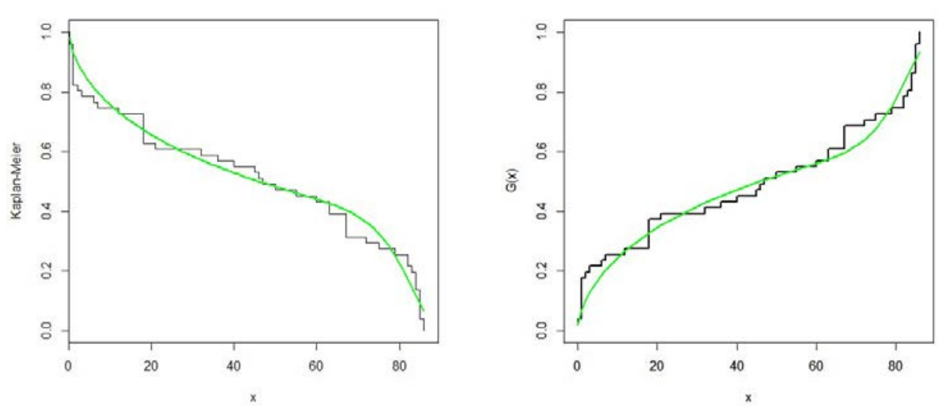

Figure 2: The estimated CDF and Kaplan-Meier survival function of the NFEW distribution for Arset data.

3.2 Meeker and Escobar data

The second data representing the failure times of a sample of 30 devices taken from Meeker and Escobar [4]. This data set is also known to have a bathtub-shaped hazard function. Table 2 provides goodness of fit measures and maximum likelihood estimates (MLEs) of parameters of the NFEW and other competing distributions along with standard errors in brackets. Again, the proposed distribution provides a better fit than the other competing distributions, as can be seen from Table 2. From Figure 3, it can easily be detected that the cdf and Kaplan–Meier of NFEW fits the data well.

Table 2: MLEs with their standard errors in brackets for second data.

Dist.	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\sigma}$	AIC	KS	P-value
NFEW	0.019	3.940			0.372	0.973	341.09	0.131	0.876
EW	1.086	0.003	1.076				375.49	0.224	0.104
MOW	1.013	0.009		3.458			371.93	0.223	0.098
NMW	5.9910-8	0.024	0.012	0.629	0.056		344.49	0.148	0.482
AW	0.019	1.3210-7	0.604	2.830			364.28	0.191	0.197
APW		5.410-12	0.088	0.011	0.807		343.82	0.134	0.655

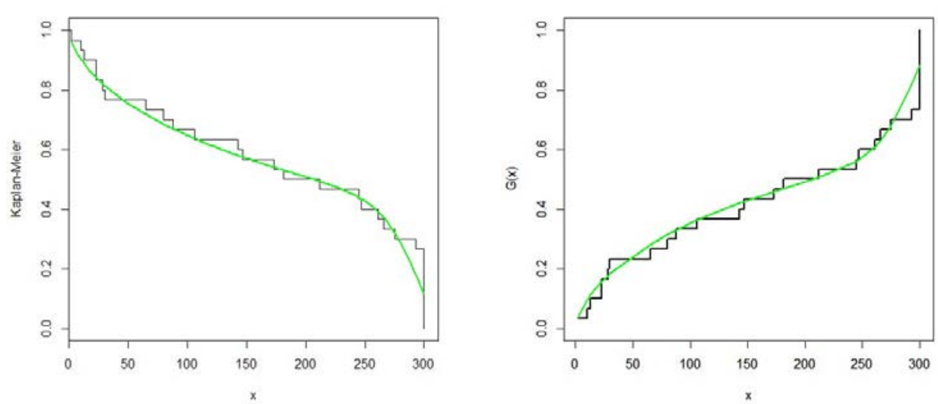

Figure 3: The estimated cdf and survival function of the NFEW distribution for Meeker and Escobar data.

4 Concluding Remarks

In this study, a new flexible extended Weibull distribution with non-monotone hazard rate function is proposed and investigated by taking into account a system of two logarithms of cumulative hazard functions. The resulting hazard rate function of the proposed model is cable of accommodating different shapes including bathtub-shape to describe the failure behaviour of a variety of real life data. Finally, two real data sets having bathtub shape hazard rate functions, have been analyzed for illustrative purposes. For these data sets, some accuracy measures along with the p-values are calculated to compare the goodness of fit of the proposed model to the other competing distributions. These measures reveal that the proposed distribution provides best fit to these bathtub shaped data than that for the other distributions considered. To support these accuracy measures, empirical cdf and Kaplan–Meier plots are also sketched which show that the cdf of NFEW model fits the data well and its survival function follows the Kaplan–Meier estimate very closely. We hope that the proposed model will attract wider applications in reliability engineering and other related fields.

References

[1] Aarset, M. V. (1987), How to identify bathtub hazard rate, *IEEE Transactions on Reliability*, **36**, 106-118.

[2] Almalki, S. J. and Yuan, J. (2013), A new modified Weibull distribution, *Reliability Engineering and System Safety* **111**, 164-170.

[3] Marshall, A. W. and Olkin, I. (1997), A new method for adding a parameter to a family of distributions withapplication to the exponential and Weibull families, *Biometrika*, **84**, 641-652.

[4] Meeker, W. Q. and Escobar, L. A. (1998), Statistical methods for reliability data, *John Wiley*, Vol. 78, New York.

[5] Mudholkar, G. S. and Srivastava, D. K. (1993), Exponentiated Weibull family for analyzing bathtub failure rate data, *IEEE Transactions on Reliability*, **42**, 299-302.

[6] Singh, B. (2016), An Additive Perks-Weibull Model with Bathtub-Shaped Hazard Rate Function, *Communication in Mathematics and Statistics*, **4**(4), 473-493.

[7] Xie, M. and Lai, C. D. (1995), Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function, *Reliability Engineering System Safety*, **52**, 87-93.

An New Lifetime Performance Index

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Abstract: In this paper, we consider a generalization of the lifetime performance index C_L introduced by Montgomery (5), for processes with multiple quality characteristics. The new index is appropriate for mutually independent and exponentially distributed characteristics. The relationship between the extended index and overall lifetime-conforming rate is also established.
Keywords Lifetime Performance Index, Process Capability Indices, Exponential Distribution.
Mathematics Subject Classification (2010) : 62P30**.**

1 Introduction

Process capability indices (PCIs) have been proposed for the manufacturing industry to provide numerical measures on how well a process is capable of reproducing items within the preset specification limits in the factory. Numerous PCIs, including C_p, C_{pk}, C_{pm}, C_{pmk}, and S_{pk} for target(nominal)-the-better type quality characteristics and C_{pl} (C_{pu}) for larger(smaller)-the-better quality characteristics, have been used to evaluate process performance for cases with single quality characteristics, (see Wu et al. (10)). The mentioned indices are only appropriate for normal or near-normal processes. Since the lifetime of products is a larger-the-better type quality characteristic which often follows a right-skew distribution, the unilateral index C_L as an extension of the index C_{pl} is suggested by Montgomery (5) to assess the performance of lifetime with one-parameter exponential distribution. Tong et al. (9) constructed the uniformly minimum variance unbiased estimator (UMVUE) of C_L and built a hypothesis testing procedure under the assumption of exponential distribution for the complete sample. The C_L has become the most popular capability (performance) index and is widely used in the industry to assess the capability of processes whose underlying characteristic follows a lifetime (right-skew) distribution. However, to date, the existing literature associating the performance of a process is still limited to the discussion of a single quality characteristic (see Ahmadi et al. (1)), no

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research work has been done on the performance index C_L for a processes with multiple characteristics. The main objective of this paper is to develop a new tool for processes involving multiple characteristics by proposing a new overall lifetime performance index C_L^T, which is a generalization of the most widely used index C_L.

2 The overall lifetime performance index

The lifetime of products is a larger-the-better type quality characteristic since products with longer lifetime tend to be more competitive in the nowaday's markets. Suppose that the lifetime variable X has a lower specification limit L and follows an exponential distribution with parameter λ with the following probability density function $f_X(x)$, cumulative distribution function $F_X(x)$, and failure rate function $h_X(x)$:

$$f_X(x) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}}, \quad F_X(x) = 1 - e^{-\frac{x}{\lambda}}, \quad h_X(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{1}{\lambda}, \quad \lambda > 0 \tag{2.1}$$

Montgomery (5) developed the lifetime performance index C_L to measure the larger-the-better type quality characteristics with a known lower specification limit L as follows:

$$C_L = \frac{\mu - L}{\sigma}, \quad -\infty < C_L \leq 1, \tag{2.2}$$

where μ and σ represent the mean and standard deviation of quality variable, respectively. For lifetime of products following the distribution defined in (1.2), the lifetime performance index C_L can be reduced as:

$$C_L = 1 - \frac{L}{\lambda} \tag{2.3}$$

Observe that when $\lambda > L$, we have the index $C_L > 0$, when $\lambda < L$, we have the index $C_L < 0$. It is also observed that the smaller the failure rate $\frac{1}{\lambda}$ the larger the lifetime performance index C_L. Therefore, the index C_L can accurately assess the performance of lifetime of products. To determine whether the lifetime of products X are consistently achieved by manufacturers and delivered within their required specification L preset by customers, we define the lifetime-conforming rate (which is also known as process yield in literature) p and lifetime-nonconforming rate r as:

$$p = Pr(X > L) = e^{-\frac{L}{\lambda}} = e^{C_L - 1}, \tag{2.4}$$

$$r = Pr(X \leq L) = 1 - e^{-\frac{L}{\lambda}} = 1 - e^{C_L - 1}. \tag{2.5}$$

A strictly increasing (decreasing) relation holds between p (r) and C_L. Because of this one-to-one mathematical correspondence, we can use the index C_L to assess p or r. For example, $C_L \geq 0.99$ means that the p would be at least 0.99 or that the lifetime-nonconforming rate is < 0.01 (parts per millions (ppm) of non-conformities is less that 10000). Hence, the C_L values provide useful lifetime-performance information when practitioners test the product lifetimes under an exponential distribution. Process performance analysis for non-exponential distributions has been considered in the literature, too. However, most the cases result by a transformation of the data.

Processes in factories are commonly described in multiple characteristics. Performance measures for processes with a single characteristic has been investigated extensively. However, performance measures for processes with multiple characteristics are comparatively neglected. There are several approaches to define multivariate PCIs (MPCIs) to evaluate the whole capability/performance of a process with more than one interested characteristics which are classified in a broad sense in de-Felipe and Benedito (4). A simple way to introduce MPCIs, is based on the exact (or approximate) relation of exist univariate PCIs with the process yield that are studied for example in Chen et al. (3), Pearn et al. (7), and Pearn et al. (8) to extend the univariate indices S_{pk}, C_{pl} and C_{pu}, and C_{pk}, respectively. For processes with multiple characteristics, Bothe (2) considered a simple measure by taking the minimum measure of each single characteristic. For example, consider a m-characteristic process with m yield measures (lifetime-conforming rates in our study) $p_1, p_2, ...,$ and p_m. The overall process yield (overall lifetime-conforming rate) would be measured as $P = min\{p_1, p_2, ..., p_m\}$. We note that this approach does not reflect the real situation accurately. Suppose the process has five characteristics ($m = 5$), with equal characteristic yield measures $p_1 = p_2 = p_3 = p_4 = p_5 = 0.9973$. Using the approach considered by Bothe (2), the overall process yield is calculated as $P = min\{p_1, p_2, p_3, p_4, p_5\} = 99.73$ (or 2700 ppm of non-conformities). Assuming that the five characteristics are mutually independent, then the actual overall process yield should be calculated as:

$$P = p_1 \times p_2 \times p_3 \times p_4 \times p_5 = 0.9866 \tag{2.6}$$

(or 134.273 ppm of non-conformities), which is significantly less than that calculated by Bothe (2). Based on the the considered approach in Chen et al. (3) and the relations (2.4) and (2.6) for a process with m multiple characteristics that are mutually independent and exponentially distributed, we propose the following overall lifetime performance index, referred to as C_L^T:

$$C_L^T = \sum_{j=1}^{m} C_{Lj} - (m-1), \quad -\infty < C_L^T \leq 1, \tag{2.7}$$

Table 1: The overall lifetime performance index C_L^T and its P and $NCPPM$

C_L^T	P	*ncppm*	C_L^T	P	*ncppm*	C_L^T	P	*ncppm*
$-\infty$	0.0000	1000000	-1.4	0.0907	909282	0.2	0.4493	550671
-3	0.0183	981684.4	-1.2	0.1108	889196.8	0.3	0.4966	503414.7
-2.8	0.0224	977629.2	-1	0.1353	864664.7	0.4	0.5488	451188.4
-2.6	0.0273	972676.3	-0.8	0.1653	834701.1	0.5	0.6065	393469.3
-2.4	0.0334	966626.7	-0.6	0.2019	798103.5	0.6	0.6703	329680
-2.2	0.0408	959237.8	-0.4	0.2466	753403	0.7	0.7408	259181.8
-2	0.0498	950212.9	-0.2	0.3012	698805.8	0.8	0.8187	181269.2
-1.8	0.0608	939189.9	0	0.3679	632120.6	0.9	0.9048	95162.58
-1.6	0.0743	925726.4	0.1	0.4066	593430.3	1	1.0000	0.0000

where C_{Lj} denotes the C_L value of the j-th characteristic for $j = 1, 2, ..., m$. The new index, C_L^T, may be viewed as a generalization of the single characteristic lifetime performance index, C_L, considered by Montgomery (5). More specifically, we can establish the relationship between the index C_L^T and the overall lifetime-conforming rate P by $P = e^{C_L^T - 1}$. Thus, the new index C_L^T also provides an exact measure of the overall process conforming rate, similar to the one-characteristic case. Viewing from the aspect of non-conformities, the exact measure of non-conformities in *ppm* (*ncppm*) for a well-controlled exponentially distributed process with mutually independent characteristics can then be calculated as $ncppm = (1 - e^{C_L^T - 1}) \times 10^6$. TABLE 1 displays the values of *ncppm* and P for some common values of C_L^T.

For a process with m characteristics, if the requirement for the overall process capability is $C_L^T \geq c_0$, a sufficient condition (which is minimal) for the requirement to each single characteristic can be obtained by the following. Let $c^{'}$ be the minimum C_L value required for each single characteristic, if:

$$C_L^T = \sum_{j=1}^{m} C_{Lj} - (m-1) \geq c_0, \tag{2.8}$$

then we have:

$$c^{'} \geq \frac{c_0 + (m-1)}{m}. \tag{2.9}$$

Thus, the overall lifetime performance requirement $C_L^T \geq c_0$ would be satisfied, if the capability of j-th characteristic satisfies $C_{Lj} \geq c_l$ for all $j = 1, 2, ..., m$, where the lower bound c_l on

Table 2: Lower bound of various lifetime performance levels

m	c_0				
	0.7	0.8	0.9	0.95	0.98
1	0.7000	0.8000	0.9000	0.9500	0.9800
2	0.8500	0.9000	0.9500	0.9750	0.9900
3	0.900	0.9334	0.9667	0.9833	0.9933
4	0.9250	0.9500	0.9750	0.9875	0.9950
5	0.9400	0.9600	0.9800	0.9900	0.9960
6	0.9500	0.9667	0.9833	0.9917	0.9967
7	0.9571	0.9714	0.9857	0.9929	0.9971
8	0.9625	0.9750	0.9875	0.9938	0.9975
9	0.9667	0.9778	0.9888	0.9944	0.9978
10	0.9700	0.9800	0.9900	0.9950	0.9980

each C_{Lj} can be calculated, as:

$$c_l = \frac{c_0 + (m-1)}{m}. \tag{2.10}$$

TABLE 1 displays the lower bound c_l of C_{Lj}, if the requirement of the overall process capability C_L^T are 0.7, 0.8, 0.9, 0.95 and 0.98 for $m = 1(1)10$ characteristics. For example, if c_0 is set to be 0.95 with $m = 5$, i.e., the overall lifetime-conforming rate is set to be no less than 0.9512. The overall lifetime performance $C_L^T \geq 0.95$ would be satisfied, if each single characteristic conforming rate is no less than $(0.9512)^{1/5} = 0.9900$ (equivalent to 10000 *ncppm*), and the lifetime performance for all the five characteristics be at least:

$$C_{Lj} = \frac{0.95 + (5-1)}{5} = 0.9900. \tag{2.11}$$

As it is mentioned earlier, due to the exist exact one-to-one relations between the exponential distribution and the other widely-used life distributions, the proposed index could be implemented for the mentioned distributions with the transformation technique.

3 Estimation of C_L^T

In practice, sample data must be collected in order to calculate the individual indices in (2.3) since the process means λ_j for $j = 1, 2, ..., m$ are usually unknown. Tong et al. (9) derived the

UMVUE of C_L based on the complete sample $X_1, X_2, ..., X_n$ as:

$$\widehat{C}_L = 1 - \frac{(n-1)L}{\sum_{i=1}^{n} X_i}. \tag{3.1}$$

Consider the complete samples $X_{1j}, X_{2j}, ..., X_{nj}$ for the j-th characteristics which follows an exponential distribution with mean λ_j. Also, suppose that L_j is the lower specification limit for the j-th characteristic. The UMVUE of the overall lifetime performance index C_L^T can be written as:

$$\widehat{C}_L^T = \sum_{j=1}^{m} \widehat{C}_{Lj} - (m-1), \tag{3.2}$$

where $\widehat{C}_{Lj}$ is the UMVUE of C_{Lj} based on j-th sample by (3.1). The variance of $\widehat{C}_L^T$ can be obtained as (Tong et al. (9)):

$$Var(\widehat{C}_L^T) = \frac{1}{n-2} \sum_{j=1}^{m} \left(\frac{L_j}{\lambda_j}\right)^2, \quad n_j > 2. \tag{3.3}$$

4 An application

In this section, an example is presented to demonstrate the applicability of C_L^T in manufacturing industries. The numerical example is concerned to two-components systems in Murthy et al. (6). The systems consists of two components which their lifetimes X_1 and X_2 are statistically independent and follow a two-parameter Weibull distribution. Suppose each system has a lower specification limit as $L_{X_1} = 9.7456$ and $L_{X_2} = 12.5848$. The failure times of components 1 and 2 for nine system failures and the maximum likelihood estimates of parameters are given in TABLE 3. It is known that the transformed variables $T_1 = X_1^{(1.33)}$ and $T_2 = X_2^{(1.35)}$ follow the exponential distributions with pdf and cdf in (1.2) with parameters $\lambda_1 = 112.05$ and $\lambda_2 = 110.38$, respectively. In addition the transformed lower specification limits are calculated as $L_{T_1} = (9.7456)^{1.33} = 20.6593$ and $L_{T_2} = (12.5848)^{1.35} = 30.5346$, respectively. Hence, the individual and overall indices are estimated as:

$$\widehat{C}_{L1} = 1 - \frac{20.6593}{112.05} = 0.8156, \quad \widehat{C}_{L2} = 1 - \frac{30.5346}{110.38} = 0.7234 \tag{4.1}$$

$$\widehat{C}_L^T = 0.8156 + 0.7234 - (2-1) = 0.5390. \tag{4.2}$$

The results show that for the mentioned products, the overall lifetime-conforming rate $\widehat{P} = 0.6305$ or equivalently 369500 *ppm* of nonconformities are expected.

Table 3: Failure times of components for two-component systems

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	shape	scale
System 1	77.2	74.3	9.6	251.6	134.9	115.7	195.7	42.2	27.8	1.33	112.05
System 2	156.6	108.0	12.4	108.0	84.1	51.2	289.8	59.1	35.5	1.35	110.38

Conclusions

In this paper, an overall lifetime performance index denoted by C_L^T was introduced to assess the performance of a process with multiple lifetime characteristics that are distributed exponentially. The proposed index provides and exact measure of overall process performance and overall lifetime-conforming rate. The UMVUE of the index C_L^T are derived based on complete samples. Finally, an example based on a real data set is presented.

References

[1] Ahmadi, M.V. and Doostparast, M. (2018), Pareto analysis for the lifetime performance index of products on the basis of progressively first-failure-censored batches under balanced symmetric and asymmetric loss functions, *Journal of Applied Statistics*, 1-32.

[2] Bothe, D.R. (1992), A capability study for an entire product, *In ASQC Quality Congress Transactions*, **46**, 172-178.

[3] Chen, K.S., Pearn, W.L. and Lin, P.C. (2003), Capability measures for processes with multiple characteristics, *Quality and Reliability Engineering International*, **19**(2), 101-110.

[4] de-Felipe, D. and Benedito, E. (2017), A review of univariate and multivariate process capability indices, *The International Journal of Advanced Manufacturing Technology*, **92**(5-8), 1687-1705.

[5] Montgomery, D.C. (1985). *Introduction to statistical quality control*, John Wiley & Sons (New York).

[6] Murthy, D.P., Xie, M. and Jiang, R. (2004), *Weibull models*, Vol. 505, John Wiley & Sons.

[7] Pearn, W.L., Shiau, J.J., Tai, Y.T. and Li, M.Y. (2011), Capability assessment for processes with multiple characteristics: a generalization of the popular index Cpk, *Quality and Reliability Engineering International*, **27**(8), 1119-1129.

[8] Pearn, W.L., Wu, C.H. and Tsai, M.C. (2013), A note on capability assessment for processes with multiple characteristics: a generalization of the popular index Cpk, *Quality and Reliability Engineering International*, **29**(2), 159-163.

[9] Tong, L.I., Chen, K.S. and Chen, H.T. (2002), Statistical testing for assessing the performance of lifetime index of electronic components with exponential distribution, *International Journal of Quality* & *Reliability Management*, **19**(7), 812-824.

[10] Wu, C.W., Pearn, W.L. and Kotz, S. (2009), An overview of theory and practice on process capability indices for quality assurance, *International journal of production economics*, **117**(2), 338-359.

Stochastic Properties of Generalize Finite Mixture Models with Dependent Components

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Abstract: The purpose of this talk is to present some stochastic ordering results for the lifetimes of two classical finite mixture models with dependent components in the sense of the hazard rate order and the reversed hazard rate order.

Keywords Stochastic orders, Mixture model, Coherent system, Copula function.

Mathematics Subject Classification (2010) : 60E15**,** 60K10**.**

1 Introduction

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a vector of non-negative dependent and identically distributed (d.i.d) random variables with absolute continuance distribution F, survival function $\overline{F} = 1 - F$ and density function f. The joint survival (or reliability) function of $\mathbf{X}$ has the following form, $\overline{F}_{\mathbf{X}}(\mathbf{x}) = P(X_1 > x_1, \dots, X_n > x_n) = \hat{\mathbb{C}}(\overline{F}(x_1), \dots, \overline{F}(x_n)), where \mathbf{x}=(x_1, \dots, x_n)$ and $\hat{\mathbb{C}}$ is the survival copula, which is the multivariate distribution copula on $[0,1]^n$ with uniformly distributed margins on $[0,1]$. In the literature, $\hat{\mathbb{C}}$ is called a reliability copula (see, Nelsen (2006)). Let $\mathbb{K}_i(\overline{F}(x)) = \hat{\mathbb{C}}(\overline{F}(x)\mathbf{1}_i, \mathbf{1}_{n-i})$ denote i-dimensional margins of the multivariate distribution copula, $\hat{\mathbb{C}}$, where the entries of both $\mathbf{1}_i$ and $\mathbf{1}_{n-i}$ are all ones, with $\mathbb{K}_1(\overline{F}(x)) = \overline{F}(x)$ and $\mathbb{K}_n(\overline{F}(x)) = \hat{\mathbb{C}}(\overline{F}(x), \dots, \overline{F}(x))$. We are defined a survival function of generalize finite mixture models (the mixing proportions may be negative) from i-dimensional margins of $\hat{\mathbb{C}}$ as follows $\overline{H}_{\mathbf{X},\mathbf{a}_n}(\overline{F}(x)) = \sum_{i=1}^{n} a_n(i)\mathbb{K}_i(\overline{F}(x)), where \mathbf{a}_n = (a_n(1), \dots, a_n(n))$ are some real numbers (weights) such that $\sum_{i=1}^{n} a_n(i) = 1$. If all the weights are positive then the classical model (1) reduce to the ordinary positive finite mixture models. If some of the weights are negative then, we have a negative mixture. In (1), if we put $u = \overline{F}(x)$ for all $u \in [0,1]$, then we have $\overline{H}_{\mathbf{a}_n}(u) = \sum_{i=1}^{n} a_n(i)\mathbb{K}_i(u), where \overline{H}_{\mathbf{a}_n}(u)$ is a proper survival function from $[0,1]^n$ to $[0,1]$ and $\overline{H}_{\mathbf{a}_n}(0) = 1$ and $\overline{H}_{\mathbf{a}_n}(1) = 0$. The distribution function corresponding to generalize

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mixture model given as in (1) is $\mathrm{H}_{\mathbf{X},\mathbf{a}_n}(F(x)) = \sum_{i=1}^{n} a_n(i)\mathbb{K}_i(F(x)), where \mathrm{H}_{\mathbf{X},\mathbf{a}_n}(F(x)) = 1 - \overline{H}_{\mathbf{X},\mathbf{a}_n}(F(x))$ and $\mathbb{K}_i(u) = 1 - \overline{\mathbb{K}}_i(1-u)$.

Several researchers have been vastly studied, stochastic comparisons of two mixture models (see Amini-Seresht and Khaledi, 2015; Khaledi and Shaked, 2010; Belzunce et al., 2009; Gupta et al., 2011; Gupta and Gupta, 2009; Li and Da, 2010; Li and Zhao, 2011; Misra et al., 2009; Amini-Seresht and Zhang, 2017). In the present paper, firstly, we consider two statistical models $H_{\mathbf{X},\mathbf{a}_n}$ and $H_{\mathbf{Y},\mathbf{a}_n}$ having the different components and the same weights and obtain the ordering results between them in the sense of the hazard rate, reversed hazard rate and likelihood orders. Next, we consider $H_{\mathbf{X},\mathbf{a}_n}$ and $H_{\mathbf{X},\mathbf{b}_n}$ having the same components and different weights. Several results that compare $H_{\mathbf{X},\mathbf{a}_n}$ and $H_{\mathbf{X},\mathbf{b}_n}$ with respect to various stochastic orders, are established.

There are many statistical models in the literature which are the special cases of the model as given in (1), for example the lifetime distribution of the k-out-of-n systems and coherent systems with dependent components are the special case of such statistical model. The purpose of this paper is to compare two statistical models having distributions of the above form, in the sense of various stochastic orders like, the hazard rate, reversed hazard rate and likelihood ratio orders. For more comprehensive discussions details of the above stochastic orderings, one may refer to Shaked and Shanthikumar (2007) and Müller and Stoyan (2002).

2 Main results

Here, we obtain some general results to compare the statistical models in (1) with the following two cases: two mixture models formed from two sets of random vectors of components, $\mathbf{X}$ and $\mathbf{Y}$ with the same weights and two mixture models formed from of a set of random vector of components, $\mathbf{X}$ with different weights. These results may also be of independent interest.

Let $H_{\mathbf{X},\mathbf{a}_n}$ and $H_{\mathbf{Y},\mathbf{a}_n}$ be two generalize finite mixture models with d.i.d components $\mathbf{X}$ and $\mathbf{Y}$, respectively. If

(i) $\frac{u\overline{H}'_{\mathbf{a}_n}(u)}{\overline{H}_{\mathbf{a}_n}(u)}$ is decreasing in u for all $u \in (0,1)$, and

(ii) $X_1 \leq_{hr} Y_1$,

Then, it holds that $H_{\mathbf{X},\mathbf{a}_n} \leq_{hr} H_{\mathbf{Y},\mathbf{a}_n}$. In the next theorem, we consider the reversed hazard rate order to compare the lifetimes of coherent systems with the different homogeneous dependent component lifetimes. Let $H_{\mathbf{X},\mathbf{a}_n}$ and $H_{\mathbf{Y},\mathbf{a}_n}$ be two generalize finite mixture models with d.i.d components $\mathbf{X}$ and $\mathbf{Y}$, respectively. If

(i) $\frac{(1-u)\overline{H}'_{\mathbf{a}_n}(u)}{1-\overline{H}_{\mathbf{a}_n}(u)}$ is increasing in u for all $u \in (0,1)$, and

(ii) $X_1 \leq_{rh} Y_1$,

Then, it holds that $H_{\mathbf{a}_n,\mathbf{X}} \leq_{rh} H_{\mathbf{a}_n,\mathbf{Y}}$. Next, some sufficient conditions under which two classical models with the same components are compared stochastically with respect to the hazard rate ordering and the reversed hazard rate ordering are provided. Let $H_{\mathbf{X},\mathbf{a}_n}$ and $H_{\mathbf{X},\mathbf{b}_n}$ be two generalize finite mixture models with d.i.d components $\mathbf{X}$ and the vector of weights $\mathbf{a}_n$ and $\mathbf{b}_n$, respectively. If

(i) $\frac{u\overline{\mathbb{K}}'_j(u)}{\overline{\mathbb{K}}_j(u)}$ is increasing in j for all $1 \leq j \leq n$, and

(ii) $a_n(i)b_n(j) \leq a_n(j)b_n(i)$ for all $1 \leq i \leq j \leq n$.

Then, it holds that $H_{\mathbf{X},\mathbf{a}_n} \leq_{hr} H_{\mathbf{X},\mathbf{b}_n}$.

Let $H_{\mathbf{X},\mathbf{a}_n}$ and $H_{\mathbf{X},\mathbf{b}_n}$ be two generalize finite mixture models with d.i.d components $\mathbf{X}$ and the vector of weights $\mathbf{a}_n$ and $\mathbf{b}_n$, respectively. If

(i) $\frac{(1-u)\overline{\mathbb{K}}'_j(u)}{1-\overline{\mathbb{K}}_j(u)}$ is decreasing in j for all $1 \leq j \leq n$, and

(ii) $a_n(i)b_n(j) \leq a_n(j)b_n(i)$ for all $1 \leq i \leq j \leq n$.

Then, it holds that $H_{\mathbf{X},\mathbf{a}_n} \leq_{rh} H_{\mathbf{X},\mathbf{b}_n}$.

References

[1] Amini-Seresht, E. and Khaledi, B-E. (2015), Multivariate stochastic comparisons of mixture models, *Metrika*, **78**, 1015-1034.

[2] Amini-Seresht, E. and Zhang, Y. (2017), Stochastic comparisons on two finite mixture models, *Operations Research Letters*, **45**, 475-480.

[3] Belzunce, F., Mercader, J.A., Ruiz, J.M. and Spizzichino, F. (2009), Stochastic comparisons of multivariate mixture models, *Journal of Multivariate Analysis,* **100**, 1657-1669

[4] Gupta, R.C. (2008), Reliability of a k out of n System of Components Sharing a Common Environment, *Applied Mathematics Letters*, **15**, 837-844.

[5] Müller A. and Stoyan, D. (2002), *Comparison methods for stochastic models and risks*, Chichester, Wiley.

[6] Navarro, J., Guillamon, A. and Ruiz, M.C. (2009), Generalized mixtures in reliability modelling: applications to the construction of bathtub shaped hazard models and the study of systems, *Applied Stochastic Models in Business and Industry*, **25**, 323-337.

[7] Navarro, J. (2016), Stochastic comparisons of generalized mixtures and coherent systems, *TEST*, **25**, 150-169.

[8] Nelsen, R.B. (2006), *An Introduction to Copulas*, Springer, New York.

[9] Shaked, M. and Shanthikumar, J.G. (2007), *Stochastic orders*, Springer, New York.

A Graphical Method to Determine the Maximum Likelihood Estimation of Parameters of the New Pareto-Type Distribution: Complete and Censored Data

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Abstract: In this paper, a graphical method is used to determine the maximum likelihood estimation of parameters of the new Pareto-Type distribution based on complete and censored data. Using this graphical method, we will also discuss the existence and uniqueness of the maximum likelihood estimates.

Keywords New Pareto-Type distribution, Maximum Likelihood Estimation, Graphical Method, Censored Data.

Mathematics Subject Classification (2010) : 62F10**,** 62N01**,** 62N02**.**

1 Introduction

The new Pareto- type distribution was recently proposed by Bourguignon et al. (3) to model reliability and income data. It is a generalization of the well-known Pareto distribution. The two-parameter new Pareto- type distribution (denoted by $NP(\alpha, \beta)$) has the cumulative distribution function (cdf)

$$F(x; \alpha, \beta) = 1 - \frac{2\beta^{\alpha}}{x^{\alpha} + \beta^{\alpha}}, \quad x \geq \beta, \tag{1.1}$$

and the probability density function (pdf)

$$f(x; \alpha, \beta) = \frac{2\alpha\,(\beta/x)^{\alpha}}{x[1 + (\beta/x)^{\alpha}]^2}, \quad x \geq \beta, \tag{1.2}$$

where α and β are shape and scale parameters, respectively.

For the NP(α, β) distribution, the maximum likelihood method does not provide an explicit estimator for the shape parameter based on complete and censored data. For the maximum likelihood estimation (MLE), the corresponding likelihood equation needs to be solved numerically. In this article, we use a simple graphical solution for the determination of the MLE of

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the shape parameter. This graphical approach also shows the existence and uniqueness of the MLEs. Some related work on graphical approach are Dodson (4) and Balakrishnan and Kateri (2) who discussed this method for estimating the parameters of a Weibull distribution.

The paper is organized as follows. In Section 2, the graphical estimation method is described in detail for the case of a complete sample from the NP distribution. In Section 3, the method is extended to censored data (simple and progressive Type II). Finally, in Section 4, we illustrate the method with an example.

2 MLEs for the complete sample case

If $x_1, \cdots, x_n$ is a random sample from $NP(\alpha, \beta)$, then the likelihood function is

$$L(\alpha,\beta) = \prod_{i=1}^{n} f(x_i,\alpha,\beta) = (\frac{2\alpha}{\beta})^n \prod_{i=1}^{n} \frac{(\beta/x_i)^{\alpha+1}}{(1+(\beta/x_i)^{\alpha})^2}. \tag{2.1}$$

The log-likelihood function is

$$\begin{aligned} l(\alpha,\beta) = \ln L(\alpha,\beta) =& n\log(2\alpha) - n\log(\beta) + (\alpha+1)\sum_{i=1}^{n}\log(\frac{\beta}{x_i}) \\ & - 2\sum_{i=1}^{n}\log\left[1+(\frac{\beta}{x_i})^{\alpha}\right] \end{aligned} \tag{2.2}$$

The MLEs of the unknown parameters are obtained by maximizing the log-likelihood function in (2.2) with respect to α and β. It can be seen that $l(\alpha,\beta)$ is monotonically increasing with β. Since $x \geq \beta$, we conclude that the MLE of β is $\widehat{\beta} = x_{(1)}$. Substituting $\widehat{\beta}$ in (2.2), we obtain the profile log-likelihood function of α without the additive constant as

$$\begin{aligned} l(\alpha, x_{(1)}) = & n\log(2\alpha) - n\log(x_{(1)}) + (\alpha+1)\sum_{i=1}^{n}\log(\frac{x_{(1)}}{x_i}) \\ & - 2\sum_{i=1}^{n}\log\left[1+(\frac{x_{(1)}}{x_i})^{\alpha}\right]. \end{aligned} \tag{2.3}$$

Therefore, the MLE of α, say $\widehat{\alpha}$, can be obtained by maximizing (2.3) with respect to α. Consequently, the MLE $\widehat{\alpha}$ of α is obtained as the solution to the following equation

$$h(\alpha) = \frac{\partial l(\alpha, x_{(1)})}{\partial\alpha} = -2\sum_{i=1}^{n}\frac{(\frac{x_{(1)}}{x_i})^{\alpha}\log\left(\frac{x_{(1)}}{x_i}\right)}{1+(\frac{x_{(1)}}{x_i})^{\alpha}} + \sum_{i=1}^{n}\log\left(\frac{x_{(1)}}{x_i}\right) + \frac{n}{\alpha} = 0. \tag{2.4}$$

This equation has to be solved by using some numerical methods such as Newton-Raphson iterative method to compute $\widehat{\alpha}$.

We now use an alternative approach based on a very simple and easy-to-apply graphical method to compute the MLE $\widehat{\alpha}$. We can rewrite (2.4) as

$$\frac{1}{\alpha} = \frac{2}{n}\sum_{i=1}^{n} \frac{(\frac{x_{(1)}}{x_i})^{\alpha}\, \log\left(\frac{x_{(1)}}{x_i}\right)}{1+(\frac{x_{(1)}}{x_i})^{\alpha}} - \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{x_{(1)}}{x_i}\right). \tag{2.5}$$

We denote the RHS of (2.5) by $H(\alpha;x)$ and show that for a given sample $\underline{x} = (x_1, ..., x_n)$, $H(\alpha;x)$ is a monotone increasing function of α with a finite and positive limit as $\alpha \to \infty$. We have

$$\frac{\partial H(\alpha;x)}{\partial \alpha} = \frac{2}{n}\sum_{i=1}^{n} \frac{(\frac{x_{(1)}}{x_i})^{\alpha}\, \log^2\left(\frac{x_{(1)}}{x_i}\right)}{\left(1+(\frac{x_{(1)}}{x_i})^{\alpha}\right)^2} \geq 0 \tag{2.6}$$

which establishes the required property that $H(\alpha, x)$ is indeed a monotone increasing function of α. Further, it can be shown that

$$\lim_{\alpha\to\infty} H(\alpha,x) = -\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{x_{(1)}}{x_i}\right) > 0,$$

$$\lim_{\alpha\to 0} H(\alpha,x) = 0,$$

and

$$\lim_{\alpha\to 0} H(\alpha,x) < \lim_{\alpha\to\infty} H(\alpha,x).$$

Therefore, a plot of the LHS and the RHS of (2.5) gives a simple graphical method of determining the MLE of the shape parameter α. The above three equations, combined with the fact that $1/\alpha$ is monotone decreasing (to 0) and $H(\alpha, x)$ is monotone increasing, ensures the existence and uniqueness of the MLE of α.

3 MLEs for the censored sample case

Let $X_{1:n} < X_{2:n} < \cdots < X_{m:n}$ be a Type- II right censored sample from the NP distribution, where m $(1 < m < n)$ is the number of observed failures. Then, the log-likelihood becomes

$$\begin{aligned} l = \ln L(x,\alpha,\beta) = const + m\log\alpha - m\log\beta + (n-m)\log\left[\frac{2\beta^{\alpha}}{x_{m:n}^{\alpha}+\beta^{\alpha}}\right] \\ + (\alpha+1)\sum_{i=1}^{m}\log(\frac{\beta}{x_{i:n}}) - 2\sum_{i=1}^{m}\log(1+(\frac{\beta}{x_{i:n}})^{\alpha}) \end{aligned} \tag{3.1}$$

It can be shown that $l(\alpha, \beta)$ is monotonically increasing with β, thus the MLE of β, is $\hat{\beta} = x_{1:n}$. The MLE $\hat{\alpha}$ of α is obtained as the solution to the following equation

$$\begin{aligned}\frac{\partial l}{\partial \alpha} =& \frac{m}{\alpha} + \sum_{i=1}^{m} \log(\frac{x_{1:n}}{x_{i:n}}) + (n-m)\frac{\log(\frac{x_{1:n}}{x_{m:n}})}{1+(\frac{x_1}{x_{m:n}})^{\alpha}} \\ &- 2\sum_{i=1}^{m} \frac{(\frac{x_{1:n}}{x_{i:n}})^{\alpha}\log(\frac{x_{1:n}}{x_{i:n}})}{1+(\frac{x_{1:n}}{x_{i:n}})^{\alpha}} = 0.\end{aligned} \tag{3.2}$$

There is no closed- form expression for the MLE of α and its computation has to be performed numerically. Again, we can use the graphical method to compute the MLE $\hat{\alpha}$. In this case, we have

$$\frac{1}{\alpha} = \frac{2}{m}\sum_{i=1}^{m} \frac{(\frac{x_{1:n}}{x_{i:n}})^{\alpha}\log(\frac{x_{1:n}}{x_{i:n}})}{1+(\frac{x_{1:n}}{x_{i:n}})^{\alpha}} - \frac{1}{m}\sum_{i=1}^{m}\log(\frac{x_{1:n}}{x_{i:n}}) - \frac{(n-m)}{m}\frac{\log(\frac{x_{1:n}}{x_{m:n}})}{1+(\frac{x_{1:n}}{x_{m:n}})^{\alpha}} \tag{3.3}$$

Therefore, in this case,

$$H(\alpha; \mathbf{x}) = \frac{2}{m}\sum_{i=1}^{m} \frac{(\frac{x_{1:n}}{x_{i:n}})^{\alpha}\log(\frac{x_{1:n}}{x_{i:n}})}{1+(\frac{x_{1:n}}{x_{i:n}})^{\alpha}} - \frac{1}{m}\sum_{i=1}^{m}\log(\frac{x_{1:n}}{x_{i:n}}) - \frac{(n-m)}{m}\frac{\log(\frac{x_{1:n}}{x_{m:n}})}{1+(\frac{x_{1:n}}{x_{m:n}})^{\alpha}}. \tag{3.4}$$

The function $H(\alpha; \mathbf{x})$ is a monotone increasing function of α for a given sample $\mathbf{x}$, since

$$\frac{\partial H(\alpha, x)}{\partial \alpha} = \frac{2}{m}\sum_{i=1}^{m} \frac{(\frac{x_{1:n}}{x_{i:n}})^{\alpha}\log^2(\frac{x_{1:n}}{x_{i:n}})}{\Big(1+(\frac{x_{1:n}}{x_{i:n}})^{\alpha}\Big)^2} + \frac{(n-m)}{m}\frac{(\frac{x_{1:n}}{x_{m:n}})^{\alpha}\log^2(\frac{x_{1:n}}{x_{m:n}})}{\Big(1+(\frac{x_{1:n}}{x_{m:n}})^{\alpha}\Big)^2} \geq 0. \tag{3.5}$$

Moreover, we have

$$\lim_{\alpha\to\infty} H(\alpha; x) = -\frac{1}{m}\sum_{i=1}^{m}\log(\frac{x_{1:n}}{x_{i:n}}) - \frac{(n-m)}{m}\log(\frac{x_{1:n}}{x_{m:n}}) > 0,$$

$$\lim_{\alpha\to 0} H(\alpha; x) = -\frac{(n-m)}{2m}\log(\frac{x_{1:n}}{x_{m:n}}) > 0,$$

and

$$\lim_{\alpha\to 0} H(\alpha; x) < \lim_{\alpha\to\infty} H(\alpha; x).$$

Thus, again a plot of the LHS and the RHS of (3.3) gives a simple graphical method to determine the MLE $\hat{\alpha}$. This plot also shows the existence and uniqueness of the MLE $\widehat{\alpha}$.

In case of progressive Type- II censoring, let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$, $(1 \leq m \leq n)$ be a progressive Type-II censored sample observed from a life test involving n units taken from the NP distribution and $(R_1, \ldots, R_m)$, where each $R_i \geq 0$ and $\sum_{i=1}^{m} R_i = n - m$, is the censoring scheme. For notation simplicity, we denote the observed progressively Type-II censored sample as $x_{1:n}, x_{2:n}, \dots, x_{m:n}$. The log- likelihood function in case of NP distribution is

$$
\begin{aligned}
l = const + m \log \alpha - m \log \beta + (\alpha+1) \sum_{i=1}^{m} \log(\frac{\beta}{x_{i:n}}) \\
- 2 \sum_{i=1}^{m} \log(1 + (\frac{\beta}{x_{i:n}})^{\alpha}) + \sum_{i=1}^{m} R_i \log\left(\frac{2(\frac{\beta}{x_{i:n}})^{\alpha}}{1 + (\frac{\beta}{x_{i:n}})^{\alpha}}\right),
\end{aligned} \tag{3.6}
$$

which gives $\hat{\beta} = x_{1:n}$ and

$$
\begin{aligned}
H(\alpha, \mathbf{x}) = \frac{2}{m} \sum_{i=1}^{m} \frac{(\frac{x_{1:n}}{x_{i:n}})^{\alpha} \log(\frac{x_{1:n}}{x_{i:n}})}{1 + (\frac{x_{1:n}}{x_{i:n}})^{\alpha}} - \frac{1}{m} \sum_{i=1}^{m} \log(\frac{x_{1:n}}{x_{i:n}}) \\
- \frac{1}{m} \sum_{i=1}^{m} R_i \frac{\log(\frac{x_{1:n}}{x_{i:n}})}{1 + (\frac{x_{1:n}}{x_{i:n}})^{\alpha}}.
\end{aligned} \tag{3.7}
$$

In this case, again we have

$$
\frac{\partial H(\alpha, x)}{\partial \alpha} = \frac{2}{m} \sum_{i=1}^{m} \frac{(\frac{x_{1:n}}{x_{i:n}})^{\alpha} \log^2(\frac{x_{1:n}}{x_{i:n}})}{\left(1 + (\frac{x_{1:n}}{x_{i:n}})^{\alpha}\right)^2} + \frac{1}{m} \sum_{i=1}^{m} R_i \frac{(\frac{x_{1:n}}{x_{i:n}})^{\alpha} \log^2(\frac{x_{1:n}}{x_{i:n}})}{\left(1 + (\frac{x_{1:n}}{x_{i:n}})^{\alpha}\right)^2} \geq 0, \tag{3.8}
$$

$$
\lim_{\alpha \to \infty} H(\alpha, x) = -\frac{1}{m} \sum_{i=1}^{m} \log(\frac{x_{1:n}}{x_{i:n}}) - \frac{1}{m} \sum_{i=1}^{m} R_i \log(\frac{x_{1:n}}{x_{i:n}}) > 0,
$$

$$
\lim_{\alpha \to 0} H(\alpha, x) = -\frac{1}{2m} \sum_{i=1}^{m} R_i \log(\frac{x_{1:n}}{x_{i:n}}) > 0,
$$

and $\lim_{\alpha \to 0} H(\alpha, x) < \lim_{\alpha \to \infty} H(\alpha, x)$. Therefore, the plot of the LHS and the RHS of the equation $1/\alpha = H(\alpha, x)$ gives a simple graphical method of determining the MLE $\hat{\alpha}$. Moreover, by arguments as before, the existence and uniqueness of the MLE of α is ensured.

4 Example

In this section, we illustrate the graphical estimation method discussed in this paper with an example. The following data set (see Table 1) from Murthy et al. (5) represents the failure times of 20 mechanical components. This data set has been analyzed recently by Bourguignon et al. (3). They showed that the use of the NP distribution for fitting this data set is reasonable. They computed the MLEs of α and β as $\hat{\alpha} = 2.871$ and $\hat{\beta} = 0.067$. The MLEs determined by the R software (using "uniroot" method) are as $\hat{\alpha} = 2.97$ and $\hat{\beta} = 0.067$.

The graphical estimation method described in Section 2 leads to a graphical solution of $\hat{\alpha} = 2.99$ (shown in Figure 1) and $\hat{\beta} = 0.067$.

Let us know consider the case of censored data. For the case of Type-II censoring, it is supposed that the life test ended when the 8-th observation is observed. Therefore, we observe

Table 1: The Failure Times Data

DATA SET	0.067	0.068	0.076	0.081	0.084	0.085	0.085	0.086	0.089	0.098
	0.098	0.114	0.114	0.115	0.121	0.125	0.131	0.149	0.160	0.485

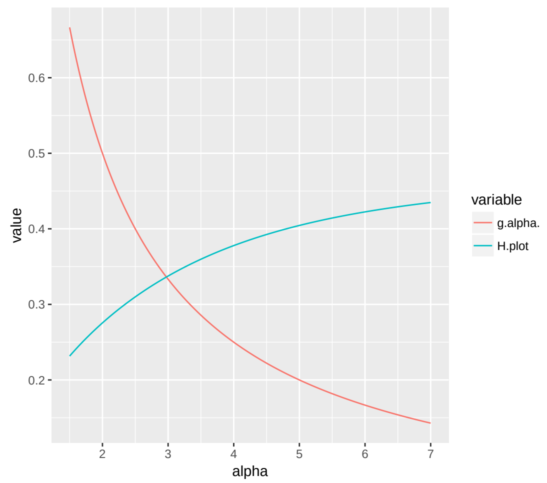

Figure 1: Plot of the $1/\alpha$ and $H(\alpha, x)$ functions for the complete data.

a Type II censored sample with $n = 20$ and $m = 8$. For the case of progressive Type-II censoring, we consider $m = 8$ and $R = (2, 0, 3, 0, 0, 2, 0, 5)$. We then generated a progressive Type-II censored sample using the algorithm presented in Balakrishnan and Cramer (1). Table 2 shows the generated progressive Type-II censored sample.

The graphical estimation procedure described in Section 3 leads to a graphical solution of $\hat{\alpha} = 3.15$ for Type-II censored data and $\hat{\alpha} = 3.11$ for progressive Type-II censored data (shown in Figures 2 and 3).

Table 2: Progressive Type-II censored data

i	1	2	3	4	5	6	7	8
$x_{i:n}$	0.067	0.068	0.076	0.081	0.084	0.085	0.089	0.098
R_i	2	0	3	0	0	2	0	5

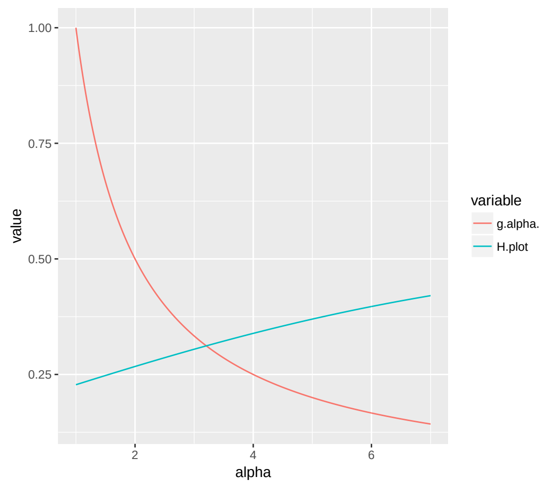

Figure 2: Plot of the $1/\alpha$ and $H(\alpha, x)$ functions for the Type II censored data.

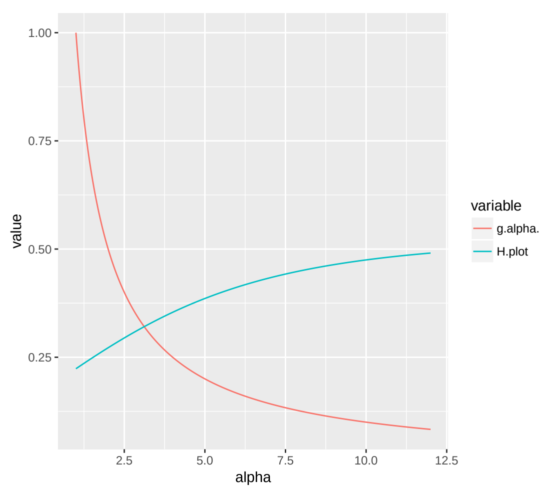

Figure 3: Plot of the $1/\alpha$ and $H(\alpha, x)$ functions for the progressive censored data.

References

[1] Balakrishnan, N. and Cramer, E. (2014), *The art of progressive censoring*, Springer, Berlin.

[2] Balakrishnan, N. and Kateri, M. (2008), On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data, *Statistics and Probability Letters*, **78**(17), 2971-2975.

[3] Bourguignon, M., Saulo, H. and Fernandez, R.N. (2016), A new Pareto-type distribution

with applications in reliability and income data, *Physica A*,**A-457**, 166-175.

[4] Dodson, B. (2006), *The Weibull Analysis Handbook*, 2nd ed.ASQ Quality Press, Milwaukee.

[5] Murthy, D.N.P., Xie, M. and Jiang, R. (2004), *Weibull Models*, Hoboken: John Wiley and Sons, Inc.

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Preventive Maintenance for Systems Subject to Marshal-Olkin Type Shock Models

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Abstract: In this paper, we consider a binary system that is subject to Marshall-Olkin type shocks. We study an age-based preventive maintenance model for this system. The optimal preventive maintenance time that minimizes the mean cost per unit of time is investigated. The efficiency of the proposed model is computed. some examples are illustrated as the applications of the proposed model.
Keywords Reliability, Optimal PM time, Emergency repair.

1 Introduction

In real-world, there are many situations in which a technical system which operates in an environment may be subject to shocks that cause a reduction in the performance of the system. For example, earthquakes may affect the road networks or changes in power voltage may affect the electrical systems. Motivated by this, many researchers in recent years studied different scenarios that the failure of the systems occurs based on shocks models. A well-known type of shock models that are considered by authors in the literature is the Marshall-Olkin (MO) shock models. In the classical MO shock model, a system consisting of two components is assumed to be subject to shocks that arrive from three different sources. A shock from the first source affects the first component, the shock from the second source affects the second component and a shock from the third source affects both components. Marshall and Olkin (5) assumed that the times of occurring the shocks in each source are independent of the exponential distribution and derived the joint reliability function of the components of the system. After their work, the extensions of MO shock models are considered in numerous papers. Recently, Bayramoglu and Ozkut (2) investigated the systems that are subject to MO type of shocks. They considered systems composed of n components which are subject to shocks that arrive from different sources at random times. A shock coming at random time T_i, $i = 1, \ldots, n$ destroys the ith component;

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and the shock coming at time $T_{1\dots n}$ destroys all components. They computed the reliability function of the system under different assumptions on the random times of occurring shocks. They also, extended the MO model for a system subjected to shocks coming at random times and destroying one, two, three, or more components. Bayramoglu and Ozkut (3) studied the mean residual life and the mean inactivity time functions of coherent systems subject to MO shock models. Matus et. al (6) proposed an optimization approach to define the shocks parameters in the MO shock model in order to obtain the correlations between the failure times of the components of the system. Ozkut and Eryilmaz (8) studied a shock model which combines MO and run shock models.

The preventive maintenance (PM) is one of the noteworthy areas in the reliability engineering. The PM policy is used to care and preserve the systems and avoid the sudden or gradual failure of the system that imposes some costs. The aim of PM policy is to provide a systematic model for inspecting the system at predetermined times to fix and correct the primitive failures. The optimal PM models first introduced by Barlow and Hunter (1). After this work, many papers and books devoted to PM problems; see, for examples, (4), (7) and (9).

In this paper, we consider a binary $(n-k+1)$-out-of-n system. We assume that the components of this system are subject to shocks that come from $(n+1)$ different sources. A shock coms from ith source affects only on the ith component, $i = 1, \dots, n$ and destroys it. The shock that arrives from $(n+1)$th source affects on all components and distroys them. The aim of this paper is to present an optimal PM policy for such a system. We consider three types of costs that the system may impose, the cost of replacing a failed component with new one, the cost of applying emergency repair (ER), the cost for employing PM action. The mean cost per unit of time is obtained and the optimal PM time that minimizes it is investigated. We also compute the efficiency of applying PM. Some illustration examples are also provided.

2 Preventive maintenance model

Consider an $(n-k+1)$-out-of-n system. Assume that the components of system are subject to shocks that come from independent sources at random times $T_1, \dots, T_n, T_{1,\dots,n}$. The shock that occurs at time T_i destroys ith component, $i = 1, \dots, n$ and the shock that occurs at $T_{1,\dots,n}$ destroys all the components. In such a situation, the lifetime of the ith component is $X_i = \min\{T_i, T_{1,\dots,n}\}$ and the lifetime of the system is $T = X_{k:n} = \min\{T_{k:n}, T_{1,\dots,n}\}$ where $T_{k:n}$ is the kth order statistics among $T_1, \dots, T_n$. Let $T_1, \dots, T_n$ be exchangeable random variables with reliability function $\bar{F}_{T_1}(t)$ and $T_{1,\dots,n}$ have reliability function $\bar{F}_{T_{1\dots n}}(t)$. Suppose that $T_{1,\dots,n}$ is independent of $T_1,\dots,T_n$ and $P(T_{1,\dots,n}=T_i)=0$, $i=1,\dots,n$. In this following, we investigate a preventive maintenance policy for this system.

Assume that the PM is performed at t_{PM} or emergency repair (ER) is performed at failure time of the system, whichever comes first. Suppose that at t_{PM} or at the failure time of the system, that is a renewal cycle, the failed components are replaced with new ones and the system becomes as good as new one. Suppose that c_0 denotes the cost of replacing a failed component with new one, c_{ER} and c_{PM} are the costs of applying ER and PM, respectively. In what follows, we obtain the mean cost and the mean length of one renewal cycle. Then, we compute the optimal PM time t^*_{PM} that minimizes the mean cost per unit of time. In computing the mean cost per cycle one of the following cases may happen.

(I) Assume the system has not failed until t_{PM}. In such a situation, it is clear that the shock that destroys all components comes after t_{PM},($T_{1,\dots,n}>t_{PM}$) and $T_{i:n}\le t_{PM}<T_{i+1:n}$, $i=0,1,\dots,n$. Thus, at time t_{PM}, i components have failed. The expected cost can be obtained as

$$\begin{aligned}
C_1(t_{PM}) &= \sum_{i=1}^{n} P(T_{i:n}\le t_{PM}<T_{i+1:n},T>t_{PM})(ic_0+c_{PM})\\
&= \sum_{i=1}^{k-1} P(T_{i:n}\le t_{PM}<T_{i+1:n},T_{k:n}>t_{PM},T_{1\dots n}>t_{PM})(ic_0+c_{PM})\\
&= \sum_{i=1}^{k-1} P(T_{i:n}\le t_{PM}<T_{i+1:n})P(T_{1\dots n}>t_{PM})(ic_0+c_{PM})\\
&= \bar{F}_{T_{1\dots n}}(t_{PM})\sum_{i=1}^{k-1}\binom{n}{i}F^i_{T_1}(t_{PM})\bar{F}^{n-i}_{T_1}(t_{PM})(ic_0+c_{PM}).
\end{aligned}$$

in which the third equality follows from the fact that $T_{1,\dots,n}$ is independent of $T_1,\dots,T_n$.

(II) Assume that the system has failed before t_{PM}. In such a situation if $T=T_{1\dots n}$ then the number of failed components is n. Therefore, the mean cost can be obtained as

$$\begin{aligned}
C_2(t_{PM}) &= P(T\le t_{PM},T=T_{1\dots n})(nc_0+c_{ER})\\
&= P(T_{1\dots n}\le t_{PM},T_{1,\dots,n}<T_{k:n})(nc_0+c_{ER})\\
&= (nc_0+c_{ER})\int_0^\infty F_{T_{1\dots n}}(\min\{t,t_{PM}\})f_{T_{k:n}}(t)dt
\end{aligned}$$

If $T = T_{k:n}$ then one can get the mean cost as

$$\begin{aligned} C_3(t_{PM}) &= P(T \le t_{PM}, T = T_{k:n})(kc_0 + c_{ER}) \\ &= P(T_{k:n} \le t_{PM}, T_{k:n} < T_{1\ldots n})(kc_0 + c_{ER}) \\ &= (kc_0 + c_{ER}) \int_0^\infty F_{T_{k:n}}(\min\{t, t_{PM}\}) f_{T_{1\ldots n}}(t) dt. \end{aligned}$$

Therefore, mean cost per cycle is written as

$$\begin{aligned} C(t_{PM}) &= C_1(t_{PM}) + C_2(t_{PM}) + C_3(t_{PM}) \\ &= \bar{F}_{T_{1\ldots n}}(t_{PM}) \sum_{i=1}^{k-1} \binom{n}{i} F_{T_1}^i(t_{PM}) \bar{F}_{T_1}^{n-i}(t_{PM})(ic_0 + c_{PM}) \\ &+ (nc_0 + c_{ER}) \int_0^\infty F_{T_{1\ldots n}}(\min\{t, t_{PM}\}) f_{T_{k:n}}(t) dt \\ &+ (kc_0 + c_{ER}) \int_0^\infty F_{T_{k:n}}(\min\{t, t_{PM}\}) f_{T_{1\ldots n}}(t) dt, \end{aligned}$$

where for exchangeable random variables $T_1, \ldots, T_n$, using Theorem 1 of Bayramoglu and Ozkut (2),

$$\bar{F}_{T_{k:n}}(t) = \sum_{j=n-k+1}^{n} \binom{n}{j} \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r} P(T_1 > t, \ldots, T_{j+r} > t).$$

If $T_1, \ldots, T_n$ are i.i.d. it can be easily seen that $P(T_1 > t, \ldots, T_{j+r} > t) = \bar{F}_{T_1}^{j+r}(t)$. The length of one cycle can be derived as

$$\begin{aligned} L(t_{PM}) &= \int_0^{t_{PM}} P(T > t) dt \\ &= \int_0^{t_{PM}} \bar{F}_{T_{1\ldots n}}(t) \bar{F}_{T_{k:n}}(t) dt. \end{aligned}$$

Therefore, the mean cost per unit of time in each renewal cycle can be achieved as

$$\mu(t_{PM}) = \frac{C(t_{PM})}{L(t_{PM})}.$$

The aim is to obtain t_{PM}^* that minimizes $\mu(t_{PM})$.

Without PM, the mean cost is obtained as

$$C = (nc_0 + c_{ER}) \int_0^\infty F_{T_{1\ldots n}}(t) f_{T_{k:n}}(t) dt + (kc_0 + c_{ER}) \int_0^\infty F_{T_{k:n}}(t) f_{T_{1\ldots n}}(t) dt.$$

Then, without PM, the mean cost per unit of time is written as

$$\mu(\infty) = \frac{C}{\int_0^\infty \bar{F}_{T_{1\ldots n}}(t) \bar{F}_{T_{k:n}}(t) dt}.$$

The efficiency of the proposed PM model is obtained as

$$\eta = \frac{\mu(\infty)}{\mu(t_{PM}^*)}.$$

3 Illustrative examples

In this section, we illustrate the proposed PM model using some examples. In following, we assume an $(n-k+1)$-out-of-n system when $n = 10$ and suppose that $T_1, \ldots, T_{10}$ are i.i.d. from Weibull distribution with survival function $\bar{F}_{T_1}(t) = e^{-0.1t^{a_1}}$ and $T_{1,\ldots,10}$ have Weibull distribution with survival function $\bar{F}_{T_{1,\ldots,10}}(t) = e^{-0.5t^{a_2}}$. Two cases are discussed. In the first case, we assume that a_1 and a_2 are fixed and k, c_{PM} and c_{ER} have different values. In the second case, it is assumed that a_1 and a_2 get different values while k, c_{PM} and c_{ER} are considered fix.

The table 1, presents the optimal times to apply PM policy and its efficiencies when $a_1 = a_2 = 2$ and $c_0 = 1$ for different values of k, c_{PM} and c_{ER}. It can be seen from the table that when c_{PM} increases, while c_{ER} and k are fixed, t^*_{PM} increases and the efficiency decreases. This is so because by increasing c_{PM} the system needs PM later. Also, it can be seen that when c_{ER} increases, while c_{PM} and k are fixed, t^*_{PM} decreases and the efficiency increases; i.e. by increasing c_{ER}, the system needs PM earlier. By increasing k, when k is small enough, for fixed values of c_{PM} and c_{ER}, t^*_{PM} increases. When k is not small, by increasing the value of k, t^*_{PM} does not change significantly.

In this example, the common distribution of $T_1, \ldots, T_{10}$ and also the distribution of $T_{1,\ldots,10}$ are IFR. Thus the distribution of lifetime of the system is also IFR because $P(T > t) = P(T_{1,\ldots,10} > t)P(T_{k:n} > t)$. As expected, it can be seen from table 1 that the efficiencies of applying PM are significant. Therefore, we should apply the PM policy, for different values of costs.

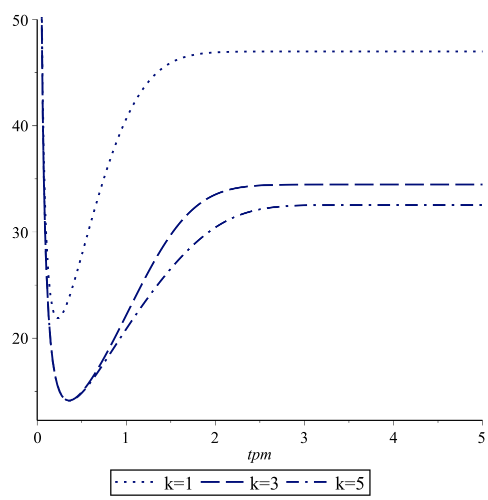

Figure 1: The plots of mean cost per unit of time when $c_0 = 1$, $c_{PM} = 2.5$ and $c_{ER} = 30$

Table 1: Optimal PM times for $(n-k+1)$-out-of-n system when $n=10$, and $c_0=1$.

k	c_{PM}	c_{ER}	$E(T)$	t^*_{PM}	$C(t^*_{PM})$	$\eta(t^*_{PM})$
1	2.5	15	0.7236	0.3219	15.9335	1.64794
		30		0.2315	21.8819	2.1473
	4	15		0.4312	19.4056	1.3531
		30		0.3015	27.1346	1.7316
3	2.5	15	1.1070	0.4541	11.1606	1.8740
		30		0.3565	14.1319	2.4388
	4	15		0.5911	13.7799	1.5178
		30		0.4583	17.6161	1.9564
5	2.5	15	1.21599	0.4592	11.1355	1.8153
		30		0.3587	14.1146	2.3061
	4	15		0.6072	13.7035	1.4751
		30		0.4653	17.5614	1.8535
7	2.5	15	1.2463	0.4592	11.1354	1.7973
		30		0.3587	14.1145	2.2706
	4	15		0.6073	13.7033	1.4605
		30		0.4653	17.5613	1.8250

Figure 1 depicts the plots of mean cost per unit of time when $c_0 = 1$, $c_{PM} = 2.5$ and $c_{ER} = 30$ for different values of k.

Consider a 4-out-of-10 system. Table 2 presents the optimal times for applying PM when $c_0 = 1$, $c_{PM} = 4$ and $c_{ER} = 40$ for different values of a_1 and a_2. The first row is related to the case that T_i's are DFR ($a_1 = 0.5$). In this case when $T_{1,\ldots,10}$ is DFR ($a_2 = 0.5$) or has exponential distribution ($a_2 = 1$), the system does not need the PM. If $T_{1,\ldots,10}$ is IFR ($a_2 = 1.5$), the system needs PM. From the second row, when $T_{1,\ldots,10}$ is DFR ($a_2 = 0.5$ or $a_2 = 1$), the efficiency of applying PM is not significant. The third row shows that the system needs PM when T_1 is IFR.

Figure 2 depicts the plots of mean cost per unit of time when $c_0 = 1$, $c_{PM} = 4$, $c_{ER} = 40$ and $a_1 = 1.5$ for different values of a_2.

Table 2: Optimal PM times for 4-out-of-10 system when $c_0 = 1$, $c_{PM} = 4$ and $c_{ER} = 40$.

a_1	a_2	$E(T)$	t^*_{PM}	$\eta(t^*_{PM})$
0.5	0.5	7.4030	∞	1
	1	1.9994	∞	1
	1.5	1.4330	0.5464	1.3558
1	0.5	3.8110	14.7808	1.0012
	1	1.9627	7.5658	1.0006
	1.5	1.43197	0.5143	1.3733
1.5	0.5	2.3802	3.8046	1.0933
	1	1.7866	2.8777	1.047
	1.5	1.4202	0.4976	1.3969

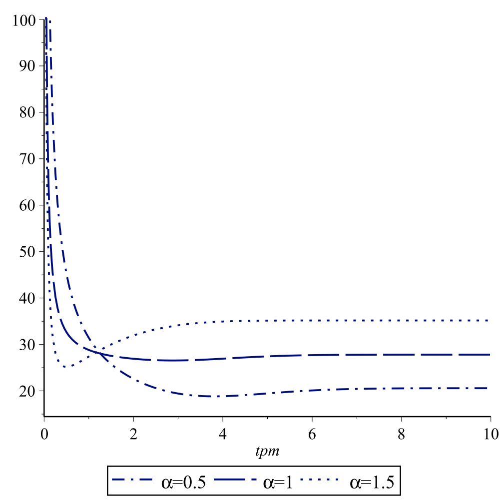

Figure 2: The plots of mean cost per unit of time when $c_0 = 1$, $c_{PM} = 4$ and $c_{ER} = 40$

References

[1] Barlow, R., and Hunter, L. (1960), Optimum preventive maintenance policies. *Operations research*, **8**(1), 90-100.

[2] Bayramoglu, I. and Ozkut, C.M., (2015), The Reliability of Coherent Systems Subjected to Marshall-Olkin Type Shocks, *IEEE Transactions on Reliability*, **64**(1), 435-443.

[3] Bayramoglu, I. and Ozkut, M., (2016), Mean Residual Life and Inactivity Time of a Coherent

System Subjected to Marshall-Olkin Type Shocks, *Journal of Computational and Applied Mathematics*, **298**, 190-200.

[4] Gertsbakh, I. (2000), *Reliability theory: with applications to preventive maintenance*, Springer.

[5] Marshall A.W. and Olkin I. (1967), A Multivariate Exponential Distribution, *Journal of the American Statistical Association*, **62**, 30-44.

[6] Matus, O., Barrera, J., Moreno, E. and Rubino, G., (2018), On the Marshall-Olkin Copula Model for Network Reliability Under Dependent Failures, *IEEE Transactions on Reliability.*

[7] Nakagawa, T. (2014), *Random maintenance policies*, London, Springer.

[8] Ozkut, M. and Eryilmaz, S., (2019), Reliability Analysis under Marshall-Olkin Run Shock Model, *Journal of Computational and Applied Mathematics*, **349**, 52-59.

[9] Wang, H., and Pham, H. (2006), *Reliability and optimal maintenance*, Springer Science & Business Media.

Stochastic Comparisons of Series and Parallel Systems From Heterogeneous Log-logistic Random Variables with Archimedean Copula

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Abstract: This paper studies the usual stochastic, star, convex transform orders of both series and parallel systems comprised of heterogeneous (and dependent) Log-logistic components. Sufficient conditions are established for the star ordering between the lifetimes of series and parallel systems consisting of dependent components having multiple-outlier Log-logistic model. Under certain conditions on Archimedean copula and the parameter, we also discuss convex transform order between the series and parallel systems. These results generalize some corresponding ones in the literature to the case of dependent scenarios.

Keywords Stochastic orders, Log-logistic distribution, Series systems, Parallel systems, Weak supermajorization order.

1 Introduction

The Log-logistic distribution is a flexible family of distributions which has been considered extensively in reliability and survival analysis. A random variable X is said to have the Log-logistic distribution distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$ (denoted by $X \sim L - Log(\alpha, \beta)$) if its cumulative distribution and sensity function are F(x; $\alpha, \beta) = \frac{x^\beta}{\alpha^\beta + x^\beta}$ $\quad x > 0$,to

f(x; $\alpha, \beta) = \frac{\frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}}{(1+(\frac{x}{\alpha})^\beta)^2}$ $\quad x > 0$.Log-logistic distribution plays a important role in survival analysis dealing with data sets. The data may be the survival times of cancer patients in which the hazard rate increases at the beginning and decreases later. In this direction we refer to Bennett (1983). In economics, it is usually known as Fisk distribution (see Fisk (1961)) and is applied as an alter- native to the log-normal distribution. We refer to Ahmad et al. (1988), Robson and Reed (1999) and Geskus (2001) for further details of the importance and applications of log-logistic distribution.

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One of the most commonly used systems in reliability is an r-out-of-n system. This system comprising of n components, works iff at least r components work, and it includes parallel, fail-safe and series systems all as special cases corresponding to $r = 1$, $r = n - 1$ and $r = n$, respectively. Let $X_1, \cdots, X_n$ denote the lifetimes of components of a system and $X_{1:n} \leq \cdots \leq X_{n:n}$ represent the corresponding order statistics. Then, $X_{n-r+1:n}$ corresponds to the lifetime of a r-out-of-n system. Due to this direct connection, the theory of order statistics becomes quite important in studying $(n - r + 1)$-out-of-n systems and in characterizing their important properties.

The comparison of important characteristics associated with lifetimes of technical systems is an interesting topic in reliability theory, since it usually enables us to approximate complex systems with simpler systems and subsequently obtaining various bounds for important ageing characteristics of the complex system. A convenient tool for this purpose is the theory of stochastic orderings. Stochastic comparisons of series and parallel systems with heterogeneous components have been discussed extensively for the various lifetimes. We refer the readers to Kochar and Xu (2014), Li and Li (2015), Li and Fang (2015), Fang et al. (2016), Amini-Seresht et al. (2016), Barmalzan et al. (2017), Ding et al. (2017), Zhang et al. (2018) for detailed discussions on this topic.

The rest of this paper is organized as follows. Section 2 reviews some basic concepts that will be used in the sequel. In Section 3, we discuss the usual stochastic order of series or parallel systems with heterogeneous and dependent Log-logistic components. Finally, the convex transform and star orders of series or parallel systems are discussed in Section 4.

2 Preliminaries

2.1 Stochastic orders

Suppose X and Y are two non-negative random variables with distribution functions F_X and F_Y, survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, right continuous inverses (quantile functions) F_X^{-1} and F_Y^{-1}, hazard rates $r_X = f_X/\bar{F}_X$ and $r_Y = f_Y/\bar{F}_Y$, respectively. Suppose X and Y are two non-negative continuous random variables. X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all $x \in \mathbb{R}^+$. This result is equivalent to saying that $\mathbb{E}(\phi(X)) \leq \mathbb{E}(\phi(Y))$ for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$ when the involved expectations exist.

(i) X is said to be smaller than Y in the convex transform order (denoted by $X \leq_c Y$) if $F_Y^{-1}F_X(x)$ is convex in $x \geq 0$. Equivalently, $X \leq_c Y$ if and only if $F_X^{-1}F_Y(x)$ is concave in $x \geq 0$;

(ii) X is said to be smaller than Y in the star order (denoted by $X \leq_* Y$) if $F_Y^{-1}F_X(x)/x$ is increasing in $x \geq 0$.

X is said to be smaller than Y in the dispersive order (denoted by $X \leq_{disp} Y$) if $\mathrm{F}_X^{-1}(\beta) - F_X^{-1}(\alpha) \leq F_Y^{-1}(\beta) - F_Y^{-1}(\alpha) \quad \text{for} \quad 0 \leq \alpha < \beta \leq 1, orequivalently,$ X$\leq_{disp} Y$ if and only if $f_Y(F_Y^{-1}(F_X(x))) \leq f_X(x)$ for all $x > 0$. Interested readers may refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for comprehensive discussions on various stochastic orderings and relations between them.

2.2 Majorization order

Consider two vectors $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ with the increasing arrangements $a_{(1)} \leq \cdots \leq a_{(1)}$ and $b_{(1)} \leq \cdots \leq b_{(1)}$, respectively. Then:

(i) A vector $\mathbf{a}$ is said to be majorized by the vector $\mathbf{b}$ (denoted by $\mathbf{a} \overset{m}{\preceq} \mathbf{b}$) if $\sum_{j=1}^{i} a_{(j)} \geq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \cdots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$;

(ii) A vector $\mathbf{a}$ is said to be weakly supermajorized by the vector $\mathbf{b}$ (denoted by $\mathbf{a} \overset{w}{\preceq} \mathbf{b}$) if $\sum_{j=1}^{i} a_{(j)} \geq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \cdots, n$.

A real-valued function ϕ, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$, is said to be Schur-convex (Schur-concave) on $\mathbb{A}$ if $\mathbf{a} \overset{m}{\preceq} \mathbf{b}$ implies $\phi(\mathbf{a}) \leq (\geq)\phi(\mathbf{b})$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{A}$. For an elaborate discussion on the theory of majorization and Schur functions, one may refer to the book by Marshall et al. (2011). Necessary and sufficient conditions for the characterization of Schur-convex and Schur-concave functions are as given in the following lemma.

(Marshall et al. (2011), p. 84) Suppose $J \subset \mathbb{R}$ is an open interval and $\phi : J^n \to \mathbb{R}$ is continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex (Schur-concave) on I^n are

(i) ϕ is symmetric on J^n;

(ii) for all $i \neq j$ and all $\mathbf{z} \in J^n$, $(\mathrm{z}_i - z_j)\left(\frac{\partial\phi(\mathbf{z})}{\partial z_i} - \frac{\partial\phi(\mathbf{z})}{\partial z_j}\right) \geq 0\,(\leq 0), where \partial\phi(\mathbf{z})/\partial z_i$ denotes the partial derivative of ϕ with respect to its i-th argument.

The following result provides some conditions for the characterization of vector functions that preserve weak supermajorization order. (Marshall et al. (2011), p. 87) Consider the real-valued function φ, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$. Then, $\boldsymbol{u} \overset{w}{\succeq} \boldsymbol{v}$ implies $\varphi(\boldsymbol{u}) \geq \varphi(\boldsymbol{v})$ if and only if φ is decreasing and Schur-convex on $\mathbb{A}$.

Archimedean copulas have been widely used in reliability theory, actuarial science and many other areas due to its mathematical tractability and the capability of capturing wide ranges of dependence. By definition, for a decreasing and continuous function $\psi : [0, \infty) \longrightarrow [0, 1]$ such that $\psi(0) = 1$ and $\psi(+\infty) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse,

$$C_\psi(u_1, \cdots, u_n) = \psi(\phi(u_1) + \cdots + \phi(u_n)) \qquad for\ all\ u_i \in [0, 1],\ \ i = 1, \cdots, n$$

is called an Archimedean copula with the generator ψ if $(-1)^k \psi^{[k]}(x) \geq 0$ for $k = 0, \cdots, n-2$ and $(-1)^{n-2}\psi^{[n-2]}(x)$ is decreasing and convex.

Recall that a function f is said to be superadditive if $f(x + y) \geq f(x) + f(y)$ for all x and y in the domain of f. For more discussions on copulas and their properties, one may refer to Nelsen (2006) and McNeil and Něslehov*á* (2009).

3 Usual Stochastic Order

Suppose $X_i \sim L - Log(\alpha_i, \beta)$ $(i = 1, \cdots, n)$ and associated Archimedean copula with generator ψ_1. Also, suppose $Y_i \sim L - Log(\mu_i, \beta)$ $(i = 1, \cdots, n)$ and associated Archimedean copula with generator ψ_2. Assume that $\phi_2 \circ \psi_1$ is superadditive. If ψ_1 or ψ_2 is log-concave, then for $0 < \beta \leq 1$

$$(\mu_1, \cdots, \mu_n) \overset{w}{\preceq} (\alpha_1, \cdots, \alpha_n) \Longrightarrow X_{n:n} \leq_{st} Y_{n:n}.$$

Proof. The distribution functions of $X_{n:n}$ and $Y_{n:n}$ are given by

$$F_{X_{n:n}}(x) = \psi_1(\sum_{k=1}^{n} \phi_1(\frac{x^\beta}{\alpha_k^\beta + x^\beta})) \qquad\qquad x > 0,$$

$$F_{Y_{n:n}}(x) = \psi_2(\sum_{k=1}^{n} \phi_2(\frac{x^\beta}{\mu_k^\beta + x^\beta})) \qquad\qquad x > 0,$$

respectively. The superadditivity of $\phi_2 \circ \psi_1$ implies that

$$\psi_1(\sum_{k=1}^{n} \phi_1(\frac{x^\beta}{\mu_k^\beta + x^\beta})) \leq \psi_2(\sum_{k=1}^{n} \phi_2(\frac{x^\beta}{\mu_k^\beta + x^\beta})).$$

Then, to prove the desired results, it is sufficient to show that

$$\psi_1(\sum_{k=1}^{n} \phi_1(\frac{x^\beta}{\alpha_k^\beta + x^\beta})) \geq \psi_1(\sum_{k=1}^{n} \phi_1(\frac{x^\beta}{\mu_k^\beta + x^\beta})).$$

Let us define $\Psi(\alpha_1,\cdots,\alpha_n)=\psi_1(\sum_{k=1}^n\phi_1(\frac{x^\beta}{\alpha_k^\beta+x^\beta}))$. According to Lemma 2.2, we need only to show that $\Psi(\alpha_1,\cdots,\alpha_n)$ is decreasing and Schur-convex in $(\alpha_1,\cdots,\alpha_n)$ for any fix $x>0$. Taking the derivative of $\Psi(\alpha_1,\cdots,\alpha_n)$ with respect to α_i, we have

$$\begin{aligned}\frac{\partial\Psi(\alpha_1,\cdots,\alpha_n)}{\partial\alpha_i} &= -\beta x^\beta\frac{\alpha_i^{\beta-1}}{(\alpha_i^\beta+x^\beta)^2}\frac{1}{\psi_1'(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))}\psi_1'(\sum_{k=1}^n\phi_1(\frac{x^\beta}{\alpha_k^\beta+x^\beta}))\\ &= -\beta\eta(x,\alpha_i,\beta)\,I(x,\alpha_i,\beta)\,\psi_1'(\sum_{k=1}^n\phi_1(\frac{x^\beta}{\alpha_k^\beta+x^\beta})),\end{aligned}$$

where $\eta(x,\alpha_i,\beta)=\frac{\alpha_i^{\beta-1}}{\alpha_i^\beta+x^\beta}$ and $I(x,\alpha_i,\beta)=\frac{x^\beta/(\alpha_i^\beta+x^\beta)}{\psi_1'(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))}$. Since ψ_1 is n-monotone, it holds that $\psi_1'\le 0$ and then $\Psi(\alpha_1,\cdots,\alpha_n)$ is decreasing with respect to α_i for any $x>0$. Note that for $i=1,\cdots,n$

$$\begin{aligned}[\psi'(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))]^2\frac{\partial I(x,\alpha_i,\beta)}{\partial\alpha_i} &= \frac{\beta\alpha_i^{\beta-1}x^\beta}{(\alpha_i^\beta+x^\beta)^2}[\psi_1'(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))]^{-1}\\ &\times\left\{\frac{x^\beta}{(\alpha_i^\beta+x^\beta)}\psi_1''(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))-[\psi_1'(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))]^2\right\}.\end{aligned}$$

Since ψ_1 is log-concave, it holds that

$$\frac{\partial^2\ln\psi_1(x)}{\partial x^2}=\frac{\psi_1''(x)\,\psi_1(x)-[\psi_1'(x)]^2}{\psi_1^2(x)}\le 0. \tag{3.1}$$

As a result, we have

$$\begin{aligned}[\psi_1'(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta})]^2 &- \frac{x^\beta}{\alpha_i^\beta+x^\beta}\psi_1''(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))\\ &= [\psi_1'(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta})]^2-\psi_1(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta}))\,\psi_1''(\phi_1(\frac{x^\beta}{\alpha_i^\beta+x^\beta})),\end{aligned} \tag{3.2}$$

which implies $\frac{\partial I(x,\alpha_i,\beta)}{\partial\alpha_i}\ge 0$. That is, $I(x,\alpha_i,\beta)$ is increasing in α_i, for $i=1,\cdots,n$. On the other hand, it is easy to show that $\eta(x,\alpha_i,\beta)$ is a decreasing in α_i for $0<\beta\le 1$. Therefore, $\eta(x,\alpha_i,\beta)\,I(x,\alpha_i,\beta)$ is is increasing in α_i for $0<\beta\le 1$. Therefore for any $i\ne j$, we have

$$\begin{aligned}A &= (\alpha_i-\alpha_j)\left(\frac{\partial\Psi(\alpha_1,\cdots,\alpha_n)}{\partial\alpha_i}-\frac{\partial\Psi(\alpha_1,\cdots,\alpha_n)}{\partial\alpha_j}\right)\\ &= -\beta\,\psi_1'(\sum_{k=1}^n\phi_1(\frac{x^\beta}{\alpha_k^\beta+x^\beta}))\,(\alpha_i-\alpha_j)\,\left(\eta(x,\alpha_i,\beta)\,I(x,\alpha_i,\beta)-\eta(x,\alpha_j,\beta)\,I(x,\alpha_j,\beta)\right).\end{aligned}$$

Since $\eta(x,\alpha_i,\beta)\,I(x,\alpha_i,\beta)$ is increasing in α_i for $0<\beta\le 1$, the right side of A is non-negative. Then, the desired result follows immediately from the Lemma 2.2. $\square$

3.1. *It is worthwhile to note that the condition "$\phi_2\circ\psi_1$ is superadditive, and ψ_1 or ψ_2 is log-concave" in Theorem 3 is quite general and easy to be constructed for many well-known*

Archimedean copulas. For example, consider the Gumbel-Hougaard copula with the generator $\psi(t) = e^{1-(1+t)^{\theta}}$ *for* $\theta \in [1, \infty)$. *It is easy to show that* $\log \psi(t) = 1 - (1+t)^{\theta}$ *is concave in* $t \in [0,1]$. *Let us set* $\psi_1(t) = e^{1-(1+t)^{\alpha}}$ *and* $\psi_2(t) = e^{1-(1+t)^{\beta}}$. *It can be observed that* $\phi_2 \circ \psi_1(t) = (1+t)^{\alpha/\beta - 1}$. *Taking the derivative of* $\phi_2 \circ \psi_1(t)$ *with respect to* t *twice, it can be seen that* $[\phi_2 \circ \psi_1(t)]'' = (\frac{\alpha}{\beta})(\frac{\alpha}{\beta} - 1)(1+t)^{\alpha/\beta - 1} \geq 0$ *for* $\alpha > \beta > 1$, *which implies the superadditivity of* $\phi_2 \circ \psi_1(t)$. $\square$

Suppose $X_i \sim L - Log(\alpha_i, \beta)$ $(i = 1, \cdots, n)$ and associated Archimedean copula with generator ψ_1. Also, suppose $Y_i \sim L - Log(\mu_i, \beta)$ $(i = 1, \cdots, n)$ and associated Archimedean copula with generator ψ_2. Assume that $\phi_2 \circ \psi_1$ is superadditive. If ψ_1 or ψ_2 is log-convex, then for $\beta > 0$

$$(\mu_1, \cdots, \mu_n) \overset{m}{\preceq} (\alpha_1, \cdots, \alpha_n) \Longrightarrow X_{1:n} \leq_{st} Y_{1:n}.$$

Proof. The proof is similar to Theorem 1 and is therefore omitted here for the sake of brevity $\square$

4 Convex Transform and Star Orders

Suppose $X_i \sim L - Log(\alpha_i, \beta)$ $(i = 1, \cdots, n)$ and $Y_i \sim L - Log(\alpha, \beta)$ $(i = 1, \cdots, n)$ share a common Archimedean copula with generator ψ. If $\psi(x)$ is log-concave and $0 < \beta \leq 1$, then we have $X_{n:n} \leq_c Y_{n:n}$. **Proof.** We note that

$$F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x)) = \alpha \left(\frac{\psi(\frac{1}{n}\sum_{i=1}^{n} \phi(\frac{x^{\beta}}{\alpha_i^{\beta} + x^{\beta}}))}{1 - \psi(\frac{1}{n}\sum_{i=1}^{n} \phi(\frac{x^{\beta}}{\alpha_i^{\beta} + x^{\beta}}))} \right)^{1/\beta}.$$

To prove the desired results, it is sufficient to show that $F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))$ is convex with respect to $x > 0$. The partial derivatives of $F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))$ with respect to x, respectively, are

$$\frac{\partial}{\partial x}\left\{F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))\right\} \overset{sgn}{=} \frac{1}{\beta} \times \frac{\psi'(x)}{(1-\psi(x))^2}\left(\frac{\psi(x)}{1-\psi(x)}\right)^{\frac{1}{\beta}-1},$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left\{F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))\right\} &\overset{sgn}{=} \frac{1}{\beta}\times\left(\frac{\psi'(x)}{(1-\psi(x))^2}\right)'\left(\frac{\psi(x)}{1-\psi(x)}\right)^{\frac{1}{\beta}-1}\\ &\quad+\frac{1}{\beta}\times\left(\frac{1}{\beta}-1\right)\times\left(\frac{\psi'(x)}{(1-\psi(x))^2}\right)^2\left(\frac{\psi(x)}{1-\psi(x)}\right)^{\frac{1}{\beta}-2}\\ &= \frac{1}{\beta}\times\left(\frac{\psi''(x)-\psi''(x)\psi(x)+2\psi'^2(x)}{(1-\psi(x))^3}\right)\left(\frac{\psi(x)}{1-\psi(x)}\right)^{\frac{1}{\beta}-1}\\ &\quad+\frac{1}{\beta}\times\left(\frac{1}{\beta}-1\right)\times\left(\frac{\psi'(x)}{(1-\psi(x))^2}\right)^2\left(\frac{\psi(x)}{1-\psi(x)}\right)^{\frac{1}{\beta}-2}.\end{aligned}$$

The assumption $0<\beta\leq 1$ implies

$$\frac{1}{\beta}\times\left(\frac{1}{\beta}-1\right)\times\left(\frac{\psi'(x)}{(1-\psi(x))^2}\right)^2\left(\frac{\psi(x)}{1-\psi(x)}\right)^{\frac{1}{\beta}-2}>0.$$

Thus $F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))$ is convex if $\psi''(x)-\psi''(x)\psi(x)+2\psi'^2(x)>0$. In this regards, if ψ is log-concave then $\psi''(x)\psi(x)-\psi'(x)^2<0$ and consequently $\psi''(x)-\psi''(x)\psi(x)+2\psi'^2(x)>0$. □

Suppose $X_i\sim L-Log(\alpha_i,\beta)$ $(i=1,\cdots,n)$ and $Y_i\sim L-Log(\alpha,\beta)$ $(i=1,\cdots,n)$ share a common Archimedean copula with generator ψ. If $\psi(x)$ is log-concave and $0<\beta\leq 1$, then we have $X_{1:n}\leq_c Y_{1:n}$. **Proof.** The proof is similar to Theorem 3 and is therefore omitted here for the sake of brevity □

The following useful lemma presents a characterization of the star order in a parametric family.

(Saunders and Moran, 1978). Suppose $\{F_\theta|\theta\in\mathbb{R}\}$ is a class of distribution functions such that F_θ is supported on some interval $(t_0,t_1)\subseteq\mathbb{R}^+$ and has a density f_θ which does not vanish on any sub-interval of (t_0,t_1). Then, $F_\theta\overset{*}{\leq}F_{\theta^*}$, $\quad for\ \theta,\theta^*\in\mathbb{R},\ \theta\leq\theta^*, ifandonlyifF'_\theta(t)/t\,f_\theta(t)$ is decreasing in t, where F'_θ is the derivative of F_θ with respect to θ.

Suppose $X_i\sim L-Log(\alpha_1,\beta)$ $(i=1,\cdots,p)$ and $X_j\sim L-Log(\alpha_2,\beta)$ $(i=p+1,\cdots,n)$ and $Y_i\sim L-Log(\mu_1,\beta)$ $(i=1,\cdots,p)$ and $Y_j\sim L-Log(\mu_2,\beta)$ $(i=1,\cdots,p)$ share a common Archimedean copula with generator ψ. If

$$(1-t)\left(2+\frac{t\phi''(t)}{\phi'(t)}\right)$$

is decreasing with respect to $t\in[0,1]$, then we have

$$(\underbrace{\alpha_1,\cdots,\alpha_1}_{p},\underbrace{\alpha_2,\cdots,\alpha_2}_{q})\overset{m}{\preceq}(\underbrace{\mu_1,\cdots,\mu_1}_{p},\underbrace{\mu_2,\cdots,\mu_2}_{q})\Longrightarrow Y_{n:n}\leq_* X_{n:n},$$

where $p+q=n$. **Proof.** Without loss of generality, let us assume that $\alpha_1 \le \alpha_2$ and $\mu_1 \le \mu_2$. Then, we observe that

$$(\underbrace{\alpha_1, \cdots, \alpha_1}_{p}, \underbrace{\alpha_2, \cdots, \alpha_2}_{q}) \overset{m}{\preceq} (\underbrace{\mu_1, \cdots, \mu_1}_{p}, \underbrace{\mu_2, \cdots, \mu_2}_{q}) \Longleftrightarrow \mu_1 \le \alpha_1 \le \alpha_2 \le \mu_2 \quad and \quad p\alpha_1 + q\alpha_2 = p\mu_1 + q\mu_2 = k.$$

Set $\alpha = \alpha_2$, $\mu_2 = \mu$, $\alpha_1 = \Big(k - q\alpha\Big)/p$ and $\mu_1 = \Big(k - q\mu\Big)/p$. Under this setting, the distribution functions of $X_{n:n}$ and $Y_{n:n}$ are

$$F_\alpha(x) = \psi\left[p\phi\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta + x^\beta}\right) + q\phi\left(\frac{x^\beta}{\alpha^\beta + x^\beta}\right)\right]; \qquad x \in \mathbb{R}^+,$$

$$F_\mu(x) = \psi\left[p\phi\left(\frac{x^\beta}{(\frac{k-q\mu}{p})^\beta + x^\beta}\right) + q\phi\left(\frac{x^\beta}{\mu^\beta + x^\beta}\right)\right]; \qquad x \in \mathbb{R}^+,$$

respectively. Now, to obtain the required result, it is sufficient to show that $\frac{F'_\alpha(x)}{xf_\alpha(x)}$ decreasing in $x \in \mathbb{R}^+$ for $\alpha \in (k/n, k/q]$. The derivative of F_α with respect to α is

$$\begin{aligned} F'_\alpha(x) &= \psi'\left[p\phi\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta + x^\beta}\right) + q\phi\left(\frac{x^\beta}{\alpha^\beta + x^\beta}\right)\right] \\ &\quad \times \left[q\frac{\beta x^\beta (\frac{k-q\alpha}{p})^{\beta-1}}{((\frac{k-q\alpha}{p})^\beta + x^\beta)^2}\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta + x^\beta}\right) - q\frac{\beta x^\beta \alpha^{\beta-1}}{(\alpha^\beta + x^\beta)^2}\phi'\left(\frac{x^\beta}{\alpha^\beta + x^\beta}\right)\right]. \end{aligned}$$

On the other hand, the density function corresponding to F_α has the form

$$\begin{aligned} f_\alpha(x) &= \psi'\left[p\phi\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta + x^\beta}\right) + q\phi\left(\frac{x^\beta}{\alpha^\beta + x^\beta}\right)\right] \\ &\quad \times \beta x^{\beta-1}\left[p\frac{(\frac{k-q\alpha}{p})^\beta}{((\frac{k-q\alpha}{p})^\beta + x^\beta)^2}\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta + x^\beta}\right) + q\frac{\alpha^\beta}{(\alpha^\beta + x^\beta)^2}\psi'\left(\frac{x^\beta}{\alpha^\beta + x^\beta}\right)\right]. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\frac{F'_\alpha(x)}{xf_\alpha(x)} &= \frac{q\frac{\beta x^\beta(\frac{k-q\alpha}{p})^{\beta-1}}{((\frac{k-q\alpha}{p})^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)-q\frac{\beta x^\beta\alpha^{\beta-1}}{(\alpha^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)}{\beta x^\beta\left[p\frac{(\frac{k-q\alpha}{p})^\beta}{((\frac{k-q\alpha}{p})^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)+q\frac{\alpha^\beta}{(\alpha^\beta+x^\beta)^2}\psi'\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)\right]} \\
&= \left(-\alpha+\frac{k\frac{(\frac{k-q\alpha}{p})^{\beta-1}}{((\frac{k-q\alpha}{p})^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)}{q\frac{(\frac{k-q\alpha}{p})^{\beta-1}}{((\frac{k-q\alpha}{p})^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)-q\frac{\alpha^{\beta-1}}{(\alpha^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)}\right)^{-1} \\
&= \left(-\alpha+\frac{k}{q}\left(1-\frac{\frac{\alpha^{\beta-1}}{(\alpha^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)}{\frac{(\frac{k-q\alpha}{p})^{\beta-1}}{((\frac{k-q\alpha}{p})^\beta+x^\beta)^2}\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)}\right)^{-1}\right)^{-1} \\
&= \left(-\alpha+\left(-\left(\frac{\alpha}{\frac{k-q\alpha}{p}}\right)^{\beta-1}\times\frac{\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)^2\phi'\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)}{\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)^2\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)}+1\right)^{-1}\right)^{-1}.
\end{aligned}
$$

Thus, it suffices to show that, for $\alpha\in[k/n,k/q)$,

$$
\Delta(x)=\frac{\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)^2\phi'\left(\frac{x^\beta}{\alpha^\beta+x^\beta}\right)}{\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)^2\phi'\left(\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}\right)}
$$

is increasing in $x\in\mathbb{R}^+$. Set $t_1=\frac{x^\beta}{\alpha^\beta+x^\beta}$, $t_2=\frac{x^\beta}{(\frac{k-q\alpha}{p})^\beta+x^\beta}$. From the fact $\alpha\in(k/n,k/q]$, we have $t_1\leqslant t_2$ for all $x\in\mathbb{R}^+$. So,

$$
\Delta(x)=\frac{t_1^2\phi'(t_1)}{t_2^2\phi'(t_2)}.
$$

The derivative of $\Delta(x)$ with respect to x is

$$
\begin{aligned}
\Delta'(x) &\overset{sgn}{=} \left[2t_1't_1\phi'(t_1)+t_1^2t_1'\phi''(t_1)\right]\times t_2^2\phi'(t_2) \\
&\quad -\left[2t_2't_2\phi'(t_2)+t_2^2t_2'\phi''(t_2)\right]\times t_1^2\phi'(t_1) \\
&\overset{sgn}{=} 2\frac{t_1'}{t_1}+\frac{t_1'\phi''(t_1)}{\phi'(t_1)}-2\frac{t_2'}{t_2}+\frac{t_2'\phi''(t_2)}{\phi'(t_2)}.
\end{aligned}
$$

It is easy to show that the derivative of t_1 and t_2 with respect to x are

$$
\begin{aligned}
t_1' &= \frac{\beta x^{\beta-1}\alpha^\beta}{(\alpha^\beta+x^\beta)^2} \\
&= \frac{\beta}{x}(1-t_1)t_1, \\
t_2' &= \frac{\beta x^{\beta-1}(\frac{k-q\alpha}{p})^\beta}{((\frac{k-q\alpha}{p})^\beta+x^\beta)^2} \\
&= \frac{\beta}{x}(1-t_2)t_2,
\end{aligned}
$$

respectively. Thus we have

$$
\begin{aligned}
\Delta'(x) \quad &\overset{sgn}{=} \quad \frac{2\beta}{x}(1-t_1) + \frac{\beta}{x}(1-t_1)\frac{t_1\phi''(t_1)}{\phi'(t_1)} \\
&\quad -\frac{2\beta}{x}(1-t_2) - \frac{\beta}{x}(1-t_2)\frac{t_2\phi''(t_2)}{\phi'(t_2)} \\
&\overset{sgn}{=} \quad 2(1-t_1) + (1-t_1)\frac{t_1\phi''(t_1)}{\phi'(t_1)} \\
&\quad -2(1-t_2) - (1-t_2)\frac{t_2\phi''(t_2)}{\phi'(t_2)}.
\end{aligned}
$$

Since $t_1 \le t_2$, thus $\Delta'(x) \ge 0$ if $(1-t)\left(2 + \frac{t\phi''(t)}{\phi'(t)}\right)$ be decreasing in $t \in [0,1]$. $\square$

Suppose $X_i \sim LL(\alpha_1, \beta)$ $(i = 1, \cdots, p)$ and $X_j \sim LL(\alpha_2, \beta)$ $(j = p+1, \cdots, n)$ and $Y_i \sim LL(\mu_1, \beta)$ $(i = 1, \cdots, p)$ and $Y_j \sim LL(\mu_2, \beta)$ $(j = 1, \cdots, p)$ are with a common Archimedean survival copula having generator ψ. If

$$(1-t)\left(2 + \frac{t\phi''(t)}{\phi'(t)}\right)$$

is decreasing with respect to $t \in [0,1]$, then we have

$$(\alpha_1 - \alpha_2)(\mu_1 - \mu_2) \ge 0 \quad \textit{and} \quad (\underbrace{\alpha_1, \cdots, \alpha_1}_{p}, \underbrace{\alpha_2, \cdots, \alpha_2}_{q}) \overset{m}{\preceq} (\underbrace{\mu_1, \cdots, \mu_1}_{p}, \underbrace{\mu_2, \cdots, \mu_2}_{q}) \Longrightarrow Y_{1:n} \le_* X_{1:n},$$

where $p + q = n$. **Proof.** The proof is similar to Theorem 5 and is therefore omitted here for the sake of brevity $\square$

The following example provides an illustration of the result in Theorems 4 and 4.

(i) For the independent case, the generator becomes $\psi(u) = e^{-u}$, $u \ge 0$. Then, we have $\phi(t) = -\ln(t), t \in [0,1]$. It can be calculated that

$$\phi'(t) = \frac{-1}{t} \quad , \quad \phi''(t) = \frac{1}{t^2}.$$

Therefore,

$$
\begin{aligned}
(1-t)\left(2 + \frac{t\phi''(t)}{\phi'(t)}\right) \quad &= \quad (1-t)\left(2 + \frac{1/t}{-1/t}\right) \\
&= \quad 1-t,
\end{aligned}
$$

which means that $(1-t)\left(2 + \frac{t\phi''(t)}{\phi'(t)}\right)$ is a decreasing function in $t \in [0,1]$.

(ii) Consider the Clayton copula with generator $\psi(t) = (\theta t + 1)^{-1/\theta}$, where $\theta \in (0,1]$. Then, we have $\phi(t) = \theta^{-1}(t^{-\theta} - 1)$. Therefore

$$\phi'(t) = -t^{-\theta-1} \qquad , \qquad \phi''(t) = (\theta+1)t^{-\theta-2},$$

and then

$$(1-t)\left(2+\frac{t\phi''(t)}{\phi'(t)}\right)=(1-t)(1-\theta)$$

which means that $(1-t)\left(2+\frac{t\phi''(t)}{\phi'(t)}\right)$ is decreasing in $t\in[0,1]$ for $\theta\in(0,1]$. $\square$

References

[1] Ahmad, M., Sinclair, C. and Werritty, A. (1988), Log-logistic flood frequency analysis, *Journal of Hydrology*, **98**, 205224.

[2] Amini-Seresht, E., Qiao, J., Zhang, Y. and Zhao, P. (2016), On the skewness of order statistics in multiple-outlier PHR models, *Metrika*, **79**, 817-836.

[3] Barmalzana, G., Payandeh Najafabadib, A.T. and Balakrishnan, N. (2017), Ordering results for series and parallel systems comprising heterogeneous exponentiated Weibull components, *Communications in Statistics-Theory and Methods*, Online.

[4] Bennett, S. (1983), Log-logistic regression models for survival data, *Applied Statistics*, 165171.

[5] Ding, W., Yang, J. and Ling, X. (2017), On the skewness of extreme order statistics from heterogenous samples, *Communication in Statistics-Theory and Methods*, **46**, 2315-2331.

[6] Fang, R., Li, C. and Li, X. (2016). Stochastic comparisons on sample extremes of dependent and heterogenous observations. *Statistics*, **50**, 930-955.

[7] Fisk, P.R. (1961), The graduation of income distributions, *Econometrica: journal of the Econometric Society,* 171-185.

[8] Geskus, R.B. (2001), Methods for estimating the aids incubation time distribution when date of seroconversion is censored, *Statistics in Medicine*, **20**, 795-812.

[9] Li, X. and Fang, R. (2015), Ordering properties of order statistics from random variables of Archimedean copulas with applications, *Journal of Multivariate Analysis,* **133**, 304-320.

[10] Li, C. and Li, X. (2015), Likelihood ratio order of sample minimum from heterogeneous Weibull random variables, *Statistics and Probability Letters*, **97**, 46-53

[11] Marshall, A.W., Olkin, I. and Arnold, B.C. (2011), *Inequalities: Theory of Majorization and its Applications*, Second edition, Springer, Verlag, New York.

[12] Marshall, A.W. and Olkin, I. (2007), *Life Distributions*, Springer-Verlag, New York.

[13] McNeil, A.J. and Něslehová, J. (2009), Multivariate Archimedean copulas, D-monotone functions and l_1-norm symmetric distributions, *The Annals of Statistics*, **37**, 3059-3097.

[14] Müller, A. and Stoyan, D. (2002), *Comparison Methods for Stochastic Models and Risks,* John Wiley & Sons, New York.

[15] Nelsen, R.B. (2006), *An Introduction to Copulas*, New York, Springer.

[16] Oja, H. (1981), On location, scale, skewness and kurtosis of univariate distributions, *Scandinavian Journal of Statistics,* **8**, 154-168.

[17] Robson, A. and Reed, D. (1999), Procedures for flood freequency estimation, *Statistical procedures for flood freequency estimation*, Institute of Hydrology.

[18] Saunders, I.W. and Moran, P.A. (1978), On the quantiles of the gamma and F distributions, *Journal of Applied Probability*, **15**, 426-432.

[19] Shaked, M. and Shanthikumar, J.G. (2007), *Stochastic Orders,* Springer- Verlag, New York.

[20] Zhang, Y., Cai, X., Zhao, P. and Hairu Wang, H. (2018), Stochastic comparisons of parallel and series systems with heterogeneous resilience-scaled components, *Statistics*, https://doi.org/10.1080/02331888.2018.1546705.

Measures of Income Inequality Based on Quantile Function

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Abstract: Quantile functions are equivalent alternatives to distribution functions in analysis and modelling of statistical data. Curves that measure inequality in incomes have been a topic of immense interest for more than a century ever since the work of Lorenz in 1905. A measure of income inequality is designed to provide an index that can abridge the variations prevailing in income among the individuals in a group. In the present paper, we study more aspects on the income inequality measures using quantile function approach. Aging concepts such as IFR, IFRA, NBU, HNBUE, ... have an important role in reliability. The inequality curves and indices and some links with reliability aging using quantile function approach are concentrated in this talk. We will focus on income inequality, quantile function and aging consepts and their links. We look into possible functional relationships between the income inequality measures and quantile function. We examine the possible relationships of the inequality measures as well as reliability concepts like mean residual life function and reversed mean residual life function. Then functional relationships enable us to establish characterization results for probability distributions.

Keywords Quantile function, Order statistics, Lorenz order, Income inequality measures.

Mathematics Subject Classification (2010) : 62P20, 91B82, 91B70.

1 Introduction

Recently, several inequality curves have been made or investigated as the descriptors of income inequality. The Bonferroni curve and the Zenga-2007 curve are main the functions of the Lorenz curve. The Lorenz curve is inspected as a very advantageous tool of economic suitable to its important role in the evaluation of the inequality of income distributions and wealth. An

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approach for modelling statistical data is to use quantile function. In many cases, quantile functions are more convenient as they are less influenced by extreme observations and thus provide a straightforward analysis with a limited amount of information. However, quantile functions have several properties that are not shared by distributions, which makes it more convenient for analysis. There are explicit general distribution forms for the quantile function of order statistics. There are many simple quantile functions which are very good in empiricalmodel building where distribution functions are not effective. In such situations, conventional methods of analysis using distribution functions are not appropriate. For various properties and applications of quantile functions, we refer to Parzen (15), Gilchrist (5), Sarabia (17), and Sarabia et al. (18). In reliability, a single long term survivor can have a marked effect on mean life, especially in the case of heavy tailed models which are very common. In such cases, quantile based estimates are generally found to be more precise and robust against outliers. For more properties and applications of quantile functions in reliability analysis, one could refer to Nair et al. (12), Nair and Vineshkumar (13), Nair and Vineshkumar (14), Midhu et al. (8), Midhu et al. (9), Prendergast and Staudte(16) and Nair et al. (11). The rest of the paper is organized as follows. After the present introductory section, in Section 2 we give a brief review of the background materials needed for the paper. In addition to a discussion on the definition and properties of quantile functions, we also provide discussions on basic reliability concepts such as hazard rate, mean residual life, reversed hazard rate and reversed mean residual life. We also provide a brief review of the widely used income inequality measures, their interrelationships and their properties. In Section 3, we provide some relations between quantile function and inequality indices and reliability based on quantile. Finally, conclusions are noted in Section 4.

2 Definitions and notations

Throughout this article we assume that, X and Y are two non-negative continuous random variables with positive and finite means. We propound F and G for the distribution functions and apply the symbolisms f and g to score respective probability (density) distributions. The survival function of F is signified by $\overline{F} = 1 - F$, and use similar notation for all other function. We say that $r_F(x) = \dfrac{f(x)}{\overline{F}(x)}$ ($\tilde{r}_F(x) = \dfrac{f(x)}{F(x)}$) is the hazard rate (reversed hazard rate) function of F, and the mean residual life function (reversed mean residual lifetime) is given by $m_F(x) = \int_x^\infty \overline{F}(t)dt/\overline{F}(x), x \geq 0$ ($\dfrac{\int_0^x F(t)dt}{F(x)}$). All other functions are notated by similar notations.

Let's provide some definitions which will be utilized in this article.

As pointed out in the introduction, representation of a probability distribution in terms of quantile functions has the advantage that it can be used in situations where conventional distribution function approach fails. In several instances, further analysis using this approach is mathematically more tractable. A study based on quantile functions thus provides simpler and clearer perspective for solving problems in statistical modelling. Let X be a non-negative continuous random variable defined over $-\infty < x < \infty$ with distribution function $F(x)$ and density function $f(x)$. The quantile function, denoted by $Q(u)$ is defined as

$$Q(u) = \inf\{x : F(x) \geq u,\, u \in (0,1)\}. \tag{2.1}$$

It may be noted that $Q(u)$ is same as $F^{-1}(u)$. The lower and upper bounds of the support of F (S_F) are $Q(0)$ and $Q(1)$ respectively. Also by the strict monotonicity of $F(x)$, we have $x = Q(u)$. The quantile density function associated with a probability distribution is defined as $q(u) = Q'(u)$. The quantile density function is non-negative and can be interpreted as the slope of quantile function. Setting $x = Q(u)$ in the probability density function, the density quantile function turns out to be $f(Q(u))$. It may be noticed that the quantile density function and the density quantile function are connected through the relationship $f(Q(u))q(u) = 1$. The generalized failure rate (GFR) and the generalized reversed failure rate (GRFR) were introduced as the variant extension of the failure rate (FR) and the reversed failure rate (RFR) functions. The concept of GFR can be useful in stochastic models of service systems and also in supplying chain models. The GFR and RGFR are introduced by the following definitions: The random variable X with pdf f and cdf F, the generalized failure rate (GFR) of X is the function

$$h(x) = xr(x) = \frac{xf(x)}{1-F(x)},$$

and the generalized reversed failure rate (GRFR) of X is

$$\tilde{h}(x) = x\tilde{r}(x) = \frac{xf(x)}{F(x)}.$$

Lorenz curve presents a graphical tool to investigate income inequality. The Gini coefficient has been found helpful to analysis the inequality of incomes. The value of the Gini coefficient shows the extent of income inequality. The Lorenz curve was first defined by Lorenz (1905) ((7)). The Lorenz curve of X, a non-negative random variable with positive and finite mean, is given by

$$L(p) = \frac{1}{E(X)} \int_0^{Q(p)} uf(u)du, \quad 0 \leq p \leq 1. \tag{2.2}$$

Lorenz curve is a distribution function, twice differentiable, convex, increasing. $L(0) = 0$ and $L(1) = 1$ on $[0;1]$. $lim_{p\to 1}L'(p)(1-p) = 0$, $L_X(p) \leq p$.

The Gini coefficient is the most famous criterion for income inequality. This is corresponding to twice the region between the equality line and the Lorenz curve, which is exactly a relative measure of income inequality $G = 1 - 2\int_0^1 L(y)dy$. A relatively minor adjustment of the Lorenz curve is the Bonferroni curve $B_X(p)$. It was written as, $B_X(p) = \dfrac{L(p)}{p}, 0 < p < 1$. The Bonferroni curve is could be concave in some parts and convex in the others and strictly increasing ((6)).

The Zenga index is the ratio of the mean income of the poorest $100p$ in the distribution to that of the rest of the distribution, namely the $100(1-p)$ richest is the Zenga curve $Z(p)$. The Zenga curve can be written as, $Z_X(p) = 1 - \dfrac{L(p)}{p} \cdot \dfrac{1-p}{1-L(p)} \quad p \in (0,1)$. The Zenga index, is defined by, $Z = \int_0^1 Z(p)dp$. X is smaller than Y in the Lorenz order ($X \leq_L Y$), Bonferroni order ($X \leq_B Y$) or Zenga order ($X \leq_Z Y$) iff $L_Y(p) \leq L_X(p)$, $B_Y(p) \leq B_X(p)$ or $Z_X(p) \leq Z_Y(p)$ respectively. These orders are invariant with respect to scale transformation ((1)). From definitions of the Zenga and Bonferroni curves and definitions of the Zenga, Bonferroni and Lorenz orders immediately conclude that $X \leq_L Y \Longleftrightarrow X \leq_B Y \Longleftrightarrow X \leq_Z Y$.

3 Main results

Nair and Sankaran (10) introduced the basic concepts in reliability theory in terms of quantile functions. We refolmulate the concepts of generalized failure rate and generalized reversed failure rate using the quantile function approach which are produced below. The generalized failure rate quantile and generalized reversed failure rate quantile function can be written as

- $H(p) = h(Q(p)) = \dfrac{Q(p)}{(1-p)q(p)}$.
- $\tilde{H}(p) = \tilde{h}(Q(p)) = \frac{Q(p)}{pq(p)}$.

$H(p)$ and $\tilde{H}(p)$ uniquely determine the distribution through the relationships

- $Q(p) = \exp\{\int_0^p \dfrac{(1-t)}{H(t)}dt\}$.
- $Q(p) = \exp\{\int_0^p \dfrac{(t)}{\tilde{H}(t)}dt\}$.

According to the role and importance of the quantile function, we rewrite the relations and concepts of economic inequality using the quantile function. Let X be a non-negative random variable with finite mean. The Lorenz curve, Gini index, Bonferroni curve, Zenga curve, Canbra curve, Pietra coefition and Right spread function can be written as bellow:

$$
\begin{aligned}
L(p) &= \frac{1}{\int_0^1 Q(t)dt}\int_0^p Q(t)dt = \frac{pE(X|X \leq Q(p))}{E(X)}, \\
G &= 1 - \frac{2}{\mu}\int_0^1 \int_0^p Q(t)dtdp, \\
B(p) &= \frac{\int_0^p Q(t)dt}{\mu p} = \frac{E(X|X \leq Q(p))}{\mu}, \\
Z(p) &= 1 - \frac{(1-p)\int_0^p Q(t)dt}{p\int_p^1 Q(t)dt} \\
C(u) &= \frac{E(X) - E(X|X \leq Q(u)}{E(X) + E(X|X \leq Q(u)} \\
B &= 1 + \frac{\int_0^1 \log t.Q(t)du}{\mu}, \\
P &= \frac{1}{\mu}\int_0^{F(\mu)} (\mu - Q(t))dt, \\
EW &= E[\max\{X - Q(p), 0\}].
\end{aligned}
$$

It is clear that can be determined the distribution with the introduction of any of the functions. for example

$$
\begin{aligned}
Q(p) &= \mu(B(p) + pB'(p)), \\
Q(p) &= \mu\frac{d}{dp}\left[\frac{p(1-C(p)}{C(p)-1}\right], \\
Q(p) &= \frac{d}{dp}\frac{\mu p(1-Z(p))}{1-pZ(p)}.
\end{aligned}
$$

The distribution function can be determined easily using these equations, along with having information about inequality indices and especially by estimating a functional form for each of these indices. Following, we look into the problem of characterizing probability distributions using possible relationships between $L(p)$ and certain reliability concepts. For a non-negative continuous random variable X, the relationship

$$
L(u) = \frac{A - uB - (1-u)M(u)}{A - B}, \tag{3.1}
$$

holds if and only if X follows the distribution specified by the quantile function ($M(u) = m(Q(u))$ is thr mean residual quantile function).

$$
Q(u) = \frac{\mu B}{B - A} + C(1-u)^{\frac{B-A}{\mu}}, \tag{3.2}
$$

provided $C(A-B)>0$. . Using the definitions of $L(u)$ and $M(u)$ and replacing in equation (3.1) we have

$$\frac{1}{\mu}\int_0^u Q(t)dt = \frac{A-uB}{A-B} + \frac{1-u}{A-B}\left(\frac{1}{1-u}\int_u^1 Q(t)dt - Q(t)\right).$$

Differentiating the above expression with respect to u and rearranging the terms, we get

$$q(u) = \frac{A-B}{(1-u)\mu}Q(u) - \frac{B}{1-u} = 0,$$

The solution of the above differential equation is,

$$Q(u) = \frac{\mu B}{B-A} + C(1-u)^{\frac{B-A}{\mu}}.$$

For $Q(u)$ is an increasing function, $C(A-B)>0$. The proof of the converse is straight forward and hence omitted □

Setting $B=0$ in equations (3.1) and (3.2) we get

$$L(u) = \frac{A-(1-u)M(u)}{A},$$

and $Q(u) = C(1-u)^{\frac{-A}{\mu}}$. Put $C=k$ and $A=\frac{\mu}{\alpha}$ in the above expression, we get Pareto distribution of first kind with quantile function, $Q(u)=k(1-u)^{\frac{-1}{\alpha}}$ For a non-negative random variable X with reversed mean residual quantile function $\tilde{M}(p)$, the relationship

$$L(u) = \beta u\tilde{M}(p) \tag{3.3}$$

holds if and only if X follows power distribution specified by the quantile function

$$Q(u) = \sigma u^{\frac{1}{\phi}};\ \sigma,\ \phi>0. \tag{3.4}$$

. For the quantile function given in (3.4), direct calculations give $\tilde{M}(p) = \dfrac{\sigma}{\phi+1}u^{\frac{1}{\phi}}$ and $L(u) = \frac{\phi+1}{\sigma}u\tilde{M}(p)$ where $\beta = \frac{\phi+1}{\sigma}$.

Conversely, suppose that $L(u)=\beta u\tilde{M}(p)$ holds, we have

$$\frac{1}{\mu}\int_0^u Q(t)dt = \beta u\left(Q(u) - \frac{1}{u}\int_0^u Q(t)dt\right). \tag{3.5}$$

Differentiating the above equation with respect to u, we get

$$\frac{q(u)}{Q(u)} = \frac{1}{u\mu\beta},$$

The solution to the above differential equation is

$$Q(u) = Cu^{\frac{1}{\beta\mu}}. \tag{3.6}$$

Put $C=\sigma$ and $\beta=\frac{\phi}{\mu}$, we get the quantile function given in (3.4) and the theorem follows. □

3.1 Aging concepts based on quantile function and income inequality measures

Concept of ageing is an important notion not only in the field of Reliability theory but also in Economics. Bhattacharjee (4) has observed that the distribution of land holdings obey anti ageing properties like DFR, DFRA, IMRL, NWUE etc. In this section we explain the problem from another point of view. The ageing properties are examined using quantile function and Lorenz curve. To facilitate a quantile based analysis, Nair and Vineshkumar (14) expressed the basic ageing concepts in terms of quantile functions. Various ageing concepts like increasing (decreasing) hazard rate-IHR(DHR), increasing(decreasing) average hazard rate-IHRA(DHRA), new better than used in hazard rate(NBUHR), new better than used in hazard rate average(NBUHRA), increasing(decreasing) mean residual life-IMRL(DMRL),increasing (decreasing) variance residual life IVRL(DVRL), new better (worse) than used-NBU(NWU) etc are presented in the paper. Mainly the ageing concepts are studied under three broad heads, those based on hazard functions, residual quantile functions and survival functions. We list the ageing concepts based on these broad heads in the distribution function setup as well as quantile setup. [(10)] In the quantile framework, a random variable X is said to have

- Increasing hazard quantile function IHR (decreasing hazard quantile function DHR) if and only if for all $0 < u_1 < u_2 < 1$

$$R(u_2) \geq (\leq) R(u_1).$$

- $IMRL\,(DMRL)$ if and only if (with $E(X) < \infty$) for all $0 < u_1 < u_2 < 1$ $M(u_1) \geq (\leq)M(u_2)$ or equivalently $\int_0^1 [Q(u + (1-u)t) - Q(u)]dt$ is a increasing (decreasing) in u.

- $HNBUE\,(HNWUE)$ if and only $\int_u^1 (1-t)q(t)dt \leq (\geq) \mu e^{\frac{-Q(u)}{\mu}}$.

Let X be a random variable with absolutely continuous distribution. Then X is said $IMRL\,(DMRL)$ if

$$L(u) \geq (\leq) Q(u)(1-u) + (1-u)q(u).$$

. (10) showed that X be $IMRL\,(DMRL)$ then $M(u) \geq (\leq)\frac{1}{H(u)}$. It is clear that

$$M(x) = \mu \frac{\overline{L}(x)}{1-x} - Q(x), \tag{3.7}$$

we have

$$\mu \left[\frac{1 - L(u)}{1-u} - \frac{d}{du} L(u) \right] \geq (\leq) \frac{1}{H(u)}.$$

On the other hand, we can write the relation between the mean resdiual quantile function and Lorenz curve by the following equation:

$$L(u) - Q(u)(1-u) = M(u).$$

So

$$L(u) \geq (\leq) Q(u)(1-u) + \frac{1}{H(u)}.$$

Finally, we put $H(u) = \frac{1}{(1-u)q(u)}$ and the theorem follows. $\square$

Let X be non-negative and continuous random variable with distribution function F and Lorenz curve L. If X is IFR (DFR) then $L(p) \geq (\leq) Q(p)(p-1) - \dfrac{pQ(p)}{\ln(1-p)}$ . . (2) showed that if F is IFR (DFR) then

$$\begin{cases} \overline{F}(t) \geq (\leq) e^{-at} & ; \quad t \leq Q(p), \\ \overline{F}(t) \leq (\geq) e^{-at} & ; \quad t \geq Q(p), \end{cases}$$

where $a = -\dfrac{\ln(1-p)}{Q(p)}$. Considering the definition of the Lorenz curve we have

$$\begin{aligned} L(p) = \int_0^{Q(p)} tf(t)dt &= Q(p)(p-1) + \int_0^{Q(p)} \overline{F}(t)dt, \\ &\geq (\leq) Q(p)(p-1) + \int_0^{Q(p)} e^{-at}dt, \\ &= Q(p)(p-1) + \frac{Q(p)(\exp(\ln(1-p)-1))}{\ln(1-p)}, \\ &= Q(p)(p-1) - \frac{pQ(p)}{\ln(1-p)}, \end{aligned}$$

and the proof is complete. $\square$

4 Conclusions

In the present work, the definitions of income inequality measures are reformulated using quantiles. In addition to examining the connection between the measure and other existing inequality measures, the relationship of the concept with certain reliability concepts are exploited to obtain characterization results for probability distributions. Further some results on some aging concepts using Lorenz curve are also established.

References

[1] Arnold, B.C. (1991), Preservation and attenuation of inequality as measured by Lorenz order, *Stochastic Orders and Decision under Risk, IMS Lectures Notes - Monograph*, Series, 25-37.

[2] Barlow, R.E. and Proschan, F. (1996), *Mathematical theory of reliability*, Society for Industrial and Applied Mathematics.

[3] Belzunce, F., Pinar, J.F., Ruiz, J.M., and Sordo, M.A. (2012), Comparison of risks based on the expected proportional shortfall, *Insurance: Mathematics and Economics*, **51**(2), 292-302.

[4] Bhattacharjee, M. (1993), How rich are the rich? modeling affluence and inequality via reliability theory, *Sankhya: The Indian Journal of Statistics*, Series B, **55**, 1-26.

[5] Gilchrist, W. (2000), *StatisticalModelling with Quantile Functions*, CRC Press, London, UK.

[6] Giorgi G.M. and Crescenzi M. (2001), A look at the Bonferroni inequality measure in a reliability framework, *Statistica LXL*, **4**, 571-583.

[7] Lorenz, M.O. (1905), Methods of measuring the concentration of wealth, *Publications of the American Statistical Association*, **9**(70), 209-219.

[8] Midhu, N.N., Sankaran, P.G. and Nair, N.U. (2014), A class of distributions with linear hazard quantile function, *Communications in Statistics-Theory and Methods*, **43(17)**, 3674-3689.

[9] Midhu, N.N., Sankaran, P.G. and Nair, N.U. (2013), A class of distributions with linear mean residual quantile function and its generalizations, *Statistical Methodology*, **15**, 1-24.

[10] Nair, N.U. and Sankaran, P.G. (2009). Quantile-based reliability analysis, *Communications in Statistics*, **38**(1-2), 222-232.

[11] Nair, N.U. and Sankaran, P.G. and Balakrishnan, N. (2013), *Quantile-Based Reliability Analysis*, Birkhauser, Basel, Switzerland.

[12] Nair, N.U., Sankaran, P.G. and Kumar, B.V. (2008), Total time on test transforms of order and their implications in reliability analysis, *Journal of Applied Probability*, **45(4)**, 1126-1139.

[13] Nair, N.U. and Vineshkumar, B. (2010), L-moments of residual life, *Journal of Statistical Planning and Inference*, **140**(9), 2618-2631.

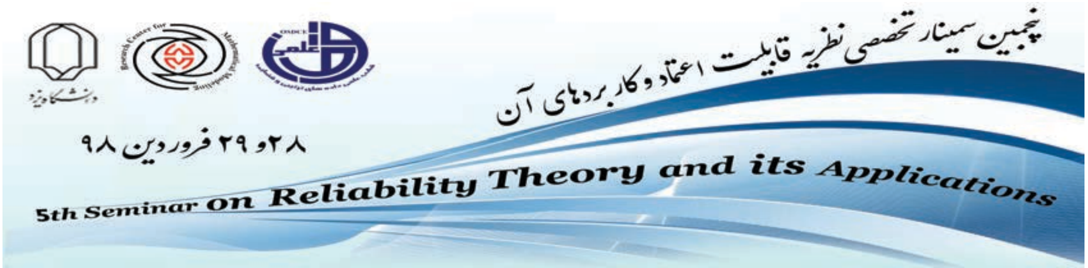
[14] Nair, N.U. and Vineshkumar, B. (2011), Ageing concepts: an approach based on quantile function, *Statistics & Probability Letters*, **81**(12), 2016-2025.

[15] Parzen, E. (1979), Nonparametric statistical datamodeling, *Journal of the American Statistical Association*, **74**(365), 105-131.

[16] Prendergast, L.A. and Staudte, R.G. (2016), Quantile versions of the Lorenz curve, *Electronic Journal of Statistics*, **10**(2), 1896-1926.

[17] Sarabia, J.M. (2008), A general definition of the Leimkuhler curve, *Journal of Informetrics*, **2**(2), 156-163.

[18] Sarabia, J.M., Prieto, F. and Sarabia, M. (2010), Revisiting a functional formfor the Lorenz curve, *Economics Letters*, **107**(2), 249-252.



Some Aging Properties of a Repairable Coherent System

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Abstract: We consider a repairable system with n independent and identically distributed components which begins to operate at time 0. If the system fails, then it undergoes minimal repair and begins to operate again. We find a general representation of the failure rate of the system based on its components failure rate. The reliability, aging and stochastic properties of the system lifetime are also investigated.

Keywords Aging properties, Minimal repair, Repairable system, Stochastic ordering.

Mathematics Subject Classification (2010) : 47A55**,** 39B52**,** 34K20**,** 39B82**.**

1 Introduction

The most common models for repairable systems are perfect repair and minimal repair. After a perfect repair the system is restored to an as-good-as-new state; i.e. the reliability is restored to its original state. After a minimal repair, the system is only restored to the state prior to failure, i.e. to a same-as-old state. There are some situations that the effectiveness of repair may be different from perfect and minimal repairs. The imperfect repair models treat the repair more generally than the perfect and minimal, see (6), (11) and (13). In this paper, we focus on the repairable systems that after each failure undergoes minimal repair. We find a general representation for the survival and failure rate functions of the system after the $(n-1)$th minimal repair. These representations are useful in studying the aging properties of the system lifetime when we have some information about the system component lifetimes. These results may be extended to the imperfect repair case.

Consider a system with lifetime T, survival function $\bar{F}_T(t)$ and failure rate $r_T(t)$. Throughout the paper, we will denote by IFR (DFR), IFRA (DFRA), NBU (NWU), IMRL (DMRL), NBUFR and NBUFRA the increasing (decreasing) failure rate, increasing (decreasing) failure rate average, new better (worse) than used, increasing (decreasing) mean residual life, new better than used in failure rate ($r_T(0) \leq r_T(t)$, for all $t \geq 0$) and new better than used in failure

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rate average ($r_T(0) \leq \frac{1}{t}\int\limits_0^t r_T(x)dx$), respectively. For more details and applications of these concepts, we refer the reader to (4) and (8).

The next lemma is useful in our derivations. The proof is given by (12) and (1). For $x > 0$, $\phi_n(x) = \frac{\sum\limits_{j=0}^{n-1} x^{j+1}/j!}{\sum\limits_{j=0}^{n} x^j/j!}$ and $g_n(x) = \frac{x^{n-1}/(n-1)!}{\sum\limits_{j=0}^{n-1} x^j/j!}$
are nondecreasing in x.

2 Main results

Let $T(n)$ denote the lifetime of a coherent system after the $(n-1)$th minimal repair. Then by using the connection between the minimal repair process and record values ((9), (3) and (1)), the survival function and density function of $T(n)$ are given by

$$\bar{F}_{T(n)}(t) = \bar{F}_T(t) \sum_{k=0}^{n-1} \frac{[-\log \bar{F}_T(t)]^k}{k!}, \tag{2.1}$$

and

$$f_{T(n)}(t) = f_T(t) \frac{[-\log \bar{F}_T(t)]^{n-1}}{(n-1)!}, \tag{2.2}$$

respectively. Consequently, the hazard function of $T(n)$ can be obtained as

$$r_{T(n)}(t) = g_n(-\log \bar{F}_T(t)) r_T(t). \tag{2.3}$$

The following properties are satisfied for a coherent system with lifetime T that undergoes minimal repair at each failure.

i) If T is IFR, IFRA, NBU or DMRL then $T(n)$ has the corresponding property.

ii) If $T(n)$ is DFR, DFRA, NWU, or IMRL, then T has the corresponding property.

iii) If T is NBUFR or NBUFRA, then $T(n)$ has the same property.

. The proofs of i) and ii) can be easily found from Eq. (2.3) and Theorem 3.1 of (1).
iii) From Eq. (2.3) and definition of NBUFR, we have

$$\begin{aligned} r_{T(n)}(0) &= g_n(-\log \bar{F}_T(0)) r_T(0) \\ &\leq g_n(-\log \bar{F}_T(t)) r_T(t) \\ &= r_{T(n)}(t). \end{aligned}$$

To show NBUFRA property of $T(n)$, note that

$$
\begin{aligned}
r_{T(n)}(0) &= g_n(-\log \bar{F}_T(0))r_T(0) \\
&\leq g_n(-\log \bar{F}_T(0))\frac{1}{t}\int_0^t r_T(x)dx \\
&\leq \frac{1}{t}\int_0^t g_n(-\log \bar{F}_T(x))r_T(x)dx \\
&= \frac{1}{t}\int_0^t r_{T(n)}(t)dt,
\end{aligned}
$$

where the second inequality is obtained from nondecreasing property of $g_n(.)$. $\square$

If a coherent system after the $(n-1)$th minimal repair is IFR, IFRA or NBU, then it has the same property after the nth minimal repair. . After some manipulations, we have $\frac{r_{T(n+1)}(t)}{r_{T(n)}(t)} = \phi_n(-\log(\bar{F}_T(t)))$. The proof is now completed by Lemma 1 and the results of (5). $\square$

Consider a coherent system with lifetime T and independent and identically distributed (i.i.d.) component lifetimes $X_1, \ldots, X_n$ from a common cumulative distribution function (cdf) F. Then, the reliability function of T may be represented as

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \tag{2.4}$$

where $s_i = \Pr(T = X_{i:n})$, known as Samaniego's signature (14), and $\bar{F}_{i:n}(t)$ is the reliability function of the ith order statistics of the component lifetimes.

In the next lemma, we find a new representation for the survival and failure rate functions of $T(n)$. These representations are useful to study the aging and stochastic properties of the lifetime of a repairable system based on the corresponding properties of its component lifetimes. Consider a repairable coherent system with signature $\mathbf{s} = (s_1, \ldots, s_n)$, and assume that the system component lifetimes $X_1, \ldots, X_n$ are i.i.d. with distribution F. The survival and the failure rate functions of the system after the $(n-1)$th minimal repair can be expressed as

i) $\bar{F}_{T(n)}(t) = q_n(\bar{F}(t))$,

and

ii) $r_{T(n)}(t) = \alpha_n(\bar{F}(t))r(t)$,

respectively, where $q_n(u) = e^{-\Lambda(u)} \sum_{k=0}^{n-1} \frac{(\Lambda(u))^k}{k!}$,

$e^{-\Lambda(u)} = \sum_{j=0}^{n} \bar{S}_{j+1}\binom{n}{j}u^{n-j}(1-u)^j$, $\bar{S}_{j+1} = \sum_{i=j+1}^{n} s_i$ and $\alpha_n(u) = u\Lambda'(u)g_n(\Lambda(u))$.

Under the assumptions of Theorem 2, if $u\Lambda'(u)$ is decreasing in u and the component lifetimes are according to IFR, IFRA or NBU distribution F, then the distribution of the system lifetime after the $(n-1)$th minimal repair has the same property. . Since $\Lambda(u)$ is a decreasing function of u, the proof is straightforward from Theorem 2 part ii) and the results of (5). $\square$

It is not difficult to verify that $u\Lambda'(u)$ is decreasing if and only if $k(x) = \frac{\sum_{i=0}^{n-1}(n-i)s_{i+1}\binom{n}{i}x^i}{\sum_{i=0}^{n-1}\bar{S}_{j+1}\binom{n}{i}x^i}$ is increasing in $x > 0$, where is equivalent to Eq. (4.11) in (15). For a k-out-of-n system with signature vector $\mathbf{s} = (0, \ldots, 0, 1, 0, \ldots, 0)$, with a "1" as its kth element, $k(x)$ can be simplified as $k_0(x) = \frac{(n-k+1)\binom{n}{k-1}x^{k-1}}{\sum_{i=0}^{k-1}\binom{n}{i}x^i}$, where is increasing in x. Therefore, if the component lifetimes of a repairable k-out-of-n system with minimal repairs have an IFR, IFRA or a NBU property, then the system lifetime after the $(n-1)$th minimal repair also have the same property. Consider a bridge system in 5 components with $\mathbf{s} = (0, 1/5, 3/5, 1/5, 0)$. After some manipulations, we obtain $k(x) = \frac{4x^3+18x^2+4x}{2x^3+8x^2+5x+1}$, where is an increasing function of x. Thus, the lifetime of the bridge structure after the $(n-1)$th minimal repair will be IFR, IFRA or NBU when its components have i.i.d. IFR, IFRA or NBU lifetimes. In the next theorem, we compare two repairable coherent systems with the same structures and different component lifetimes. Consider two repairable coherent systems with the same structures and i.i.d. component lifetimes $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with distributions F and G, respectively.

i) If $X \le_{st} Y$, then $T_X(n) \le_{st} T_Y(n)$.

ii) If $X \le_{hr} Y$ and $k(x) = \frac{\sum_{i=0}^{n-1}(n-i)s_{i+1}\binom{n}{i}x^i}{\sum_{i=0}^{n-1}\bar{S}_{j+1}\binom{n}{i}x^i}$ is increasing in $x > 0$, then $T_X(n) \le_{hr} T_Y(n)$.

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i) From $X \le_{st} Y$, we obtain $\Lambda_X(u) \ge \Lambda_Y(u)$. On the other hand $q_n(u) = P_{\Lambda(u)}(n-1)$, where $P_{\Lambda(u)}(n-1)$ denotes the distribution function of a Poisson random variable with mean value function $\Lambda(u)$. From the stochastic ordering between two Poisson random variables, the result is obvious.

ii) Theorem 2 part ii) concludes $r_{T_X(n)}(t) = \alpha_n(\bar{F}_X(t))r_X(t)$. From the assumption $X \le_{hr} Y$, we have $r_X(t) \ge r_Y(t)$ and $\bar{F}_X(t) \le \bar{F}_Y(t)$. The proof is now immediate from the fact that $\alpha_n(u)$ is a decreasing function of u.

$\square$

Now, consider two coherent systems with different structures and the same component lifetimes. The following results are straightforward from (1) and (14).

- If $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$, then $T_1(n) \leq_{st} T_2(n)$.

- If $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$, then $T_1(n) \leq_{hr} T_2(n)$,

  where $T_i(n);\ i = 1, 2$ denotes the lifetime of a repairable coherent system with signature $\mathbf{s}_i;\ i = 1, 2$, after the $(n - 1)$th minimal repair.

The residual lifetime and inactivity time of coherent systems are important measures in relaibility theory. Several authors have studied various types of these measures. Suppose that $(T(n) - t|T(j) > t);\ 1 \leq j < n$ denotes the residual lifetime of a repairable system after the $(n-1)$th minimal repair, under the condition that the system is repaired at most $k-1; 1 \leq k \leq j$, times before $t > 0$. From the results of (2) and (10), we have

$$P(T(n) - t > x|T(j) > t) = \sum_{\ell=0}^{j-1} p_{F_T}(\ell) P(Y_{n-\ell} \geq -\log \bar{F}_T(x|t)),$$

where $p_{F_T}(\ell) = P(W = \ell|W \leq j - 1)$, W is a Poisson random variable with parameter $-\log \bar{F}_T(x|t) = -\log \frac{\bar{F}_T(x+t)}{\bar{F}_T(t)}$, and $Y_{n-\ell}$ is a gamma random variable with shape and scale parameters $n - \ell$ and 1, respectively.

In the next theorem, we show that the hr-ordering between the signature vectors of two repairable coherent systems concludes the st-ordering between their residual lifetimes. Let $\mathbf{s}_i;\ i = 1, 2$ be the signature vector of a repairable coherent system with lifetime $T_i(n);\ i = 1, 2$. If $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$, then $(T_1(n)-t|T_1(j) > t) \leq_{st} (T_2(n)-t|T_2(j) > t)$. . Samaniego (14) showed that for two systems with the same components and different structures, if $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$, then $T_1 \leq_{hr} T_2$. It is also known that hr-ordering between two random variables concludes st-ordering between their residual lifetimes, i.e. $\bar{F}_{T_1}(x|t) \leq \bar{F}_{T_2}(x|t)$. After some manipulations and following the same approach in Theorem 6 of (7), we find that $(T_1(n)-t|T_1(j) > t) \leq_{st} (T_2(n)-t|T_2(j) > t)$. $\square$

3 Conclusion

We studied some reliability properties of repairable coherent systems with minimal repairs by using the connection of record values and minimal repairs. Then, we obtained a relation between the failure rate of the repairable system and its components failure rate by using the signature notion. These results can be extended to the different types of the residual lifetimes and inactivity times. One may also extend the results for a repairable system with dependent or heterogeneous component lifetimes under different types of repair.

References

[1] Ahmadi, J. and Arghami, N.R. (2001), Some univariate stochastic orders on record values, *Communications in Statistics-Theory and Methods*, **30**, 69-74.

[2] Asadi, M. and Raqab, M.Z. (2010), The Mean Residual of Record Values at the Level of Previous Records, *Metrika*, **72**, 51-64.

[3] Balakrishnan, N., Kamps, U. and Kateri, M. (2009), Minimal repair under a step-stress test, *Statistics & Probability Letters*, **79**, 1548–1558.

[4] Barlow, R.E. and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing: Probability Models,* To Begin With, Silver Spring.

[5] Block, H.W., Borges, W.S. and Savits, T.H. (1985), Age-dependent minimal repair, *Journal of Applied Probability*, **22**, 370-385.

[6] Brown, M. and Proschan, F. (1983), Imperfect repair, *Journal of Applied probability*, **20**(4), 851-859.

[7] Chahkandi, M. (2018), Some properties of repairable k-out-of-n systems, *Journal of Statistical Sciences*. Available from: http://jss.irstat.ir/article-1-504-fa.html. Accessed January 26, 2019.

[8] Deshpande, J.V., Kochar, S.C. and Singh, H. (1986), Aspects of positive ageing, *Journal of Applied Probability*, **23**, 748-758.

[9] Gupta, R.C. and Kirmani, S.N.U.A. (1988), Closure and monotonicity properties of nonhomogeneous Poisson processes and record values, *Probability in the Engineering and Informational Sciences*, **2**, 475-484.

[10] Khaledi, B., Shojaei, R. (2007), On Stochastic Orderings Between Residual Record Values, *Statistics and Pobability Letters*, **77**, 1467-1472.

[11] Kijima, M. (1989), Some results for repairable systems with general repair, *Journal of Applied probability*, **26**(1), 89-102.

[12] Kochar, S.C. (1990), Some partial ordering results on record values, *Communication in Statistics-Theory and Methods*, **19**, 299-306.

[13] Last, G. and Szekli, R. (1998), Stochastic comparison of repairable systems by coupling, *Journal of Applied probability*, **35**(2), 348-370.

[14] Samaniego, F.J. (1985), On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability*, **34**, 69-72.

[15] Samaniego, F.J. (2007), *System Signatures and their Applications in Engineering Reliability*, Springer, New York.

Measures of Inaccuracy for Concomitants of Generalized Order Statistics

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Abstract: In this paper, we obtain a measure of inaccuracy between rth concomitant of generalized order statistic and the parent random variable in Morgenstern family. Applications of this result are given for concomitants of order statistics and record values. We also study some results of cumulative past inaccuracy between the distribution function of rth concomitant of order statistic (record value) and the distribution function of parent random variable.

Keywords Measure of inaccuracy, Cumulative inaccuracy, Concomitants, Generalized order statistics.

Mathematics Subject Classification (2010) : 62B10, 62G30.

1 Introduction

The concept of generalized order statistics (GOS) was introduced by Kamps (1995) as a unified approach to a variety of models of ordered random variables such as ordinary order statistics, sequential order statistics, progressive type-II censoring, record values and Pfeifers records. The random variables $X(1,n,m,k), X(2,n,m,k),$
$\cdots, X(n,n,m,k)$ are called generalized order statistics based on the absolutely continuous distribution function F with density function f, if their joint density function is given by

$$\begin{aligned} f^{X(1,n,m,k),\ldots,X(n,n,m,k)}(x_1,\ldots,x_n) &= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^m f(x_i)\right) \\ &\quad \times(1-F(x_n))^{k-1}f(x_n), \\ &\quad F^{-1}(0)\le x_1\le x_2\le\ldots\le x_n\le F^{-1}(1), \end{aligned}$$

with parameters $n\in\mathbb{N}, k>0, m\in\mathbb{R}$, such that $\gamma_r = k+(n-r)(m+1)>0$, for all $1\le r\le n$.

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Let X and Y be two non-negative random variables with distribution functions $F(x)$ and $G(x)$, respectively. If $f(x)$ is the actual probability density function (pdf) corresponding to the observations and $g(x)$ is the density assigned by the experimenter, then the inaccuracy measure of X and Y is defined by Kerridge (1961) as follows:

$$I(f,g) = -\int_0^{+\infty} f(x)\log g(x)dx.$$

Analogous to this measure of inaccuracy, Thapliyal and Taneja (2015a) proposed a cumulative inaccuracy measure as

$$I(F,G) = -\int_0^{+\infty} F(x)\log G(x)dx.$$

Morgenstern (1956) defined a class of bivariate distributions with the probability density function (pdf) given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)\right], \quad |\alpha| \leq 1, \tag{1.1}$$

where α is the association parameter. For the Morgenstern family with pdf given by (1.1), the density function and distribution function of the concomitant of r-th GOS's $Y_{[r,n,m,k]}, 1 \leq r \leq n$, are given by Beg and Ahsanullah (2008), as follows:

$$g_{[r,n,m,k]}(y) = f_Y(y)\left[1+\alpha C^*(r,n,m,k)(1-2F_Y(y))\right], \tag{1.2}$$

$$G_{[r,n,m,k]}(y) = F_Y(y)\left[1+\alpha C^*(r,n,m,k)(1-F_Y(y))\right], \tag{1.3}$$

where $C^*(r,n,m,k) = \frac{2\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r(\gamma_i+1)} - 1$.

Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be independent and identically distributed random variables from Morgenstern distribution. If $X_{(r:n)}$ denotes the rth order statistic, then the Y's associated with $X_{(r:n)}$ denoted by $Y_{[r:n]}$ is called the concomitant of rth order statistic. The pdf and cdf of $Y_{[r:n]}$ are given by

$$f_{Y_{[r:n]}}(y) = f_Y(y)\left[1+\alpha\left(\frac{n-2r+1}{n+1}\right)(1-2F_Y(y))\right],$$

and

$$F_{Y_{[r:n]}}(y) = F_Y(y)\left[1+\alpha\left(\frac{n-2r+1}{n+1}\right)(1-F_Y(y))\right],$$

respectively. We refer the reader to Arnold (1992) for more details.

Let $(X_1, Y_1), (X_2, Y_2), \cdots$ be a sequence of bivariate random variables from a continuous distribution. If $\{R_n, n \geq 1\}$ is the sequence of upper record values in the sequence of X's, then the Y which corresponds with the nth-record will be called the concomitant of the nth-record, denoted by $R_{[n]}$. The concomitants of record values arise in a wide variety of practical experiments such as industrial stress testing, life time experiments, meteorological analysis, sporting matches and some other experimental fields. For other important applications of record values and their concomitants see Arnold (1992). The pdf and cdf for $R_{[n]}$ has obtained as follows:

$$f_{R_{[n]}}(y) = f_Y(y)[1 + \alpha_n(1 - 2F_Y(y))], \quad n \geq 1, \tag{1.4}$$

$$F_{R_{[n]}}(y) = F_Y(y)[1 + \alpha_n(1 - F_Y(y))], \tag{1.5}$$

where $\alpha_n = \alpha(2^{1-n} - 1)$.

Several authors have worked on measures of inaccuracy for ordered random variables. Thapliyal and Taneja(2013) proposed the measure of inaccuracy between the ith order statistic and the parent random variable. Thapliyal and Taneja (2015a) developed measures of dynamic cumulative residual and past inaccuracy. They studied characterization results of these dynamic measures under proportional hazard model and proportional reversed hazard model. Recently Thapliyal and Taneja (2015b) have introduced the measure of residual inaccuracy of order statistics and prove a characterization result for it. Motivated by some of the articles mentioned above, in this paper we aim to present some results on inaccuracy for concomitants of generalized order statistics in Morgenstern family.

2 Main Results

If $Y_{[r,n,m,k]}$ is the concomitant of rth-generalized order statistics from (1.2), then the inaccuracy measure between $g_{[r,n,m,k]}(y)$ and $f_Y(y)$ for $1 \leq r \leq n$, $\alpha \neq 0$ is given by

$$I(g_{[r,n,m,k]}, f_Y) = [1 + \alpha C^*(r, n, m, k)]\, H(Y) + 2\alpha C^*(r, n, m, k)\phi_f(u), \tag{2.1}$$

where $\phi_f(u) = \int_0^1 u \log f(F^{-1}(u))du$ and

$$H(Y) = -\int_0^\infty f_Y(y) \log f_Y(y) dy$$

is the Shannon entropy of the random variable Y.

As an application of the representation (2.1), we consider the following special cases.

Case 1: If we put $m = 0$ and $k = 1$, then an inaccuracy measure between $f_{Y_{[r:n]}}$ (density function of rth concomitant of order statistic) and f_Y in Morgenstern family is obtained as follows:

$$I(f_{Y_{[r:n]}}, f_Y) = H(Y) + \frac{3(n-2r+1)}{2(n+1)} \left[I(f_{Y_{[1:2]}}, f_Y) - I(f_{Y_{[2:2]}}, f_Y) \right]. \tag{2.2}$$

In the following, we present some examples and properties of $I(f_{Y_{[r:n]}}, f_Y)$. Let (X_i, Y_i), $i = 1, 2, ..., n$ be a random sample from Gumbels bivariate exponential distribution (GBED) with cdf

$$F(x,y) = \left(1 - \exp\left(\frac{-x}{\theta_1}\right)\right)\left(1 - \exp\left(\frac{-y}{\theta_2}\right)\right)\left[1 + \alpha \exp\left(\frac{-x}{\theta_1} - \frac{y}{\theta_2}\right)\right]. \tag{2.3}$$

From (2.2), we find

$$I(f_{Y_{[r:n]}}, f_Y) = [1 + \log\theta_2] - \frac{\alpha}{2}\left(\frac{n-2r+1}{n+1}\right). \tag{2.4}$$

By using (B.1), we get

$$A_\alpha(n) = I(f_{Y_{[n:n]}}, f_Y) - I(f_{Y_{[1:n]}}, f_Y) = \alpha\left(\frac{n-1}{n+1}\right),$$

which is positive, negative or zero whenever $(0 < \alpha \leq 1, n > 1)$, $(-1 \leq \alpha < 0, n > 1)$ or $(n = 1 \; or \; \alpha = 0)$, respectively. Also, the difference between $I(f_{Y_{[r:n]}}, f_Y)$ and $H(Y)$ is

$$B_{\alpha,n}(r) = I(f_{Y_{[r:n]}}, f_Y) - H(Y) = -\frac{\alpha}{2}\left(\frac{n-2r+1}{n+1}\right).$$

$B_{\alpha,n}(r)$ is positive for $-1 \leq \alpha < 0$, $1 \leq r < \frac{n+1}{2}$ (or $0 < \alpha \leq 1$, $\frac{n+1}{2} < r \leq n$). Also, it is negative for $-1 \leq \alpha < 0$, $\frac{n+1}{2} < r \leq n($ or $0 < \alpha \leq 1$,$1 \leq r < \frac{n+1}{2})$.

Now, if n is odd, then numerical computations indicate that $I(f_{Y_{[r:n]}}, f_Y)$ is increasing (decreasing) in r for $1 \leq r < \frac{n+1}{2}$, $0 < \alpha \leq 1$ ($\frac{n+1}{2} < r \leq n$, $-1 \leq \alpha < 0$). Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample from Morgenstern type bivariate Logistic distribution with cdf

$$F(x,y) = (1 + \exp(-x))^{-1}(1 + \exp(-y))^{-1}\left(1 + \frac{\alpha e^{-x-y}}{(1+e^{-x})(1+e^{-y})}\right).$$

Computation shows that

$$I(f_{Y_{[r:n]}}, f_Y) = 1 - 0.6\alpha\left(\frac{n-2r+1}{n+1}\right). \tag{2.5}$$

By using (2.5), we get

$$D_\alpha(n) = I(f_{Y_{[n:n]}}, f_Y) - I(f_{Y_{[1:n]}}, f_Y) = 1.2\alpha\left(\frac{n-1}{n+1}\right),$$

which is positive, negative or zero whenever $(0 < \alpha \leq 1, n > 1)$, $(-1 \leq \alpha < 0, n > 1)$ or $(n = 1 \; or \; \alpha = 0)$, respectively. Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample of size n with pdf (1.1). Then, we have

$$H(Y) = \frac{I(f_{Y_{[n:n]}}, f_Y) + I(f_{Y_{[1:n]}}, f_Y)}{2}.$$

We consider the concomitants of order statistics whenever $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are independent but otherwise arbitrarily distributed. Let us consider the Morgenstern family with cdf

$$F_{X_i,Y_i}(x, y) = F_{X_i}(x)F_{Y_i}(y)\left[1 + \alpha_i(1 - F_{X_i}(x))(1 - F_{Y_i}(y))\right]. \tag{2.6}$$

Now, suppose that $F_{X_i}(x) = F_X(x)$, $F_{Y_i}(y) = F_Y(y)$ and $|\alpha_i| \leq 1$. Then in this particular case, the pdfs of $Y_{[1:n]}$ and $Y_{[n:n]}$ are given by Eryilmaz (2005) as follows:

$$f_{[1:n]}(y) = f_Y(y)\left[1 + \frac{n-1}{(n+1)n}\sum_{j=1}^{n}\alpha_j(1 - 2F_Y(y))\right], \tag{2.7}$$

$$f_{[n:n]}(y) = f_Y(y)\left[1 - \frac{n-1}{(n+1)n}\sum_{j=1}^{n}\alpha_j(1 - 2F_Y(y))\right]. \tag{2.8}$$

Now, in the following theorem, the measures of inaccuracy for concomitants of extremes of order statistics is represented. Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be independent random vectors from (2.6). If $Y_{[1:n]}$ and $Y_{[n:n]}$ are concomitants of extremes of order statistics, then

$$I(f_{[1:n]}, f_Y) = \left(1 + \frac{n-1}{(n+1)n}\sum_{j=1}^{n}\alpha_j\right)H(Y) + 2\frac{n-1}{(n+1)n}\sum_{j=1}^{n}\alpha_j\phi_f(u), \tag{2.9}$$

$$I(f_{[n:n]}, f_Y) = \left(1 - \frac{n-1}{(n+1)n}\sum_{j=1}^{n}\alpha_j\right)H(Y) - 2\frac{n-1}{(n+1)n}\sum_{j=1}^{n}\alpha_j\phi_f(u). \tag{2.10}$$

By using (2.9) and (2.10) we have

$$A_n = I(f_{[n:n]}, f_Y) - I(f_{[1:n]}, f_Y) = -\frac{2(n-1)}{n(n+1)}\Delta$$

where we have set $\Delta = H(Y) + 2\sum_{j=1}^{n}\alpha_j\phi_f(u)$. if $\Delta > 0$ $(\Delta < 0)$ then $A_n < 0$ $(A_n > 0)$. Also we get

$$I(f_{[n:n]}, f_Y) + I(f_{[1:n]}, f_Y) = 2H(Y).$$

Case 2: If we put $m = -1$ and $k = 1$, then an inaccuracy measure between $f_{R_{[r]}}$ (density function of the concomitant of rth-record value) and f_Y in Morgenstern family is obtained as follows:

$$I(f_{R_{[r]}}, f_Y) = \left(1 + \alpha(2^{1-r} - 1)\right) H(Y) + 2\alpha(2^{1-r} - 1)\phi_f(u). \tag{2.11}$$

Let (X_i, Y_i), $i = 1, 2, ..., n$ be a random sample from Gumbels bivariate exponential distribution (GBED) with cdf

$$F(x, y) = \left(1 - \exp\left(\frac{-x}{\theta_1}\right)\right)\left(1 - \exp\left(\frac{-y}{\theta_2}\right)\right)\left[1 + \alpha \exp\left(\frac{-x}{\theta_1} - \frac{y}{\theta_2}\right)\right]. \tag{2.12}$$

From (2.11), we find

$$I(f_{R_{[r]}}, f_Y) = [1 + \log\theta_2] + \frac{\alpha}{2}\left(2^{1-r} - 1\right). \tag{2.13}$$

By using (2.13), we get

$$A_\alpha(r) = I(f_{R_{[r]}}, f_Y) - I(f_{R_{[r-1]}}, f_Y) = -\alpha 2^{-r},$$

which is positive, negative or zero whenever $(-1 \leq \alpha < 0, r > 1)$, $(0 < \alpha \leq 1, r > 1)$ or $(\alpha = 0)$, respectively. Also, the difference between $I(f_{R_{[r]}}, f_Y)$ and $H(Y)$ is

$$B_{\alpha,n}(r) = I(f_{R_{[r]}}, f_Y) - H(Y) = \frac{\alpha}{2}\left(2^{1-r} - 1\right).$$

$B_{\alpha,n}(r)$ is positive, negative or zero whenever $(-1 \leq \alpha < 0, r > 1)$, $(0 < \alpha \leq 1, r > 1)$ or $(r = 1 \; or \; \alpha = 0)$, respectively.

In analogy with (2.1), a measure of inaccuracy associated with $f_Y(y)$ and $g_{[r,n,m,k]}(y)$ is given by

$$I(f_Y, g_{[r,n,m,k]}) = H(Y) - E\left[\log\left(1 + \alpha C^*(r, n, m, k)\left(1 - 2U\right)\right)\right].$$

Quantile functions are efficient alternatives to the distribution function in modelling and analysis of statistical data. The quantile function is defined by,

$$Q(u) = F^{-1}(u) = \inf\{y : F(y) \geq u\}, \qquad 0 < u < 1.$$

Noting that $F(Q(u)) = u$ and differentiating it with respect to u yields
$q(u)f(Q(u)) = 1$. Let Y be a nonnegative random variable with pdf $f(\cdot)$ and quantile function

$Q(\cdot)$, then $f(Q(u))$ is called the density quantile function and $q(u) = Q'(u)$ is known as the quantile density function of Y. Now using (2.1), the corresponding quantile based $I(g_{[r,n,m,k]}, f_Y)$ is defined as

$$I(g_{[r,n,m,k]}, f_Y) = E(\log q(U)) + \alpha C^*(r,n,m,k) E\left[(1-2U)\log q(U)\right]. \tag{2.14}$$

2.1 Cumulative past inaccuracy measure for concomitants

If $Y_{[r,n,m,k]}$ is the concomitant of rth-generalized order statistics from (1.3), then the cumulative past inaccuracy measure between $G_{[r,n,m,k]}(y)$ and $F_Y(y)$ for $1 \leq r \leq n$, $\alpha \neq 0$ is given by

$$I(G_{Y_{[r,n,m,k]}}, F_Y) = \left[1 + \alpha C^*(r,n,m,k)\right] \mathcal{CE}(Y) - \frac{\alpha}{2} C^*(r,n,m,k) \mathcal{CE}(Y_{(2:2)}), \tag{2.15}$$

In analogy with (2.15), a measure of inaccuracy associated with F_Y and $G_{[r,n,m,k]}$ is given by

$$I(F_Y, G_{[r,n,m,k]}) = \mathcal{CE}(Y) - E\left[\frac{U \log\left(1 + \alpha C^*(r,n,m,k)\left(1-U\right)\right)}{f(F^{-1}(U))}\right].$$

Case 1: If we put $m = 0$ and $k = 1$, then a measure of inaccuracy between $F_{Y_{[r:n]}}$ (distribution function of rth concomitant of order statistic) and F_Y is presented as

$$I(F_{Y_{[r:n]}}, F_Y) = \left[1 + \alpha\left(\frac{n-2r+1}{n+1}\right)\right] \mathcal{CE}(Y) - \alpha\left(\frac{n-2r+1}{2(n+1)}\right)] \mathcal{CE}(Y_{(2:2)}), \tag{2.16}$$

where $\mathcal{CE}(Y) = -\int_0^\infty F_Y(y) \log F_Y(y) dy$ is the cumulative entropy of the random variable Y (see Di Crescenzo and Longobardi (2009)). Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample from Morgenstern type bivariate uniform distribution with cdf

$$F(x,y) = \frac{xy}{\theta_1\theta_2}[1 + \alpha(1 - \frac{x}{\theta_1})(1 - \frac{y}{\theta_2})], \quad 0 < x < \theta_1, \quad 0 < y < \theta_2.$$

Computation shows that

$$I(F_{Y_{[r:n]}}, F_Y) = \frac{\theta_2}{4} + \alpha\left(\frac{n-2r+1}{n+1}\right)(\frac{5\theta_2}{36}). \tag{2.17}$$

Using (B.6), we have

$$D_{\alpha,\theta_2}(n) = I(F_{Y_{[n:n]}}, F_Y) - I(F_{Y_{[1:n]}}, F_Y) = \frac{5\alpha\theta_2(-n+1)}{18(n+1)}.$$

which is positive, negative or zero whenever $(-1 \leq \alpha < 0)$, $(0 < \alpha \leq 1)$ or $(\alpha = 0)$, respectively. Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample from GBED . Then, computation shows that

$$I(F_{Y_{[r:n]}}, F_Y) = [\frac{\pi^2}{6} - 1]\theta_2 + \frac{\alpha\theta_2}{4}\left(\frac{n - 2r + 1}{n + 1}\right). \tag{2.18}$$

Using (B.8), we have

$$Q_{\alpha,\theta_2}(n) = I(F_{Y_{[n:n]}}, F_Y) - I(F_{Y_{[1:n]}}, F_Y) = \frac{\alpha\theta_2(-n + 1)}{2(n + 1)},$$

which is positive, negative or zero whenever $(-1 \leq \alpha < 0)$, $(0 < \alpha \leq 1)$ or $(\alpha = 0)$, respectively.

Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample from Morgenstern family. Then for $1 \leq r \leq \frac{n+1}{2}$, we have

$$I(F_{Y_{[r:n]}}, F_Y) \leq (\geq)\mathcal{CE}(Y), \quad -1 \leq \alpha < 0 \; (0 < \alpha \leq 1). \tag{2.19}$$

Proof. The proof follows by recalling Proposition 4.8 of Di Crescenzoand Longobardi (2009).

Case 2: If we put $m = -1$ and $k = 1$, then a measure of inaccuracy between $F_{R_{[r]}}$ (distribution function of nth concomitant of upper record value) and F_Y is presented as

$$I(F_{R_{[r]}}, F_Y) = [1 + \alpha(2^{1-r} - 1)]\mathcal{CE}(Y) + \alpha(2^{1-r} - 1)\int_0^\infty F_Y^2(y)\log F_Y(y)dy.$$

where $\mathcal{CE}(Y) = -\int_0^\infty F_Y(y)\log F_Y(y)dy$ is the cumulative entropy of the random variable Y. Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample from Morgenstern family. Then, we have

$$I(F_{R_{[r]}}, F_Y) \leq (\geq)\mathcal{CE}(Y), \quad 0 < \alpha \leq 1(-1 \leq \alpha < 0). \tag{2.20}$$

Proof. The proof follows by recalling Proposition 4.8 of Di Crescenzo and Longobardi (2009).

References

[1] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992), *A first course in order statistics*, John Wiley, New York.

[2] Beg, M.I., Ahsanullah, M. (2008), Concomitants of generalized order statistics from Farlie-Gumbel-Morgenstern distributions, *Statistical Methodology*, **5** (1), 1-20.

[3] Di Crescenzo, A. and Longobardi, M. (2009), On cumulative entropies, *Journal of Statistical Planning and Inference*, **139**, 4072-4087.

[4] Eryilmaz, S. (2005), Concomitants in a sequence of independent nonidentically distributed random vectors,*Communications in Statistics-Theory and Methods*, **34**(9-10), 1925-1933.

[5] Kamps, U. (1995), *A Concept of Generalized Order Statistics*, Teubner Skripten zur Mathematischen Stochastik. Teubner, Stuttgart.

[6] Kerridge, D.F. (1961), Inaccuracy and inference, *Journal of the Royal Statistical Society*, Series B (Methodological), **23**, 184-194.

[7] Morgenstern D. (1956), Einfache Beispiele zweidimensionaler verteilungen,*Mitteilungsblatt für Mathematische Statistik*, **8**, 234-235.

[8] Thapliyal, R. and Taneja, H.C. (2013), A measure of inaccuracy in order statistics, *Journal of Statistical Theory and Applications*, **12** (2), 200-207.

[9] Thapliyal, R. and Taneja, H.C. (2015a), Dynamic cumulative residual and past inaccuracy measures, *Journal of Statistical Theory and Applications*, **14** (4), 399-412.

[10] Thapliyal, R. and Taneja, H.C. (2015b), On residual inaccuracy of order statistics, *Statistics & Probability Letters*, **97**, 125-131.

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Aspects of Convexity and Concavity for Multivariate Copulas

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Abstract: In this talk we review recent results on different types of convexity and concavity of multivariate copulas and the relationships among them.

Keywords Componentwise concavity, Copula, Quasi-concavity, Schur-concavity.

Mathematics Subject Classification (2010) : 60E05, 62H20.

1 Introduction

Following Sklar (7) the joint distribution function H of a vector $(X_1, ..., X_n)$ of continuous random variables with the marginal distribution functions F_i, $i = 1, ..., n$, can then be expressed as $H(x_1, ..., x_n) = C\{F_1(x_1), ..., F_n(x_n)\}$, in terms of a unique multivariate copula $C : [0,1]^n \to [0,1]$, which is itself the joint distribution function of the vector $(U_1, ..., U_n) = (F_1(X_1), ..., F_n(X_n))$ of uniform (0,1) random variables. Let $\Pi^n(u_1, ..., u_n) = \prod_{i=1}^{n} u_i$ denote the copula of independent continuous random variables. Any copula C satisfies that $W^n(u_1, ..., u_n) \leq C(u_1, ..., u_n) \leq M^n(u_1, ..., u_n)$ for each $(u_1, ..., u_n) \in [0,1]^n$, where $W^n(u_1, ..., u_n) = \max(\sum_{i=1}^{n} u_i - n + 1, 0)$ and $M^n(\mathbf{u}) = \min(u_1, \cdots, u_n)$. For every $n \geq 2$, M^n is an copula; however W^n is a copula if and only if $n = 2$. For a complete discussion of copulas, see (7). The class of copulas will be denoted by $\mathcal{C}$. Given two copulas C_1 and C_2, let $C_1 \leq C_2$ denote the inequality $C_1(u_1, ..., u_n) \leq C_2(u_1, ..., u_n)$ for all $(u_1, ..., u_n) \in [0,1]^n$. Recently, investigations on various notions of concavity (convexity) for copulas such as the componentwise concavity (convexity), Schur concavity (convexity), Quasi concavity (convexity) have been considered, especially because of their application in the construction of asymmetric stochastic models, for example we can mention (1; 3; 4; 5). In this talk we review recent results on different aspects of convexity/concavity of copulas and the relationships among them.

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2 Different notions of concavity (convexity) of multivariate copulas

An n-dimensional copula C is called concave (convex) if for all $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n) \in [0,1]^n$ and $\lambda \in [0,1]$, $C(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) \geq (\leq) \lambda C(\mathbf{u}) + (1-\lambda)C(\mathbf{v})$, where $\lambda\mathbf{u} + (1-\lambda)\mathbf{v} = (\lambda u_1 + (1-\lambda)v_1, ..., \lambda u_n + (1-\lambda)v_n)$ (3). For the case $n = 2$ the concavity of a copula means that

$$C(\lambda u_1 + (1-\lambda)v_1, \lambda u_2 + (1-\lambda)v_2) \geq \lambda C(u_1, u_2) + (1-\lambda)C(v_1, v_2),$$

for all u_1, u_2, v_1, v_2 and $\lambda \in [0,1]$. As mentioned in (7), the only convex bivariate copula is W^2 and the only concave bivariate copula is M^2. Since W^n is not an a copula for $n > 2$, then the convex n-dimensional copula may not exist. The convexity and concavity are conditions too strong to be of much interest for copulas. Weaker versions of these properties are the componentwise concavity and componentwise convexity.

An n-dimensional copula C is called componentwise concave (convex) if it is concave (convex) in each coordinate when the other coordinates are held fixed. The case $n = 2$ is already studied in (5). It is easy to see that the copula M^n is componentwise concave and the copula Π^n is both componentwise concave and convex.

If C is the copula of the vector $(V_1, ..., V_n)$ of uniform [0,1] random variables then for $i = 1, ..., n$, it follows (see (7)) $\frac{\partial C(v_1,...,v_n)}{\partial v_i} = P(V_j \leq v_j, j = 1, ..., n, j \neq i | V_i = v_i)$. For a twice differentiable copula C, the componentwise concavity (convexity) means that for each $i = 1, ..., n$, the mapping $t \to P(V_j \leq v_j, j = 1, ..., n, j \neq i | V_i = t)$, is decreasing (increasing) in t. The copula C is positive lower orthant dependent (PLOD) if $C \geq \Pi^n$. The corresponding negative lower orthant dependence (NLOD) is defined by reversing the sense of the inequality. The vector $\mathbf{V}$ is said to be positive dependent through the stochastic ordering (PDS) if $P(V_j \leq v_j, j = 1, ..., n, j \neq i | V_i = t)$ is decreasing in t. Since PDS implies PLOD, then a componentwise concave copula is PLOD.

For any n-dimensional copula C, the conditions (i) C is componentwise concave; (ii) C is PDS are equivalent.

The FGM family of copulas (7) defined by $C_\theta(u_1, ..., u_n) = \prod_{i=1}^n u_i + \theta \prod_{i=1}^n u_i(1-u_i)$, $\theta \in [-1,1]$ is componentwise convex for $\theta \in [-1,0]$ and it is componentwise concave for $\theta \in [0,1]$. A copula C is Archimedean if it is of the form $C(u_1, ..., u_n) = \phi^{-1}\{\sum_{i=1}^n \phi(u_i)\}$, where $\phi^{-1}(0) = 1$ and $\phi^{-1}(x) \to 0$, as $x \to \infty$ and ϕ^{-1} is d-monotone, i.e., $(-1)^k \frac{d^k \phi^{-1}(t)}{dt^k} \geq 0$, for all k, (7).

An Archimedean copula with the strict generator ϕ is componentwise concave if, and only if $\frac{1}{\phi'}$ is concave, where ϕ' is the derivative of ϕ.

For the Clayton family of copulas (7) which is Archimedean with the generator $\phi(t) = (t^{-\alpha} - 1)/\alpha$, since $\left(\frac{1}{\phi'(t)}\right)'' = -\alpha(\alpha+1)t^{\alpha-1} < 0$ for all $\alpha > 0$, it is a componentwise concave copula.

For the Frank family of copulas (7) which is Archimedean with the generator $\phi(t) = \ln(\frac{1-\alpha}{1-\alpha^t})$, since $\left(\frac{1}{\phi'(t)}\right)'' = \alpha^{-t}\ln\alpha < 0$ for $\alpha \in (0,1)$, it is a componentwise concave copula for $\alpha \in (0,1)$.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and let $x_{[1]}, ..., x_{[n]}$ and $y_{[1]}, ..., y_{[n]}$ the components of $\mathbf{x}$ and $\mathbf{y}$ rearranged in decreasing order. The point $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (written $\mathbf{x} \prec_m \mathbf{y}$) if $\sum_{j=1}^n x_{[j]} = \sum_{j=1}^n y_{[j]}$ and $\sum_{j=1}^k x_{[j]} \leq \sum_{j=1}^k x_{[j]}$, for $k = 1, ..., n-1$. A real valued function $g : \mathbb{A} \subset \mathbb{R}^n \to \mathbb{R}$, is Schur-concave (Schur-convex) on $\mathbb{A}$ if for all $\mathbf{x}, \mathbf{y} \in A$, $\mathbf{x} \prec_m \mathbf{y}$ implies $g(\mathbf{x}) \geq (\leq) g(\mathbf{y})$; see, Marshall and Olkin (6). Let A be an open interval in $\mathbb{R}^n$. A function $g : \mathbb{A} \to \mathbb{R}$ is said to be symmetric if for each point $(x_1, ..., x_n) \in \mathbb{A}$, $g(x_1, ..., x_n) = g(x_{i_1}, ..., x_{i_n})$, for every permutation $(i_1, ..., i_n)$ of $(1, ..., n)$.

Let $\mathbb{A}$ be an open interval in $\mathbb{R}$ and let $g : \mathbb{A} \to \mathbb{R}$ be a continuously differentiable function. Then g is Schur-concave on $\mathbb{A}$ if, and only if, (i) g is symmetric; (ii) for all $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{A}$ and $i \neq j$, $(x_i - x_j)(\frac{\partial g(\mathbf{x})}{\partial x_i} - \frac{\partial g(\mathbf{x})}{\partial x_j}) \leq 0$. We note that since g is symmetric, the Schur-concavity condition in above proposition, can be reduced to $(x_1 - x_2)(\frac{\partial g(\mathbf{x})}{\partial x_1} - \frac{\partial g(\mathbf{x})}{\partial x_2}) \leq 0$.

For the FGM family of copulas, since $(u_1 - u_2)\left(\frac{\partial C(\mathbf{u})}{\partial u_1} - \frac{\partial C(\mathbf{u})}{\partial u_2}\right) = -(u_1-u_2)^2 \Pi_{j=3}^n \{1+\theta(1-u_1-u_2+2u_1u_2)\Pi_{j=3}^n(1-u_j)\} \leq 0$, holds when $|1-u_1-u_2+2u_1u_2| \leq 1$, it is a Schur-concave copula.

The next result characterizes the Schur-concave copulas.

An n-dimensional copula C is Schur-concave if, and only if, for all $u_1, ..., u_n$ and $\lambda_{ij} \in [0,1]$ with $\sum_{j=1}^n \lambda_{ij} = 1$, for all $i = 1, ..., n$ and $\sum_{i=1}^n \lambda_{ij} = 1$, for all $j = 1, ..., n$, $C(u_1, ..., u_n) \leq C\left(\sum_{j=1}^n \lambda_{1j}u_j, ..., \sum_{j=1}^n \lambda_{nj}u_j\right)$.

The copula M^n Schur-concave.. For $u_1, ..., u_n \in [0,1]$, suppose that $\min(u_1, ..., u_n) = u_n$. Using the fact that $\sum_{j=1}^n \lambda_{ij} = 1$, for $i = 1, ..., n$, it is follows that $\sum_{j=1}^{n-1} \lambda_{ij}u_n \leq \sum_{j=1}^{n-1} \lambda_{ij}u_j, i = 1, ..., n$, or equivalently, $u_n \leq \sum_{j=1}^n \lambda_{ij}u_j, \quad i = 1, ..., n$ and then $u_n \leq \min\{\sum_{j=1}^n \lambda_{1j}u_j, ..., \sum_{j=1}^n \lambda_{nj}u_j\}$. By changing u_n to arbitrary u_i, one get $\min(u_1, ..., u_n) \leq \min\{\sum_{j=1}^n \lambda_{1j}u_j, ..., \sum_{j=1}^n \lambda_{nj}u_j\}$. That is, M^n is a Schur-concave n-copula.

The following result provides the Schur-concavity of the Archimedean n-dimensional copulas.

Every Archimedean n-dimensional copula is Schur-concave.

As a consequence of this result, since the copula Π^n is Archimedean with generator $\phi(t) =$

$-\log(t)$, it is Schur-concave.

When $n = 2$, as shown in (4) the copula W^2 is the only Schur-convex copula (and since W^2 is Archimedean, it is also a Schur-concave copula). Since W^n is not an n-dimensional copula for $n > 2$, then the Schur-convex n-dimensional copula may not exist.

For any $\mathbf{u} = (u_1, ..., u_n) \in [0, 1]^n$, the k-dimensional marginal C_k, $k = 2, ..., n-1$, of a symmetric n-dimensional copula C is defined by $C_k(u_1, ..., u_k) = C(u_1, .., u_k, 1, ..., 1)$. If C is Schur-concave (Schur-convex), then C_k, $k = 2, ..., n-1$, is Schur-concave (Schur-convex) as well.

A bivariate copula C is said to be quasi-concave (7) if for all $(u, v), (u', v') \in [0, 1]^2$ and all $\lambda \in [0, 1]$, $C(\lambda u + (1-\lambda)u', \lambda v + (1-\lambda)v') \geq \min\{C(u, v), C(u', v')\}$.

The n-dimensional ($n \geq 2$) extension of quasi-concavity is as follows:

An n-dimensional copula C is called quasi-concave if for all $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n)$ in $[0, 1]^n$ and $\lambda \in [0, 1]$, $C(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) \geq \min\{C(\mathbf{u}), C(\mathbf{v})\}$. This condition is equivalent to requiring that upper-level sets of C, i.e., $U_q = \{\mathbf{u} \in \mathbb{I}^n : C(\mathbf{u}) \geq q\}$, are convex for all q.

As for all q, the set $U_q = \{\mathbf{u} \in \mathbb{I}^n : u_1 \geq q, ..., u_n \geq q\}$ is convex, then the n-dimensional copula M^n turns out to be a quasi-concave.

Note that the only quasi-convex copula is W^2 (see, (1)). Since W^2 is an Archimedean copula, it is also Quasi-concave. Since W^n is not a copula for $n > 2$, the n-dimensional copulas with the quasi-convexity property does not exist.

Every Archimedean n-dimensional copula is quasi-concave.

3 Concluding remarks and questions

We provided some results on different types of concavity and convexity properties in the class of multivariate copulas. Many questions suggest themselves for further study. We present a few:

(i) Geometrical interpretations for different types of convexity/concavity concepts for bivariate copulas can be found in the literature, see, e.g, Section 3.4.3 in (7). Is it possible to provide geometric interpretations for some of these concepts in multivariate setting?

(ii) For the case $n = 2$, the relations among the considered convexity/concavity notions could be found in (1; 5). For example: Quasi-concavity and symmetry imply Schur-concavity and componentwise concavity implies quasi-concavity. Does it occur in higher dimensions?

(iii) In bivariate case, the preservation of componentwise concavity, and Schur-concavity with respect to the ordinal sum is studied in (4; 5). Does any of the introduced convexity/concavity notion preserve under multivariate ordinal sum?

References

[1] Alvoni, E., Durante, F., Papini, P.L. and Sempi, C. (2007), Different types of convexity and concavity for copulas, In: *New Dimensions in Fuzzy Logic and related Technologies Proc. 5th EUSFLAT Conference*, (Štěpnička, M., Novák, V. and Bodenhofer, U. eds.), Volume 1, University of Ostrava, 185-189.

[2] Barlow, R.E. and Spizzichino, F. (1993), Schur-concave survival functions and survival analysis, *Journal Computational and Applied Mathematics*, **46**(3), 437-447.

[3] Dolati, A. and Nezhad, A.D. (2014), Some results on convexity and concavity of multivariate copulas, *Iranian Journal of Mathematical Sciences and Informatics*, **9**(2), 87-100.

[4] Durante, F. and Sempi, C. (2003), Copulæ and Schur-concavity, *International Mathematical Journal*, **3**, 893-905.

[5] Durante, F. and Papini, P.L. (2009), Componentwise concave copulas and their asymmetry, *Kybernetika*, **45**(6), 1003-1011.

[6] Marshall, A. and Olkin, I. (1979), *Inequalities: Theory of Majorization and its Application*, Academic Press, New York.

[7] Nelsen, R.B. (2006), *An Introduction to Copulas*, Second Edition. Springer, New York.

Survival Function Estimation in Length-Biased and Right Censored Data

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Abstract: This article concerns an almost sure representation with rate for a new estimator of a survival function in the setting of length-biased and right-censored data. This representation will be the key to obtain the asymptotic properties including normality and the uniform consistency with rate of convergence for the kernel density estimator. Simulation study are drawn to illustrate the results and to show how the kernel estimator behaves for finite samples.

Keywords Asymptotic normality, Density estimation, Kernel method, Length-biased and right-censored data, Strong representation.

Mathematics Subject Classification (2010): 62G05, 62G20.

1 Introduction

Length-biased and right-censored (LBRC) data is frequently encountered in various situations, and may arise in a prevalent cohort sampling. Statistical methods for the analysis of prevalent cohort data are considered when the onset or diagnosis time of the disease is known. In the cohort study, two conditions are considered. Assumption (1) the rate of disease occurrence remains constant over time, and (2) the density function of the time from enrollment to death is independent of the time from onset disease until the recruitment time. Conditions (1) and (2) together are referred to as the equilibrium conditions. Although, it is not always possible to follow individuals who have experienced the initiating event until the final event occurs. Hence, people may be subject to censorship. Let (A, V, C) be random variables, where A is the current-age, V represents residual lifetime, and C denotes residual censoring times. In a prevalent cohort study, A can be thought of as the time between the disease occurrence and study enrollment, V be the time from recruitment to death or censoring by the termination of study or lost follow-up during the study period and C be the censoring time from the enrollment to censoring

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occurrence. An individual would be qualified to be included in the sampling population at the recruitment time only if the survival time of i-th subject (T_i) be greater than A_i. In this model it is assumed that C is independent of (A, V). Here, the censoring is never non-informative because the censoring variable $A + C$ and survival time $T = A + V$ share the same A.

Nonparametric estimator of the survival function in LBRC model that has recently received much attention has been studied by (**?**), (**?**), (**?**) and (**?**) among others. (**?**) proposed a new nonparametric estimator of the survival function of the lifetime, which do not lose much efficiency compared to the nonparametric maximum likelihood estimator and is simpler than Huang and Qin's estimator.

Let $f(\cdot)$ and $F(\cdot)$ denote the marginal density function and distribution function of T and $S(t) = 1 - F(t)$ be its survival function and for any random variable W, we denote f_W, F_W, and S_W as density function, distribution function, and survival function for W, respectively. Suppose also that the survival distribution function of C is denoted as $G(\cdot)$. Given a random sample $\{(A_i, Y_i, \Delta_i),\ i = 1, \ldots, n\}$, where $Y_i = \min(T_i, A_i + C_i) = \min(A_i + V_i, A_i + C_i) = A_i + \tilde{V}_i$ and $\Delta_i = I(V_i \leq C_i)$, the product limit estimator of survival function F defined in (**?**) is as follows

$$\hat{S}_n(t) = \prod_{u \in [0,t]} \left\{1 - \frac{dH_n(u)}{R_n(u)}\right\}, \tag{1.1}$$

where

$$\begin{aligned}
H_n(t) &= n^{-1} \sum_{i=1}^{n} \Delta_i I(Y_i \leq t), \\
R_n(t) &= (2n)^{-1} \sum_{i=1}^{n} \left\{ I(A_i \leq t \leq Y_i) + \Delta_i I(\tilde{V}_i \leq t \leq Y_i) \right\}.
\end{aligned}$$

One of the most important properties of the $\hat{S}_n(t)$, employed in our proofs, is the strong representation of this estimator as a sum of iid random variables plus a remainder term. (**?**) obtained a remainder term of order o(1) that we will improve it. This convergence rate allows us to establish the strong consistency of kernel density estimator with a rate.

The outline of this paper is as follows. In Section 2, we obtain an almost sure representation of PL estimator $\hat{S}_n$ with a remainder term of order $O\left(n^{-1} \log\log n\right)$, when observations are subject to LBRC. Section 3 deals with the kernel estimator for density function and some asymptotic results. The proof of some preliminary lemmas are relegated to the Appendix. In Section 4 we summarize some simulation results for the quality of the kernel density estimation for various forms of the underlying density based on the mean integrated squared error (MISE).

2 Strong representation for the PL estimator

The goal of this section is to establish a strong representation of $\hat{S}_n$ in (1.1) for LBRC data and to obtain the order of the remainder term. Before stating the main results of this section, we introduce some notations. Define the functions

$$R(t) = \frac{1}{2}\,\mathrm{E}[I(A \le t \le Y) + \Delta I(\tilde{V} \le t \le Y)],$$
$$H(t) = \mathrm{E}[\Delta I(Y \le t)],$$
$$w(t) = \int_0^t G(u)du,$$

which R and H can be consistently estimated by R_n and H_n. Note that $H(\cdot)$ is a sub-distribution function corresponding to $F(\cdot)$, which is the proportion of failure uncensored events before time t in the presence of length-biased. We therefore have

$$R(t) = \mu^{-1}S(t)w(t), \quad dH(t) = \mu^{-1}f(t)w(t)dt. \tag{2.1}$$

Thus, in view of (C.1), the cumulative hazard function of T can be derived as

$$\mathit{\Lambda}(t) = \int_0^t \frac{f(u)}{S(u)}du = \int_0^t \frac{\mu^{-1}f(u)w(u)}{\mu^{-1}S(u)w(u)}du = \int_0^t \frac{dH(u)}{R(u)}.$$

Hence, a natural estimator of $\mathit{\Lambda}$, based on n observations $\{(A_i, Y_i, \Delta_i),\ i = 1, \dots, n\}$ is given by

$$\hat{\mathit{\Lambda}}_n(t) = \int_0^t \frac{dH_n(u)}{R_n(u)} = \frac{1}{n}\sum_{i:\ Y_i \le t} \frac{\Delta_i}{R_n(Y_i)}. \tag{2.2}$$

The following theorems provide the i.i.d representations of $\hat{\mathit{\Lambda}}_n$ to obtain the strong representations for the estimator $\hat{F}_n(t) = 1 - \hat{S}_n(t)$.

If $b < \tau$, then we have uniformly in $0 \le t \le b$,

$$\hat{\mathit{\Lambda}}_n(t) - \mathit{\Lambda}(t) = n^{-1}\sum_{i=1}^n \phi_i(t) + L_n(t),$$

where $\sup_{0\le t\le b}|L_n(t)| = O\left(n^{-3/4}(\log n)^{3/4}\right)$ $a.s.$ and

$$\phi_i(t) = \frac{\Delta_i I(Y_i \le t)}{R(Y_i)} - \frac{1}{2}\int_0^t R^{-2}(u)\{I(A_i \le u \le Y_i) + \Delta_i I(\tilde{V}_i \le u \le Y_i)\}dH(u). \tag{2.3}$$

. See the Appendix. □

Theorem 2 below gives a rate for the strong consistency of the cumulative hazard function estimator. For $b < \tau$, we have

$$\sup_{0<t<b} |\hat{\mathit{\Lambda}}_n(t) - \mathit{\Lambda}(t)| = O\left(n^{-1/2}(\log\log n)^{1/2}\right), \quad a.s.$$

. See the Appendix. □

The following theorem is crucial to obtain convergence rate of the kernel density estimator of f, the density associated to F, in Section 3. Let $\tau = \inf\{x;\ F(x) = 1\}$. We have uniformly in $0 \le t \le b < \tau$,

$$\begin{aligned}\hat{F}_n(t) - F(t) &= (1 - F(t))(\hat{\Lambda}_n(t) - \Lambda(t)) + L_n(t) \\ &= n^{-1} \sum_{i=1}^{n} (1 - F(t))\phi_i(t) + L_n(t)\end{aligned}$$

with $\sup_{0 \le t \le b} |L_n(t)| = O\left(n^{-1} \log\log n\right) \quad a.s.$. In view of $1 - F(t) = \exp\{-\Lambda(t)\}$ and using Lemma A, Lemma A in the Appendix and Taylor's expansion, we have

$$\begin{aligned}\hat{F}_n(t) - F(t) =& \bar{F}_n(t) - F(t) + O\left(n^{-1}\right) \\ =& \exp\{-\Lambda(t)\} - \exp\{-\hat{\Lambda}_n(t)\} + O\left(n^{-1}\right) \\ =& \exp\{-\Lambda(t)\}(\hat{\Lambda}_n(t) - \Lambda(t)) \\ &- \frac{\exp\{-\Lambda_n^*(t)\}}{2}(\hat{\Lambda}_n(t) - \Lambda(t))^2 + O\left(n^{-1}\right), \quad a.s.\end{aligned}$$

where $\Lambda_n^*(t)$ is some random point between $\min\{\hat{\Lambda}_n(t), \Lambda(t)\}$ and $\max\{\hat{\Lambda}_n(t), \Lambda(t)\}$. Corollary 2 imply that $\Lambda_n^*(t) \to \Lambda(t)$ a.s., as $n \to \infty$. From the continuity of $\exp\{-x\}$, we have $\exp\{-\Lambda_n^*(t)\} \to \exp\{-\Lambda(t)\}$ $a.s.$ Thus, the proof of the theorem will be completed using Corollary 2 again. □

3 Application: density estimation

Estimation of the density function of a random variable is a fundamental problem in probability and statistics. Estimation for the density function via various methods have been discussed by some authors. For instance, (?), (?) and (?). Among the various methods of density estimation, kernel smoothing is particularly appealing for both its simplicity and its interpretability. The pioneer of the kernel density estimation was (?) and (?). The problem of estimating the density function in LBRC model using kernel method so far not available, although (?) deal with this problem in the presence of bias and right censoring using projection method.

Let $\{h_n, n \ge 1\}$ be a sequence of positive bandwidths tending to zero and $K(\cdot)$ be a smooth kernel function. In this section, considering the well-known kernel estimator

$$\hat{f}_n(t) = h_n^{-1} \int K\left(\frac{t - x}{h_n}\right) d\hat{F}_n(x), \tag{3.1}$$

for LBRC data, we obtain consistency and asymptotic normality of this estimator as an application of the strong representation given in Theorem 2. Assume that the kernel function $K(\cdot)$ is symmetric, of bounded variation on $(-1,1)$ and satisfies the following conditions

$$\int_{-1}^{1} K(u)du = 1, \quad \int_{-1}^{1} uK(u)du = 0, \quad \int_{-1}^{1} u^2K(u)du > 0. \tag{3.2}$$

According to the second equality in (C.1), $\tilde{h}(t) = \mu^{-1}f(t)w(t)$ is the density of $H(t)$ and a kernel-type estimate of $\tilde{h}_n(t)$ is $\tilde{h}_n(t) = h_n^{-1}\int K(\frac{t-x}{h_n})dH_n(x)$. It is the aim of this section to give a representation of $\hat{f}_n - \bar{f}_n$ in terms of a sum of random variables which data are assumed to be LBRC, plus a negligible remainder, where

$$\bar{f}_n(t) = h_n^{-1}\int K\left(\frac{t-x}{h_n}\right)dF(x).$$

Let $\{h_n\}$ be a sequence of positive bandwidths satisfying

$$\frac{nh_n^2}{\log\log n} \to \infty.$$

Under the assumptions of Theorem 2 and f be bounded on $[0,b]$, we have

$$\sup_{0\le t\le b}\sqrt{nh_n}|\hat{f}_n(t) - \bar{f}_n(t) - \frac{\mu}{w(t)}\{\tilde{h}_n(t) - \mathrm{E}[\tilde{h}_n(t)]\}| = C_n,$$

where $C_n = O\Big((\frac{\log\log n}{nh_n^2})^{1/2} \vee (h_n\log\log n)^{1/2}\Big)$ $a.s.$ **.** See (**?**). $\square$

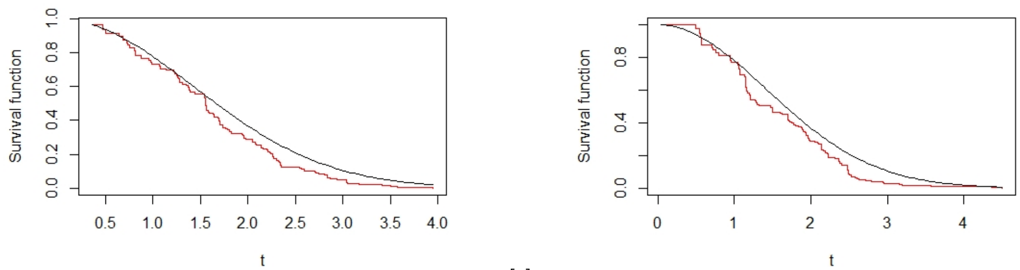
As a result of the above theorem, we can derive the strong uniform consistency and asymptotic normality of kernel density estimator and also the almost sure convergence with a rate of the kernel mode estimator.

4 Monte Carlo simulations

In this section, some numerical simulations are carried out to evaluate the performance of the kernel density estimator in LBRC scheme. The survival function $\hat{S}_n$ is computed under different levels of censoring and truncation in this simulation study.

We consider the survival variable T follows Weibull distribution with cumulative distribution function $S(t) = \exp(-t^2/4)$. The recruitment time was set to be 100 and onset variable was simulated from a $U(0,100)$ distribution. Let $C \sim U(1,2)$ and $C \sim U(0,2)$ corresponding with approximately 30% and 50% censoring. Two sample size were used, $n = 200$ and $n = 500$.

Figure 1 represents the survival estimator of (**?**) for Weibull(2,2) under LBRC sample. The left panel shows the estimator 1.1 for 30% censoring. The right panel shows the estimator for heavy censoring with about 50% observations being censored.



[t]0.5 [t]0.5

Figure 1: $\hat{S}_n(t)$ for Wiebull(2,2) with 30%(left) and 50%(right) censoring and $n = 200$

A

This section presents some preliminary lemmas that are used in the proofs of the main results. As for Theorem 2, we need a slight modification of product-limit estimator $\hat{F}_n$. Our modification of $\hat{F}_n$ is analogous to that of (**?**) for the random censorship model. This is only to safeguard against $\log 0$, when taking logarithms of $1 - \hat{F}_n(t)$. In the folglowing lemma, we show that the estimator $\bar{F}_n$ behaves in the same way as $\hat{F}_n$, where

$$\bar{F}_n(t) = 1 - \prod_{i:Y_i \le t} \left\{ 1 - \frac{\Delta_i}{nR_n(Y_i) + 1} \right\}. \tag{A.1}$$

Uniformly in $0 < t < b < \tau$, we have

$$\bar{F}_n(t) - \hat{F}_n(t) = O\left(n^{-1}\right), \quad a.s.$$

. According to (1.1), one has

$$\bar{F}_n(t) - \hat{F}_n(t) = \prod_{i:Y_i \le t} \left\{ 1 - \frac{\Delta_i}{nR_n(Y_i)} \right\} - \prod_{i:Y_i \le t} \left\{ 1 - \frac{\Delta_i}{nR_n(Y_i) + 1} \right\}.$$

Then applying $|\prod_{i=1}^n c_i - \prod_{i=1}^n d_i| \le \sum_{i=1}^n |c_i - d_i|, \ |c_i|, |d_i| \le 1$, we have

$$\begin{aligned} \bar{F}_n(t) - \hat{F}_n(t) &\le \sum_{i:Y_i \le t} n^{-2} \frac{\Delta_i}{R_n^2(Y_i)} \\ &\le n^{-1} \int_0^b \frac{1}{R_n^2(u)} dH_n(u) \\ &\le \sup_{0<u<b} \left| \frac{R^2(u)}{R_n^2(u)} \right| n^{-1} \int_0^b \frac{1}{R^2(u)} dH_n(u). \end{aligned}$$

From the SLLN, as $n \to \infty$,

$$\int_0^b \frac{1}{R^2(u)} dH_n(u) \to \int_0^b \frac{1}{R^2(u)} dH(u)$$

Thus, the desired conclusion follows. $\square$

Uniformly in $0 < t < b < \tau$, we have

$$1 - \bar{F}_n(t) = \exp\{-\hat{\Lambda}_n(t)\} + O\left(n^{-1}\right) \quad a.s.$$

. Using $|e^{-x} - e^{-y}| \le |x - y|,\ x, y \ge 0$ and expanding log expression, we have

$$\begin{aligned}
|1 - \bar{F}_n(t) - \exp\{-\hat{\Lambda}_n(t)\}| &\le |\log(1 - \bar{F}_n(t)) + \hat{\Lambda}_n(t)| \\
&= \left| \sum_{i:Y_i \le t} \log\left(1 - \frac{\Delta_i}{nR_n(Y_i) + 1}\right) + \sum_{i:\ Y_i \le t} \frac{\Delta_i}{nR_n(Y_i)} \right| \\
&= \left| \sum_{i:Y_i \le t} \frac{\Delta_i}{nR_n(Y_i)(nR_n(Y_i) + 1)} - \sum_{i:Y_i \le t} \sum_{m=2}^{\infty} \frac{\Delta_i}{m(nR_n(Y_i) + 1)^m} \right|,
\end{aligned} \tag{A.2}$$

which

$$(\text{A.2}) \le \sum_{i:Y_i \le t} \frac{\Delta_i}{n^2 R_n^2(Y_i)}.$$

Thus

$$|1 - \bar{F}_n(t) - \exp\{-\hat{\Lambda}_n(t)\}| \le n^{-1} \int_0^b \frac{dH_n(u)}{R_n^2(u)} = O\left(n^{-1}\right) \quad a.s.$$

This completes the proof of the Lemma. $\square$

$$\sup_{0<t<b} \left| \int_0^t \left(\frac{1}{R_n(u)} - \frac{1}{R(u)} \right) d[H_n(u) - H(u)] \right| = O\left(n^{-3/4} (\log n)^{3/4}\right) \quad a.s.$$

Proof of Theorem 2. It is easy to check that

$$\begin{aligned}
\hat{\Lambda}_n(t) - \Lambda(t) &= \int_0^t \frac{dH_n(u)}{R(u)} - \int_0^t \frac{R_n(u)}{R^2(u)} dH(u) + L_n(t) \\
&= \frac{1}{n} \sum_{i=1}^n \phi_i(t) + L_n(t),
\end{aligned}$$

where ϕ_i is defined in (2.3) and

$$\begin{aligned}
L_n(u) &= \int_0^t \left(\frac{1}{R_n(u)} - \frac{1}{R(u)}\right) d[H_n(u) - H(u)] + \int_0^t \frac{(R_n(u)-R(u))^2}{R_n(u)R^2(u)} dH(u) \\
&=: L_{n1}(t) + L_{n2}(t).
\end{aligned}$$

We first deal with the remainder term L_{n1}. From Lemma C, it implies that

$$\begin{aligned}
L_{n1}(t) &\leq \sup_{0<t<b} \left| \int_0^t \left(\frac{1}{R_n(u)} - \frac{1}{R(u)}\right) d[H_n(u) - H(u)] \right| \\
&= O\left(n^{-3/4}(\log n)^{3/4}\right) \quad a.s. \qquad \text{(A.3)}
\end{aligned}$$

The LIL for empirical distribution functions provides

$$\sup_{0<t<b} |R_n(t) - R(t)| = O\left(n^{-1/2}\sqrt{\log\log n}\right) \quad a.s.$$

Then

$$\begin{aligned}
L_{n2}(t) &\leq \sup_{0<u<t} (R_n(u) - R(u))^2 \int_0^t \frac{dN(u)}{R_n(u)R^2(u)} \\
&= O\left(n^{-1}\log\log n\right) \quad a.s.,
\end{aligned}$$

which together with (A.3) yields the result. $\square$

***Proof of Theorem* 2.** Applying integration by parts, one can easily get the following

$$\begin{aligned}
|\hat{\Lambda}_n(t) - \Lambda(t)| &\leq \int_0^t \left|\frac{d[H_n(u) - H(u)]}{R(u)}\right| + \int_0^t \left|\frac{1}{R(u)} - \frac{1}{R_n(u)}\right| dH_n(u) \\
&\leq \frac{|H_n(t) - H(t)|}{R(t)} + \int_0^t |H_n(u) - H(u)| d\frac{1}{R(u)} \\
&+ \int_0^t \frac{|R_n(u) - R(u)|}{R(u)R_n(u)} dH_n(u)
\end{aligned}$$

Thus we obtain the desired result by the LIL for the empirical processes.

The Odd Generalized Half-Normal Power Series Distribution

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Abstract: In this paper, a new four-parameter model called the odd generalized half-normal power series (OGHNPS) distribution is defined and studied, which is obtained by compounding odd generalized half-normal (OGHN) and power series distributions. The new distribution has several desired properties and nice physical interpretations. We derive the reliability functions, the moments, and the moment generating function of the new distribution. The method of maximum likelihood is used for estimating parameters. Two particular cases of this distribution are introduced and studied in a real example.

Keywords Compound distribution, Odd log-logistic family of distributions, Generalized half-normal distribution, Power series distribution.

Mathematics Subject Classification (2010) : 62E15**,** 62E10**,** 62N99**.**

A Introduction

The generalized half-normal (GHN) distribution (Cooray, and Ananda 2008) has a wide range of applications, including lifetime testing experiments, measurement errors, applied statistics, and clinical studies. The GHN distribution with parameters $\alpha > 0$ and $\beta > 0$ has probability density function (pdf) and cumulative distribution function (cdf) given by

$$\pi\left(y;\alpha,\beta\right)=\sqrt{\frac{2}{\pi}}\alpha\beta y^{\alpha-1}e^{-\frac{1}{2}\beta^{2}x^{2\alpha}},$$

and

$$\Pi\left(y;\alpha,\beta\right)=2\Phi\left(\beta y^{\alpha}\right)-1,$$

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respectively. On the other hand, for the GHN distribution with cdf $\Pi(x;\alpha,\beta)$ ($\Pi(x)$ for short), Cordeiro et al. (2008) proposed the cdf $G(x;\alpha,\beta,\gamma)$ of the odd generalized half-normal distribution with an additional shape parameter $\gamma > 0$ given by

$$G(x;\alpha,\beta,\gamma) = \frac{\Pi(x)^{\gamma}}{\Pi(x)^{\gamma} + \overline{\Pi}(x)^{\gamma}} \tag{A.1}$$

for $x > 0$, where $\overline{\Pi}(x) = 1 - \Pi(x)$ is the survival function of GHN distribution. This model commonly has been used in reliability studies and fatigue lifetime data. The purpose of this paper is to introduce a new lifetime distribution by compounding an OGHN distribution and the power series distribution, which refers to OGHNPS distribution. The compounding procedure follows key ideas of Marshall and Olkin (1997).

The rest of the paper is organized as follows. In Section 2, the odd generalized half-normal power series and its two well-known particular cases are introduced. Some of the mathematical properties are derived in Section 3. Estimation of the parameters of the new distribution by maximum likelihood method and a simulation study are investigated in Section 4. An illustrative example of real data set given in Section 5. The paper is concluded in Section F.

B The model definition

A discrete random variable, N is a member of power series distributions (truncated at zero) if its probability mass function is given by

$$P(n;\lambda) = \frac{a_n \lambda^n}{A(\lambda)}, \quad n = 1, 2, \ldots, \tag{B.1}$$

where a_n depends only on n and not on λ, $A(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$ and $\lambda > 0$ is such that $A(\lambda)$ is finite. In (B.1) λ is the power parameter of the distribution and $A(.)$ is the series function. This family of discrete random variables includes many of the most common distributions, including the geometric, Poisson, binomial, and logarithmic distributions.

Let N be a random variable denoting the number of failure causes which it is a member of power series distributions (truncated at zero). For given N, let $X_1, X_2, ..., X_N$ be independent identically distributed random variables from OGHN distribution and $X = \max\{X_i\}_{i=1}^{N}$. It could be shown that the cdf of X is

$$\begin{aligned}
F(x;\boldsymbol{\theta}) &= \sum_{n=1}^{\infty} F(x|N;\alpha,\beta,\gamma) P(n;\lambda) \\
&= \{A(\lambda)\}^{-1} \sum_{n=1}^{\infty} a_n \{\lambda G(x;\alpha,\beta,\gamma)\}^n \\
&= \{A(\lambda)\}^{-1} A(\lambda G(x;\alpha,\beta,\gamma)) \\
&= \{A(\lambda)\}^{-1} A\left(\frac{\lambda \Pi(x)^{\gamma}}{\Pi(x)^{\gamma} + \overline{\Pi}(x)^{\gamma}}\right) \\
&= \{A(\lambda)\}^{-1} A\left(\lambda\left[1+\left\{\frac{2\Phi(-\beta x^{\alpha})}{2\Phi(\beta x^{\alpha})-1}\right\}^{\gamma}\right]^{-1}\right) \qquad \text{(B.2)}
\end{aligned}$$

for $x > 0$, where $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)$ is the parameter vector of OGHNPS distribution.

The half-normal power series and the generalized half-normal power series (Tahmasebi, 2017) distributions are particular cases for $\alpha = \gamma = 1$ and $\gamma = 1$, respectively.

The exponentiated odd generalized half-normal family of distributions with the shape parameter c is a limiting special case of the OGH-NPS family of distributions when $\lambda \to 0^+$, where $c = \min\{n \in \mathbb{N} : a_n > 0\}$.

The pdf function of the OGHNPS class of distributions is given by

$$\begin{aligned}
f(x;\boldsymbol{\theta}) &= \frac{\sqrt{\frac{2}{\pi}}\alpha\beta\gamma\lambda x^{\alpha-1}[\{2\Phi(\beta x^{\alpha})-1\}\{2\Phi(-\beta x^{\alpha})\}]^{\gamma-1}}{A(\lambda) e^{\frac{1}{2}\beta^2 x^{2\alpha}}[\{2\Phi(\beta x^{\alpha})-1\}^{\gamma}+\{2\Phi(-\beta x^{\alpha})\}^{\gamma}]^2} \\
&\quad A'\left(\lambda\left[1+\left\{\frac{2\Phi(-\beta x^{\alpha})}{2\Phi(\beta x^{\alpha})-1}\right\}^{\gamma}\right]^{-1}\right). \qquad \text{(B.3)}
\end{aligned}$$

Two well-known particular cases of the OGHNPS distribution are as follows:

1. The pdf of odd generalized half-normal geometric (OGHNG) distribution is given by

$$f(x;\boldsymbol{\theta}) = \frac{\sqrt{\frac{2}{\pi}}\alpha\beta\gamma(1-\lambda)x^{\alpha-1}e^{-\frac{1}{2}\beta^2 x^{2\alpha}}[\{2\Phi(\beta x^{\alpha})-1\}\{2\Phi(-\beta x^{\alpha})\}]^{\gamma-1}}{[\{2\Phi(-\beta x^{\alpha})\}^{\gamma}+(1-\lambda)\{2\Phi(\beta x^{\alpha})-1\}^{\gamma}]^2}. \qquad \text{(B.4)}$$

2. The pdf of generalized half-normalPoisson(OGHNP) distribution is given by

$$\begin{aligned}
f(x;\boldsymbol{\theta}) &= \frac{\sqrt{\frac{2}{\pi}}\alpha\beta\gamma\lambda x^{\alpha-1}e^{-\frac{1}{2}\beta^2 x^{2\alpha}}[\{2\Phi(\beta x^{\alpha})-1\}\{2\Phi(-\beta x^{\alpha})\}]^{\gamma-1}}{(e^{\lambda}-1)[\{2\Phi(\beta x^{\alpha})-1\}^{\gamma}+\{2\Phi(-\beta x^{\alpha})\}^{\gamma}]^2} \\
&\quad \exp\left\{\lambda\left[1+\left\{\frac{2\Phi(-\beta x^{\alpha})}{2\Phi(\beta x^{\alpha})-1}\right\}^{\gamma}\right]^{-1}\right\}. \qquad \text{(B.5)}
\end{aligned}$$

We shall see the pdf can be decreasing and unimodal shaped for different values of parameters. Figure 1 displays the pdf of the OGHNG distribution for some selected parameter values.

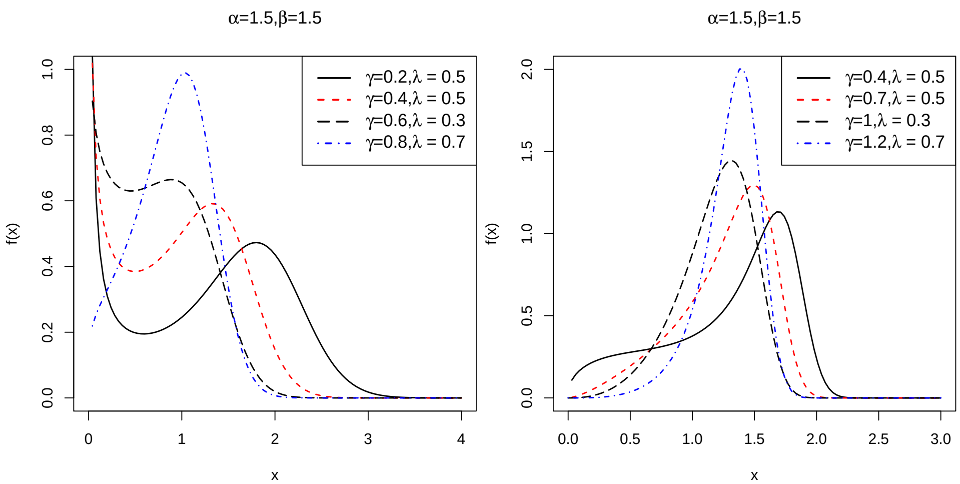

Figure 1: Graphs of the OGHNG pdf for some parameter values.

C Mathematical properties

Some basic statistical and mathematical properties of the OGHNPS distribution are provided in this section. Let X be an OGHNPS random variable with parameter vector $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)$. The survival and hazard rate functions of the BEW family are given by (for $x > 0$)

$$S(x; \boldsymbol{\theta}) = 1 - \{A(\lambda)\}^{-1} A\left(\frac{\lambda \Pi(x)^{\gamma}}{\Pi(x)^{\gamma} + \bar{\Pi}(x)^{\gamma}}\right)$$

and

$$h(x; \boldsymbol{\theta}) = \frac{\lambda \gamma \pi(x) \Pi(x)^{\gamma} \bar{\Pi}(x)^{\gamma}}{\left\{\Pi(x)^{\gamma} + \bar{\Pi}(x)^{\gamma}\right\}^{2}} \left\{ \frac{A'\left(\frac{\lambda \Pi(x)^{\gamma}}{\Pi(x)^{\gamma} + \bar{\Pi}(x)^{\gamma}}\right)}{\{A(\lambda)\} - A\left(\frac{\lambda \Pi(x)^{\gamma}}{\Pi(x)^{\gamma} + \bar{\Pi}(x)^{\gamma}}\right)} \right\}.$$

For arbitrary cdf $\Pi(x)$, Cordeiro et al. (2015) derived the following expansion

$$\left(\frac{\Pi(x)^{\gamma}}{\Pi(x)^{\gamma} + \bar{\Pi}(x)^{\gamma}}\right)^{n} = \sum_{r=0}^{\infty} c_r \Pi(x)^{r},$$

where

$$c_r = c_r(\gamma, n) = \frac{1}{\rho_0}\left(\rho_r - \frac{1}{\rho_0} \sum_{i=1}^{r} \rho_i c_{r-i}\right)$$

and ρ_r is defined by Cordeiro et al. (2015).

Using the Lemma C and the concept of power series, we derived two linear representations for the cdf and pdf of OGHNPS distribution.

$$F(x;\boldsymbol{\theta}) = \{A(\lambda)\}^{-1} \sum_{n=1}^{\infty}\sum_{r=0}^{\infty} c_{r,n}\lambda^n [2\Phi(\beta x^{\alpha}) - 1]^r$$

and

$$f(x;\boldsymbol{\theta}) = \{A(\lambda)\}^{-1}\sqrt{\frac{2}{\pi}}\alpha\beta x^{\alpha-1}e^{-\frac{1}{2}\beta^2 x^{2\alpha}} \sum_{n,r=1}^{\infty} d_{r,n}\lambda^n [2\Phi(\beta x^{\alpha}) - 1]^{r-1}, \tag{C.1}$$

where $c_{r,n} = a_n c_r$ and $d_{r,n} = r a_n c_r$.

Some mathematical properties of the proposed family, such as moments and moment generating function can be obtained by using this expansion.

The formula for the kth moment of X is obtained from (C.1) as

$$\begin{aligned} E\left[X^k;\boldsymbol{\theta}\right] &= \int_0^{\infty} x^k f(x,\theta)dx \\ &= \{A(\lambda)\}^{-1}\sqrt{\frac{2}{\pi}}\beta^{-\frac{k}{\alpha}} \sum_{n,r=1}^{\infty} d_{r,n}\lambda^n I\left(\frac{k}{\alpha}, r-1\right), \end{aligned}$$

where

$$I\left(\frac{k}{m}, r\right) = \int_0^{\infty} u^{\frac{k}{m}} \exp\left(-\frac{u^2}{2}\right)\left[\text{erf(x)}\left(\frac{u}{\sqrt{2}}\right)\right]^r du$$

and

$$\text{erf(x)} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

The moment generating function of OGHNPS distribution can be expressed as

$$\begin{aligned} M(t) &= E\left[e^{tX};\boldsymbol{\theta}\right] = \sum_{k=0}^{\infty}\frac{t^k}{k!}E\left[X^k;\theta\right] \\ &= \{A(\lambda)\}^{-1}\sum_{k=0}^{\infty}\sum_{n,r=1}^{\infty} d_{r,n}\lambda^n \frac{t^k}{\beta^{\frac{k}{\alpha}}k!} I\left(\frac{k}{\alpha}, r-1\right). \end{aligned}$$

Let $x = Q(u)$ be the OGH-NPS quantile function (qf), derived by inverting (B.2). and $Q_N(u) = \Phi^{-1}(u)$ denotes the standard normal qf. Consider $X \sim OGHNPS(\alpha, \beta, \gamma, \lambda)$. Then the qf function of OGHNPS distribution can be obtained as

$$Q(u;\boldsymbol{\theta}) = \left\{\frac{1}{\beta}Q_N\left(\left[1 + \left\{\frac{\lambda - A^{-1}(uA(\lambda))}{A^{-1}(uA(\lambda))}\right\}^{\frac{1}{\gamma}}\right]^{-1}\right)\right\}^{\frac{1}{\alpha}},$$

where $A^{-1}(.)$ is the inverse function of $A(.)$. Clearly, the OGH-NPS distribution is easily simulated by $X = Q(U)$, where U be a uniform random variable in the unit interval $(0, 1)$.

D Estimation and simulation study

Suppose $X_1, X_2, \ldots, X_n$ is a random sample with observed values $x_1, x_2, \ldots, x_n$ from the OGH-NPS family of distributions with unknown parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)$. The log-likelihood function of $\boldsymbol{\theta}$ is

$$\begin{aligned}
\ell(\boldsymbol{\theta}|\mathbf{x}) \quad = \quad & \frac{n}{2}\log\left[\frac{2}{\pi}\right] + n\log[\alpha] + n\log[\beta] + n\log[\gamma] + n\log[\lambda] + (\alpha-1)\sum_{i=1}^{n}\log[x_i] \\
& - n\log[A(\lambda)] - 2\sum_{i=1}^{n}\log\left[\{2\Phi(\beta x_i^{\alpha}) - 1\}^{\gamma} + \{2\Phi(-\beta x_i^{\alpha})\}^{\gamma}\right] \\
& - \frac{1}{2}\beta^2\sum_{i=1}^{n}x_i^{2\alpha} + (\gamma-1)\sum_{i=1}^{n}\log[2\Phi(\beta x_i^{\alpha}) - 1] + (\gamma-1)\sum_{i=1}^{n}\log[2\Phi(-\beta x_i^{\alpha})] \\
& + \sum_{i=1}^{n}\log\left[A'\left(\lambda\left[1 + \left\{\frac{2\Phi(-\beta x_i^{\alpha})}{2\Phi(\beta x_i^{\alpha}) - 1}\right\}^{\gamma}\right]^{-1}\right)\right]. \qquad \text{(D.1)}
\end{aligned}$$

The maximum likelihood estimate (MLE) of $\boldsymbol{\theta}$ called $\widehat{\boldsymbol{\theta}}$ should satisfy the following equation $U_n(\boldsymbol{\theta}) = (\partial\ell/\partial\alpha, \partial\ell/\partial\beta, \partial\ell/\partial\gamma, \partial\ell/\partial\lambda) = \mathbf{0}$. The solution of this nonlinear system of equations has no closed form. To solve this equation, it is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the log-likelihood function. In the application section, the MLEs were obtained by directly maximizing (D.1) with respect to the parameters. The `optim` routine in R was used for maximization.

We evaluate the performance of the maximum likelihood estimates of the OGHNG distribution as the special case of OGHNPS distribution with respect to sample size n. We repeated simulation study $k = 5000$ times with sample size $n = 20, 50, 100, 200$ and parameter values $I : \alpha = 1.5,\ \beta = 2,\ \gamma = 0.6,\ \theta = 0.5$ and $II : \alpha = 0.8,\ \beta = 0.7,\ \gamma = 1.5, \theta = 0.7$ then the parameters are estimated by ML method. The bias and mean squared error (MSE) of the MLE estimators are presented in Table 1.

The results indicate that the maximum likelihood estimators carry out well for estimating the parameters of the OGHNG model. According to Table 1, it can be concluded that as the sample size n increases, the MSEs decay toward zero. Furthermore, the bias of the estimated values of the parameters is greatly reduced as the sample size n is increased.

E Application

In this section, we provide illustrations to a real data set to show the importance of the OGHNPS distribution. We consider the two particular cases OGHNG and OGHNP. The MLEs of the

Table 1: The mean, bias, and MSE of the MLE estimators from 5000 samples.

	I				II			
n	$\boldsymbol{\xi}$	Average	Bias	MSE	$\boldsymbol{\xi}$	Average	Bias	MSE
20	α	2.4836	0.9836	2.2811	α	1.7111	0.9111	2.5221
	β	1.7537	-0.2463	1.7865	β	0.3497	-0.3503	0.2923
	γ	0.4797	-0.1203	0.1534	γ	0.9861	-0.50139	0.4565
	λ	0.5956	0.0956	0.1214	λ	0.6225	-0.0775	0.1272
50	α	1.8469	0.3469	0.5326	α	1.0341	0.20341	0.7534
	β	1.8564	-0.1436	0.6804	β	0.5503	-0.0.1497	0.0890
	γ	0.5235	-0.0765	0.0299	γ	1.3456	-0.1544	0.2183
	λ	0.5256	0.256	0.0486	λ	0.6709	0.0291	0.0807
100	α	1.6325	0.1325	0.1578	α	0.8974	0.0974	0.1208
	β	1.9235	0.0765	0.2035	β	0.6839	-0.0161	0.0689
	γ	0.5736	-0.0264	0.0103	γ	1.4569	-0.0431	0.1030
	λ	0.4823	-0.0177	0.0204	λ	0.7023	0.0023	0.0088
200	α	1.5356	0.0356	0.0401	α	0.8018	0.0018	0.0109
	β	2.0042	0.0042	0.0597	β	0.6981	-0.0019	0.0058
	γ	0.5950	-0.0050	0.0042	γ	1.4938	-0.0062	0.0659
	λ	0.4907	-0.0093	0.0114	λ	0.7007	0.0007	0.0034

parameters and the goodness-of-fit statistics were computed and compared with those of the popular odd Weibull (OW) (Cooray, 2006), odd generalized half-normal (OGHN) (Cordeiro et al., 2016), beta generalized exponential (BGE) (Barreto-Souza et al., 2010), beta Weibull (BW) (Famoye et al., 2005), and destructive Poisson odd generalized half-normal (DPOGHN) (Pescim et al., 2018) distributions specified by the pdfs

$$f_{OW}\left(x;\boldsymbol{\xi}_1\right)=\frac{\alpha\beta\gamma x^{\alpha}e^{\beta x^{\alpha}}\left(e^{\beta x^{\alpha}}-1\right)^{\gamma-1}}{\left\{1+\left(e^{\beta x^{\alpha}}-1\right)^{\gamma}\right\}^{-2}},$$

$$f_{OGHN}\left(x;\boldsymbol{\xi}_2\right)=\frac{\sqrt{\frac{2}{\pi}}\alpha\beta\gamma x^{\alpha-1}e^{-\frac{1}{2}\beta^2x^{2\alpha}}\left[\left\{2\Phi\left(\beta x^{\alpha}\right)-1\right\}\left\{2\Phi\left(-\beta x^{\alpha}\right)\right\}\right]^{\gamma-1}}{\left[\left\{2\Phi\left(-\beta x^{\alpha}\right)\right\}^{\gamma}+\left\{2\Phi\left(\beta x^{\alpha}\right)-1\right\}^{\gamma}\right]^2}.$$

$$f_{BW}\left(x;\boldsymbol{\xi}_3\right)=\frac{\alpha\beta x^{\alpha-1}}{B\left(a,b\right)}e^{-b\beta x^{\alpha}}\left[1-e^{-\beta x^{\alpha}}\right]^{a-1},$$

$$f_{BGE}\left(x;\boldsymbol{\xi}_4\right)=\frac{\alpha\beta e^{-\beta x}}{B(a,b)}\left(1-e^{-\beta x}\right)^{a\alpha-1}\left[1-\left(1-e^{-\beta x}\right)^{\alpha}\right]^{b-1},$$

and

$$\begin{aligned} f_{DPOGHN}\left(x;\boldsymbol{\xi}_5\right) &= \frac{\sqrt{\frac{2}{\pi}}\alpha\beta\gamma\lambda p x^{\alpha-1}e^{-\frac{1}{2}\beta^2x^{2\alpha}}\left[\left\{2\Phi\left(\beta x^{\alpha}\right)-1\right\}\left\{2\Phi\left(-\beta x^{\alpha}\right)\right\}\right]^{\gamma-1}}{\left(1-e^{-\lambda p}\right)\left[\left\{2\Phi\left(\beta x^{\alpha}\right)-1\right\}^{\gamma}+\left\{2\Phi\left(-\beta x^{\alpha}\right)\right\}^{\gamma}\right]^2} \\ &\quad \exp\left\{-\frac{\lambda p\left\{2\Phi\left(\beta x^{\alpha}\right)-1\right\}^{\gamma}}{\left\{2\Phi\left(\beta x^{\alpha}\right)-1\right\}^{\gamma}+\left\{2\Phi\left(-\beta x^{\alpha}\right)\right\}^{\gamma}}\right\}, \end{aligned}$$

for $x>0$, where $\alpha,\beta,\gamma,\lambda,a,b>0$ and $p\in(0,1)$.

The data set consists of the strength of 1.5 cm glass fibers, measured at the National physical laboratory, England (see Smith and Naylor (1987)). The MLEs, log-likelihood value, the corresponding standard errors, the Kolmogorov-Smirnov statistic, its p-value, Akaike information criterion (AIC), the corrected Akaike information criterion (AICc) and the Bayesian information criterion (BIC) are shown in Table 2, where

$$\text{AICc}=-2\log\left[\ell(\widehat{\boldsymbol{\theta}})\right]+\frac{2nk}{n-k-1}$$

and k is the number of the estimated parameters.

Table 2: Estimates and goodness-of-fit measures for the strength of 1.5 cm glass fibers dataset.

Model	$\widehat{\boldsymbol{\theta}}$	$-\ell(\widehat{\boldsymbol{\theta}})$	K-S	p-value	AIC	AICc	BIC
OW	$\alpha=6.0258,\beta=0.0539,\gamma=0.9438$	15.187	0.096	0.642	36.374	36.781	42.803
$SE\left(\widehat{\boldsymbol{\xi}}_1\right)$	(1.3333, 0.0331, 0.2667)						
OGHN	$\alpha=3.7606,\beta=0.1334,\gamma=1.2906$	14.164	0.089	0.740	34.328	34.735	40.757
$SE\left(\widehat{\boldsymbol{\xi}}_2\right)$	(0.7744, 0.0492, 0.3283)						
BW	$\alpha=7.0138,\beta=0.5533,a=0.4498,b=0.0499$	13.044	0.088	0.752	34.088	34.758	42.661
$SE\left(\widehat{\boldsymbol{\xi}}_3\right)$	(0.8896, 0.6459, 0.1810, 0.0464)						
BGE	$\alpha=22.6124,\beta=0.9227,a=0.4125,b=93.4655$	15.599	0.103	0.552	39.198	39.868	47.771
$SE\left(\widehat{\boldsymbol{\xi}}_4\right)$	(22.8153, 0.5135, 0.3152, 16.6665)						
DPOGHN	$\alpha=3.762,\beta=0.133,\gamma=1.291,\lambda=3.334,p=0.002$	13.161	0.088	0.752	36.322	37.375	47.037
$SE\left(\widehat{\boldsymbol{\xi}}_5\right)$	(0.5074, 0.5146, 0.2330, 1.9136)						
OGHNG	$\alpha=1.9156,\beta=0.9211,\gamma=0.9116,\lambda=0.9573$	11.951	0.076	0.881	31.902	32.572	40.474
$SE\left(\widehat{\boldsymbol{\theta}}_1\right)$	(0.6876, 0.6474, 0.4402, 0.0571)						
OGHNP	$\alpha=2.3641,\beta=0.8976,\gamma=0.4413,\lambda=5.9101$	12.425	0.081	0.828	32.850	33.520	41.422
$SE\left(\widehat{\boldsymbol{\theta}}_2\right)$	(0.5074, 0.5146, 0.2330, 1.9136)						

We can see that the largest log-likelihood value, the largest p-value, the smallest AIC value, the smallest AICc value, and the smallest BIC value are obtained for the OGHNG distribution.

F Conclusion

We have proposed a four-parameter of distribution referred to as the OGHNPS distribution by compounding odd generalized half-normal and power series distributions. The OGHNPS distribution contains the generalized half-normal, odd generalized half-normal and generalized half-normal power series as special cases. The mathematical properties of the OGHNPS distribution derived include quantiles, moments and moment generating function. Applications to the real data set show that the proposed distribution provides better fits than popular lifetime distributions.

References

[1] Barreto-Souza, W. and Santos, A. and Cordeiro, G.M. (2010), The beta generalized exponential distribution, *Journal of Statistical Computation and Simulation*, **80**, 159-172.

[2] Cooray, K. (2006), Generalization of the Weibull distribution: the odd Weibull family, *Statistical Modelling*, **6**, 265-277.

[3] Cooray, K. and Ananda, M.A. (2008), A generalization of the half-normal distribution with applications to lifetime data, *Communications in Statistics - Theory and Methods*, **37**, 1323 - 1337.

[4] Cordeiro, G.M. and Alizadeh, M. and Ortega, E.M. and Valdivieso Serrano, L.H. (2015), The Zografos-Balakrishnan odd log-logistic family of distributions : Properties and applications, *Hacettepe Journal of Mathematics and Statistics*, **45**.

[5] Cordeiro, G.M. and Alizadeh, M. and Pescim, R.R. and Ortega, E.M. (2017), The odd log-logistic generalized half-normal lifetime distribution: properties and applications, *Communications in Statistics - Theory and Methods*, **46**, 4195-4214.

[6] Famoye, F. and Lee, C. and Olumolade, O. (2006), The beta-Weibull distribution, *Journal of Statistical Theory and Application*, **4**, 122-136.

[7] Gleaton, J.U. and Lynch, J.D. (2004), On the distribution of the breaking strain of a bundle of brittle elastic fibers, *Advances in Applied Probability*, **36**, 98-115.

[8] Marshall, A.W. and Olkin, I. (1997), A new method for adding a parameter to a family

of distributions with application to the exponential and Weibull families, *Biometrika*, **84**, 641-652.

[9] Pescim, R. and Ortega, E. and Suzuki, A. and Cancho, V. and Cordeiro, G. (2018), A new destructive Poisson odd log-logistic generalized half-normal cure rate model, *Communication in Statistics-Theory and Methods*, 1-16.

[10] Tahmasebi, S. (2016), Notes on Generalized Half-Normal Power Series Distributions, In: Proceedings of The 13th Iranian Statistics Conference, 2016 Agust 23-25, Shahid Bahonar University of Kerman, Kerman, Iran, 662-667.

Goodness of Fit Tests for Rayleigh Distribution Based on Progressively Type II Right Censored via a New Divergence

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Abstract: In this paper, we define first a new measure of distance between two distributions based on their cumulative distribution function that is similar to Tsallis divergence. Then based on a progressively type II right censored sample, we construct goodness-of-fit tests for testing Rayleighity. Monte Carlo simulations for the power of the proposed tests are carried out under different alternatives. Finally, an illustrative example for use of the proposed tests is presented.
Keywords Goodness of fit test, Rayleigh distribution, Progressively type II right censored sample.

A Introduction

The Rayleigh distribution is a special case of the Weibull distribution with a scale parameter of 2 and a suitable model in various areas including reliability, life testing, and survival analysis. The square of a Rayleigh random variable with a shape parameter 1 is equal to a chi square random variable with 2 degrees of freedom. Also, the square root of an exponential random variable has the Rayleigh distribution. Also, the Rayleigh distribution is widely used in the physical sciences to model wind speed, wave heights and sound/light radiation and has been used in medical imaging science, to model noise variance in magnetic resonance imaging. For more information about the applications and properties of the Rayleigh distribution, we refer the interested readers to Siddiqui [13] and Johnson et al. [10]. A random variable X follows the Rayleigh distribution if and only if it has probability density function

$$f_0(x;\theta) = \frac{x}{\theta^2}\exp\left\{-\frac{x^2}{2\theta^2}\right\}, \qquad x \geq 0, \theta > 0. \tag{1.1}$$

There are several goodness of fit tests in the literature based on a complete sample for the Rayleigh distribution. Meintanis and Iliopoulos [11] proposed a class of goodness of fit tests for the Rayleigh distribution. Recently, Zamanzade and Mahdizadeh [15] based on Phi-divergence and Jahanshahi et al. [9] based on Hellinger distances suggested tests for Rayleigh distribution. Safavinejad et al. [12] by the empirical likelihood ratio method and Alizadeh et al. [1] by using the KL divergence,

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proposed goodness of fit tests for checking Rayleighity. Also, Baratpour and Khodadadi [8], based on cumulative Kullback–Leibler (KL), defined a test for Rayleigh distribution.
Type-I and Type-II censoring schemes are the most popular ones among the different censoring schemes. One of the disadvantages of these censoring schemes is the imposibility to withdraw units during the experiment. So a generalization of the classical Type-II censoring scheme, known as the progressive Type-II censoring scheme, was proposed by researchers to withdraw units during the experiment.
Progressive censoring scheme has recently received considerable attention in the statistical literature. (see Balakrishnan and Aggrawalla [3]; Balakrishnan [2]).
The progressive censoring scheme can be described as follows. Under this general censoring scheme, n units are placed on a life testing experiment and only $m(< n)$ are completely observed until failure. The censoring occurs progressively in m stages. At the time of the first failure (the first stage)$X_{1:m:n}$, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the second failure (the second stage) $X_{2:m:n}$, R_2 units are randomly removed from the remaining $n-2-R_1$ units, and so on. The procedure is continued until all the remaining surviving $R_m = n-m-R_1-\ldots-R_{m-1}$ units are removed from the experiment at the time of the $m-th$ failure (the $m-th$ stage) $X_{m:m:n}$. We will denote the m order observed failure times by $X_{1:m:n} < X_{2:m:n} < \ldots < X_{m:m:n}$ and the progressive censoring schemes with the vector $R = (R_1, \ldots, R_m)$, which is fixed previously. If $R = (0, \ldots, 0)$, then no censoring is performed at any of the m stages and corresponds to the complete sample. If $R = (0, \ldots, 0, n-m)$, we obtain the Type-II right censoring.

The present paper aims to construct a test using proposed divergence based on the progressively type II censored sample for exponentiality test. The organization of the paper is as follows. In Section 2, we construct the statistics based on extension of Tsallis divergences for testing Rayleighity. Competitor tests are expressed in Section 3. Finally, in Section 4, we use Monte Carlo simulations to evaluate the power of proposed test and competing tests for several alternatives under different sample sizes and progressive Type-II censoring schemes.

2. Test Statistic

Consider two nonnegative and absolutely continuous random variables X and Y with probability density functions (pdf) f and g, cumulative distribution functions (cdf) F and G, respectively. Then, the Tsallis divergence between f and g is defined as (see Tsallis [14])

$$D_T(f,g) = \frac{1}{\alpha-1}\left[\int_0^\infty f^\alpha(x) g^{1-\alpha}(x)dx - 1\right], \alpha(\neq 1) > 0. \tag{2.1}$$

We define new measures of distance between two distributions that are similar Tsallis divergences in the following.

Definition 2.1. Let X and Y be two non negative and absolutely continuous random variables with cdfs F and G and pdfs f and g, respectively. Then cumulative residual Tsallis (CRT) between these distributions is as follows

$$CRT(F:G) = \frac{1}{\alpha-1}\left[\int_0^\infty \bar{F}^{\alpha}(x)\bar{G}^{1-\alpha}(x)dx - \alpha E(X) - (1-\alpha)E(Y)\right], \quad 0<\alpha<1. \tag{2.2}$$

Lemma 2.2. *CRT(F:G)$\geq$ 0 and equality holds if and only if $F=G$.*

Proof. By applying the Hölder inequality, we obtain

$$\int_0^\infty \bar{F}^{\alpha}(x)\bar{G}^{1-\alpha}(x)dx \leq \left(\int_0^\infty \bar{F}(x)dx\right)^{\alpha}\left(\int_0^\infty \bar{G}(x)dx\right)^{1-\alpha}, \; 0<\alpha<1, \tag{2.3}$$

and by using the Young inequality, we get

$$\left(\int_0^\infty \bar{F}(x)dx\right)^{\alpha}\left(\int_0^\infty \bar{G}(x)dx\right)^{1-\alpha} \leq \alpha\int_0^\infty \bar{F}(x)dx + (1-\alpha)\int_0^\infty \bar{G}(x)dx. \tag{2.4}$$

Therefore, by (2.3) and (2.4) and dividing by $\alpha-1$, the desired inequality follows. In the Hölder inequality, equality holds if and only if $\bar{F}=c\bar{G}$ (c is a positive constant) and in the Young inequality, equality holds if and only if $\int_0^\infty \bar{F}(x)dx = \int_0^\infty \bar{G}(x)dx$. Thus, $c=1$ and $CRT(F:G)=0$ if and only if $F=G$. □

Mentioned properties in lemma 2.2 motivates us to use new divergence for constructing test statistic.
Suppose that $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$ is progressively Type-II right censored data with the progressive censoring scheme $R=(R_1,R_2,\ldots,R_m)$ from a continuous distribution function $F(x)$. The testing of interest is

$$H_0: F(x)=F_0(x) \quad vs \quad H_1: F(x)\neq F_0(x),$$

where $F_0(x)=1-\exp\left(-\frac{x^2}{2\theta^2}\right), \quad x\geq 0,\; \theta>0$, and θ is the unknown parameter. Based on progressively Type-II right censored data, the cumulative distribution function estimator can be written as

$$F_{m,n}(x) = \begin{cases} 0 & if \quad x < x_{1:m:n} \\ \alpha_{i:m:n} & if \quad x_{i:m:n} \leq x < x_{i+1:m:n}, \quad i=1,2,\ldots,m-1 \\ \alpha_{m:m:n} & if \quad x \geq x_{m:m:n} \end{cases} \tag{2.5}$$

where $\alpha_{i:m:n} = E(U_{i:m:n})$ is the expected value of the th Type-II progressively censored order statistic from the Uniform(0,1) distribution, given by Balakrishnan and Sandhu [?] is as

$$\alpha_{i:m:n} = 1-\prod_{j=m-i+1}^{m}\left\{\frac{j-1+R_{m-j+1}+\ldots+R_m}{j+R_{m-j+1}+\ldots+R_m}\right\}.$$

2.1. **Testing procedures based on the new divergences.** In this section, by utilizing (2.5) and estimating new divergence, we construct test statistic for testing testing Rayleighity with the progressively Type-II censored data and then consider some competing tests to compare with the proposed tests. Accordingly, by letting

$F(x)=F_{mn}(x)$ and $G(x)=F_0(x)$ in (2.2), we have

$$\begin{aligned} CRT_{mn} &= \frac{1}{\alpha-1}\Big[\int_0^{x_{m:m:n}}(1-F_{mn}(x))^{\alpha}e^{-\frac{x^2}{2\theta^2}(1-\alpha)}dx \\ &- \alpha\int_0^{x_{m:m:n}}(1-F_{mn}(x))dx-(1-\alpha)\int_0^{x_{m:m:n}}e^{-\frac{x^2}{2\theta^2}}dx\Big] \\ &= \frac{1}{(\alpha-1)}\left[\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})^{\alpha}\int_{x_{i:m:n}}^{x_{i+1:m:n}}e^{-\frac{x^2}{2\theta^2}(1-\alpha)}dx\right] \\ &- \frac{\alpha}{\alpha-1}\left[\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})\right] \\ &+ \int_0^{x_{m:m:n}}e^{-\frac{x^2}{2\theta^2}}dx, \end{aligned} \tag{2.6}$$

where $\alpha_{0:m:n}=x_{0:m:n}=0$. Dividing (2.6) by $\int_0^{x_{m:m:n}}(1-F_{mn}(x))dx$, the proposed test (that is scale-invariant) is as follows

$$\begin{aligned} CRT_{mn} &= \frac{1}{\alpha-1}\left[\frac{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})^{\alpha}\int_{x_{i:m:n}}^{x_{i+1:m:n}}e^{-\frac{x^2}{2\hat{\theta}^2}(1-\alpha)}dx}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})}\right] \\ &+ \frac{\int_0^{x_{m:m:n}}e^{-\frac{x^2}{2\hat{\theta}^2}}dx}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})}-\frac{\alpha}{\alpha-1}, \end{aligned} \tag{2.7}$$

where $\hat{\theta}^2=\frac{1}{2m}\sum_{i=1}^{m}(R_i+1)x_{i:m:n}^2$ is the maximum likelihood estimate (MLE) of based on the progressively Type-II censored sample.

3. Competitor tests

We compare the performance of the proposed test with some tests for progressively Type II censored data in the literature. These tests are provided in the following.

- The test statistic proposed by Balakrishnan et al. [5] is as follows

$$T(w,n,m) = -H(w,n,m)-\frac{1}{n}\left[\sum_{i=1}^{m}\log f_0(x_i;\hat{\theta})+\sum_{i=1}^{m}R_i\log(1-F_0(x_i;\hat{\theta}))\right],$$

where $H(w,n,m)=\frac{1}{n}\sum_{i=1}^{m}\log\left(\frac{x_{i+w:m:n}-x_{i-w:m:n}}{E(U_{i+w:m:n})-E(U_{i-w:m:n})}\right)-(1-\frac{m}{n})\log(1-\frac{m}{n})$ and $\hat{\theta}$ is an estimator of θ. If we estimate the unknown parameter by the MLE, then the test statistic for Rayleigh distribution is

$$\begin{aligned} T(w,n,m) &= -H(w,n,m) \\ &+ \frac{m}{n}\left[\log\left(\frac{1}{2m}\sum_{i=1}^{m}(1+R_i)x_i^2\right)-\frac{1}{m}\sum_{i=1}^{m}\log x_i+1\right]. \end{aligned} \tag{3.1}$$

- The test statistics proposed by Baratpour and Habibirad [7] are given by

$$T_1 = \frac{\int_0^{x_{m:m:n}}(1-F_{mn}(x))\log\frac{1-F_{mn}(x)}{1-F_0(x)}dx}{\int_0^{x_{m:m:n}}(1-F_{mn}(x))dx}+\frac{\int_0^{x_{m:m:n}}(1-F_0(x))dx}{\int_0^{x_{m:m:n}}(1-F_{mn}(x))dx}-1,$$

and

$$T_2 = \frac{\int_0^{x_{m:m:n}} F_{mn}(x) \log \frac{F_{mn}(x)}{F_0(x)} dx}{\int_0^{x_{m:m:n}} (1 - F_{mn}(x)) dx} - \frac{\int_0^{x_{m:m:n}} F_0(x) dx}{\int_0^{x_{m:m:n}} (1 - F_{mn}(x)) dx} + 1,$$

that for Rayleigh distribution are as follows

$$\begin{aligned} T_1 &= \frac{\sum_{i=1}^{m-1} (1 - \alpha_{i:m:n}) \log(1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})}{\sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})} \\ &+ \frac{\sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n}^3 - x_{i:m:n}^3)}{6\hat{\theta}^2 \sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})} \\ &+ \frac{\int_0^{x_{m:m:n}} e^{-\frac{x^2}{2\hat{\theta}^2}} dx}{\sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})} - 1, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} T_2 &= \frac{\sum_{i=1}^{m-1} \alpha_{i:m:n} \log(\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})}{\sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})} \\ &- \frac{\sum_{i=1}^{m-1} \alpha_{i:m:n} \int_{x_{i:m:n}}^{x_{i+1:m:n}} \log\left(1 - e^{-\frac{x^2}{2\hat{\theta}^2}}\right) dx}{\sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})} \\ &- \frac{x_{m:m:n} - \int_0^{x_{m:m:n}} e^{-\frac{x^2}{2\hat{\theta}^2}} dx}{\sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})} + 1. \end{aligned} \tag{3.3}$$

4. Simulation Study

In order to evaluate the performance of the proposed tests and then the comparison with the competing tests, we compare the power values of the proposed tests with the corresponding values of competing tests. We generated 50,000 random samples for different censoring schemes for the determination of the power. For this purposes, we used the 27 censoring schemes of Balakrishnan et al. [6] that are listed in Table 1.

For CRT_{mn} test, the null hypothesis will be rejected, when the test statistic is more than the corresponding critical value at a designed significance level.

The power values of the proposed tests depend on α values and type of failure rate function of alternatives. Thus, the alternatives are selected according to three types of failure rate function, increasing failure rate (IFR), decreasing failure rate (DFR) and non-monotone failure rate (NFR).

We considered the α value that maximizes power, this value is suggested to be 0.01 for all three types of failure rate function.

Tables 2 and 3 present power values of the proposed test and the competing tests tests at a %10 significance level based on the type of failure rate function. Results indicate that, almost in most cases, the CRT_{mn} statistic has higher power compared to other tests (for alternatives with IFR and DFR functions). Also, for alternatives with IFR and DFR functions, the scheme $R = (0, 0, \ldots, n - m)$ generally indicates better power than the other schemes.

Table 3 shows that the powers depend on the kind of alternatives distribution. So, a general conclusion can not be suggested.

Table 1. progressive censoring schemes used in the Monte Carlo simulations

Scheme No.	n	m	$(R_1, \ldots, R_m)$
[1]			$R_1 = 12, R_i = 0$ for $i \neq 1$
[2]		8	$R_8 = 12, R_i = 0$ for $i \neq 8$
[3]			$R_1 = R_8 = 6, R_i = 0$ for $i \neq 1, 8$
[4]			$R_1 = 8, R_i = 0$ for $i \neq 1$
[5]	20	12	$R_{12} = 8, R_i = 0$ for $i \neq 12$
[6]			$R_3 = R_5 = R_7 = R_9 = 2, R_i = 0$ for $i \neq 3, 5, 7, 9$
[7]			$R_1 = 4, R_i = 0$ for $i \neq 1$
[8]		16	$R_{16} = 4, R_i = 0$ for $i \neq 16$
[9]			$R_5 = 4, R_i = 0$ for $i \neq 5$
[10]			$R_1 = 30, R_i = 0$ for $i \neq 1$
[11]		10	$R_{10} = 30, R_i = 0$ for $i \neq 10$
[12]			$R_1 = R_5 = R_{10} = 10, R_i = 0$ for $i \neq 1, 5, 10$
[13]			$R_1 = 20, R_i = 0$ for $i \neq 1$
[14]	40	20	$R_{20} = 20, R_i = 0$ for $i \neq 20$
[15]			$R_i = 1$ for $i = 1, 2, \ldots, 20$
[16]			$R_1 = 10, R_i = 0$ for $i \neq 1$
[17]		30	$R_{30} = 10, R_i = 0$ for $i \neq 30$
[18]			$R_1 = R_{30} = 5, R_i = 0$ for $i \neq 1, 30$
[19]			$R_1 = 40, R_i = 0$ for $i \neq 1$
[20]		20	$R_{20} = 40, R_i = 0$ for $i \neq 20$
[21]			$R_1 = R_{20} = 10, R_{10} = 20, R_i = 0$ for $i \neq 1, 10, 20$
[22]			$R_1 = 20, R_i = 0$ for $i \neq 1$
[23]	60	40	$R_{40} = 20, R_i = 0$ for $i \neq 40$
[24]			$R_{2i-1} = 1, R_{2i} = 0$ for $i = 1, 2, \ldots, 20$
[25]			$R_1 = 10, R_i = 0$ for $i \neq 1$
[26]		50	$R_{50} = 10, R_i = 0$ for $i \neq 50$
[27]			$R_1 = R_{50} = 5, R_i = 0$ for $i \neq 1, 50$

References

1. Alizadeh Noughabi, R. Alizadeh Noughabi, H. and Ebrahimi Moghaddam Behabadi, A. (2014). An entropy test for the Rayleigh distribution and power comparison.*Journal of Statistical Computation and Simulation*, 84, 151–158.
2. Balakrishnan, N. (2007). Progressive censoring methodology: An appraisal (with discusssions). *Test*, 16, 211–296.
3. Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications.* Boston: Birkhauser.
4. Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulated algorithm for generating progressive Type-II censored samples. *The American Statistication*, 49, pp. 229–230.
5. Balakrishnan, N., Habibirad, A. and Arghami, N.R. (2007). Testing exponentiality based on KullbackLeibler information with progressively type-II censored data. *IEEE Transactions on Reliability*, 56, 301–307.
6. N. Balakrishnan, N., Ng, H. K. T. and Kannan, N. (2004). Goodness-of-t tests based on spacings for progressively Type-II censored data from a general location-scale distribution. *IEEE Trans. Reliability*, 53, 349-356.
7. Baratpour, S. and Habibirad, A. (2012). Testing goodness-of-fit for exponential distribution based on cumulative residual entropy. *Communications in Statistics-Theory and Methods*, 41(8), 1387–1396.
8. Baratpour, S. and Khodadadi, F. (2012). A cumulative residual entropy characterization of the Rayleigh distribution and related goodness-of-fit test. *J Statist Res Iran*, 9, 115–131.
9. Jahanshahi, S. Habibi Rad, A. and Fakoor, V. (2016). A goodness-of-fit test for Rayleigh distribution based on Helinger distance. *Annals of Data Sciences*, 3(4), 401–411.

TABLE 2. Power of the proposed and competing tests for the alternatives with the IFR and DFR functions at 10% significance level for several schemes.

Scheme No.	IFR								DFR		
	W(1.4)				G(2)				W(0.5)		
	CRT_{mm}	T_1	T_2	T	CRT_{mm}	T_1	T_2	T	CRT_{mm}	T_1	T
[1]	0.429	0.436	0.013	0.173	0.385	0.390	0.024	0.115	0.979	0.976	0.994
[2]	0.363	0.162	0.407	0.118	0.220	0.098	0.223	0.083	0.962	0.891	0.909
[3]	0.390	0.222	0.178	0.135	0.267	0.143	0.104	0.090	0.973	0.934	0.959
[4]	0.514	0.487	0.013	0.246	0.484	0.462	0.017	0.174	0.997	0.995	0.999
[5]	0.457	0.276	0.388	0.188	0.323	0.165	0.201	0.111	0.995	0.987	0.988
[6]	0.487	0.479	0.019	0.230	0.436	0.439	0.025	0.153	0.991	0.986	0.998
[7]	0.602	0.538	0.014	0.305	0.559	0.522	0.015	0.221	1.000	0.999	1.000
[8]	0.543	0.466	0.037	0.281	0.437	0.348	0.024	0.161	0.999	0.999	0.999
[9]	0.585	0.551	0.010	0.306	0.551	0.526	0.012	0.218	0.999	0.999	1.000
[10]	0.470	0.479	0.011	0.257	0.424	0.435	0.017	0.155	0.992	0.990	1.000
[11]	0.397	0.181	0.435	0.140	0.210	0.095	0.210	0.091	0.984	0.943	0.956
[12]	0.448	0.352	0.036	0.178	0.270	0.199	0.044	0.089	0.989	0.981	0.988
[13]	0.661	0.605	0.008	0.400	0.616	0.575	0.009	0.285	1.000	1.000	1.000
[14]	0.574	0.370	0.652	0.300	0.361	0.177	0.359	0.136	1.000	1.000	1.000
[15]	0.516	0.530	0.012	0.359	0.436	0.451	0.019	0.187	0.998	0.999	1.000
[16]	0.785	0.698	0.005	0.529	0.751	0.688	0.006	0.395	1.000	1.000	1.000
[17]	0.723	0.616	0.071	0.484	0.561	0.421	0.024	0.263	1.000	1.000	1.000
[18]	0.744	0.695	0.007	0.508	0.642	0.569	0.006	0.327	1.000	1.000	1.000
[19]	0.653	0.606	0.007	0.441	0.614	0.589	0.009	0.296	1.000	1.000	1.000
[20]	0.574	0.346	0.633	0.278	0.303	0.140	0.306	0.117	1.000	0.999	1.000
[21]	0.636	0.561	0.011	0.364	0.431	0.362	0.020	0.167	1.000	1.000	1.000
[22]	0.866	0.779	0.004	0.656	0.828	0.759	0.003	0.492	1.000	1.000	1.000
[23]	0.802	0.683	0.470	0.584	0.621	0.442	0.164	0.305	1.000	1.000	1.000
[24]	0.702	0.617	0.004	0.600	0.643	0.585	0.006	0.366	1.000	1.000	1.000
[25]	0.920	0.832	0.002	0.741	0.896	0.828	0.001	0.590	1.000	1.000	1.000
[26]	0.877	0.832	0.005	0.713	0.768	0.685	0.003	0.462	1.000	1.000	1.000
[27]	0.887	0.876	0.001	0.746	0.812	0.784	0.002	0.529	1.000	1.000	1.000

10. Johnson, N. Kotz, S. and Balakrishnan, N. (1994). *Continuous univariate distributions*, Wiley.
11. Meintanis, S. and Iliopoulos, G. (2003). Tests of fit for the Rayleigh distribution based on the Empirical Laplace transform. *Annals of the Institute of Statistical Mathematics*, 55(1), 137–151.
12. Safavinejad, M. Jomhoori, S., and Alizadeh Noughabi, H. (2015). A density-based empirical likelihood ratio goodness-of-fit test for the Rayleigh distribution and power comparison. *Journal of Statistical Computation and Simulation*, 85(16), 3322–3334.
13. Siddiqui, M. M. (1962). Some problems connected with Rayleigh distributions. *Journal of Research of the National Bureau of Standards*, Sect. D, 66D, 167–174.
14. Tsallis, C. (1988). Possible genealizations of Boltzmann-Gibbs statistics. *Journal of Statistical Physics*, 52, 479–487.
15. Zamanzade, E. and Mahdizadeh, M. (2017). Goodness of fit test for Rayleigh distribution based on Phi-divergence. *Revista Colombiana de Estadstica*, 40(2), 279–290.

TABLE 3. Power of the proposed and competing tests for the alternatives with the DFR and NFR functions at 10% significance level for several schemes.

Scheme No.	IFR			NFR					
	G(0.5)			L(0,0.5)			L(0,1)		
	CRT_{mm}	T_1	T	CRT_{mm}	T_1	T	CRT_{mm}	T_1	T
[1]	0.940	0.930	0.971	0.142	0.116	0.322	0.736	0.732	0.439
[2]	0.937	0.842	0.862	0.004	0.318	0.404	0.316	0.115	0.096
[3]	0.942	0.877	0.916	0.009	0.102	0.385	0.431	0.238	0.140
[4]	0.984	0.972	0.992	0.187	0.184	0.284	0.870	0.851	0.650
[5]	0.984	0.963	0.969	0.013	0.236	0.362	0.583	0.364	0.242
[6]	0.971	0.960	0.988	0.146	0.149	0.276	0.789	0.790	0.556
[7]	0.996	0.990	0.997	0.209	0.234	0.270	0.938	0.918	0.777
[8]	0.995	0.994	0.995	0.048	0.121	0.346	0.818	0.748	0.530
[9]	0.997	0.994	0.999	0.202	0.222	0.279	0.929	0.918	0.776
[10]	0.965	0.959	0.996	0.165	0.110	0.454	0.805	0.800	0.564
[11]	0.974	0.920	0.935	0.000	0.508	0.562	0.216	0.087	0.096
[12]	0.978	0.965	0.977	0.003	0.043	0.578	0.396	0.283	0.141
[13]	0.999	0.997	1.000	0.261	0.239	0.457	0.966	0.954	0.871
[14]	0.999	0.998	0.999	0.003	0.465	0.602	0.619	0.393	0.306
[15]	0.992	0.995	1.000	0.092	0.092	0.486	0.781	0.801	0.591
[16]	1.000	1.000	1.000	0.303	0.337	0.444	0.995	0.989	0.968
[17]	1.000	1.000	1.000	0.022	0.244	0.545	0.940	0.885	0.765
[18]	1.000	1.000	1.000	0.067	0.135	0.520	0.975	0.964	0.879
[19]	0.999	0.998	1.000	0.293	0.222	0.552	0.964	0.953	0.877
[20]	0.999	0.998	0.999	0.000	0.681	0.731	0.401	0.197	0.177
[21]	0.999	0.999	1.000	0.004	0.046	0.691	0.724	0.630	0.410
[22]	1.000	1.000	1.000	0.356	0.389	0.564	0.999	0.998	0.991
[23]	1.000	1.000	1.000	0.007	0.457	0.728	0.950	0.889	0.793
[24]	1.000	1.000	1.000	0.218	0.248	0.505	0.976	0.960	0.945
[25]	1.000	1.000	1.000	0.379	0.467	0.594	1.000	1.000	0.998
[26]	1.000	1.000	1.000	0.051	0.263	0.689	0.996	0.994	0.974
[27]	1.000	1.000	1.000	0.121	0.216	0.673	0.998	0.998	0.990

Comparison between Constant-Stress and Step-Stress Tests under Time and Cost Constraint Data from Weibull Distribution

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Abstract: To collect the information about the lifetime distribution of a product, a standard life testing method at normal working conditions is impractical when the product has a long lifetime. Accelerated life testing quickly yields information on product life. Test units are run at high stress and fail sooner than at design stress. The lifetime at design stress is estimated by extrapolation using a regression model. Due to constrained resources in practice, test time must be determined carefully at the design stage in order to run an accelerated life test efficiently. Test time directly affect the experimental cost as well as the estimate precision of the parameters of interest. This article investigates optimal test time based on two optimality criteria under the constraint that the total experimental cost does not exceed a pre-specified budget. The purpose is to quantify the advantage of using step-stress testing in comparison to constant-stress testing.

Keywords Accelerated life testing, Constant-stress testing, Maximum likelihood estimation, Optimal allocation, Step-stress testing.

Mathematics Subject Classification (2010) : 62N05, 62N01, 90C31.

A Introduction

Technological advances in engineering has resulted in products having high mean time to failure (MTTF). However, prolonged time to failure makes the study of lifetime characteristics difficult. To overcome this, the technique of accelerated life test (ALT) is used rather than usual life test. ALT is a technique to fasten the failure of products in order to obtain quick information about life characteristics. In ALT, products are exposed to higher levels of stress factors like temperature, pressure, humidity, voltage etc. to get quick failures and the data thus obtained is properly analyzed to infer the life characteristics under normal use. Based on stress loading

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there are three types of ALTs namely, constant stress ALT (CSALT), step-stress ALT (SSALT) and progressive stress ALT(PSALT) (4).

In CSALT, only one level of higher stress is used. Sometimes it may be difficult to run at a higher stress for too long and CSALT may not produce enough quick failures. In PSALT, a test unit is subjected continuously to increasing stress. One major drawback of PSALT is that the progressive stress cannot be controlled accurately enough for long time in order to produce enough number of failures. In SSALT, a test unit is subjected to a specified level of stress for a prefixed period of time. If it does not fail during that period of time, then the stress level is increased for further prefixed period of time. This process is continued till all test units fail or some termination criteria is met. CSALT and SSALT with two levels of stress is known as simple CSALT and simple SSALT. SSALT yields quick failures when compared to CSALT and PSALT. Also it provides reliable estimates for life characteristics. Han and Ng have described the advantage of using SSALT over CSALT. For more details about ALT one may refer to (5).

In this paper, the optimal simple CSALT and simple SSALT are compared for the Weibull failure data under Type-I censoring.

The optimal ALT design has attracted great attention in the reliability literature. Under complete sampling, Hu et al. (2) studied the statistical equivalency of a simple step-stress ALT to other stress loading designs. The optimal CSALT and SSALT are compared for the exponential failure data under Type-I censoring by Han and Ng (1).

The main focus of this article is to investigate the advantage of using SSALT relative to CSALT. Assuming a log-linear relationship between the mode lifetime parameter and stress level, with the Khamis-Higgins (KH) model for the effect of changing stress in step-stress ALT. The KH model is based on a time transformation of the cumulative exposure model. Khamis and Higins (3) have proposed this model. The optimal design point is determined under two optimality criteria. In particular, the cases of Type-I censoring is considered under Weibull lifetime distribution for units subjected to stress.

In addition, the operation cost could substantially increase with the physical constraints and limitations of the testing instruments when the stress level increases. In this paper, under the practical constraint that the total experimental cost does not exceed a pre-specified budget, we investigate the optimal ALT plans.

The rest of the article is organized as follows. The Section B presents the model assumptions and formulation. Sections C and D derive the MLEs of the model parameters and the associated Fisher information for simple CSALT and SSALT. Section E then defines the two optimality criteria based on the Fisher information matrix. Section F discusses the total experimental cost of these tests. Section G provides the results of a numerical study. Finally, Section H is devoted to some conclusions.

B Model and Assumptions

Let s_1 and s_2 be two stress levels, with s_0 being the stress under normal use. The standardized stress loading is then defined as

$$x_i = \frac{s_i - s_0}{s_2 - s_0}, \quad i = 0, 1, 2, \tag{B.1}$$

so that the range of x is $[0, 1]$.

For any stress level x_i, $i = 0, 1, 2$, lifetime of a test unit follows Weibull distribution with cumulative distribution function (c.d.f)

$$F_i(t) = 1 - e^{-(\frac{t}{\lambda_i})^k}, \qquad t \geq 0, \tag{B.2}$$

and with corresponding probability density function (p.d.f)

$$f_i(t) = \frac{k}{\lambda_i^k} t^{k-1} e^{-(\frac{t}{\lambda_i})^k}, \qquad t \geq 0, \tag{B.3}$$

where k is a shape parameter and λ is a scale parameter.

Also, it is assumed that the shape parameter k is constant for all stress levels. The scale parameter λ_i at stress x_i is given by

$$\log \lambda_i = \beta_0 + \beta_1 x_i \tag{B.4}$$

for $i = 0, 1, 2$, where β_0 and β_1 are unknown parameters depending on the nature of the product and the method of test.

C Constant-Stress Test

In this section the simple constant-stress test is considered under Type-I censoring for the Weibull distribution. There are N units placed on test. There N_1 units assigned to stress x_1, and the remaining units $N_2 = N - N_1$ to stress x_2. Under stress x_1, units will be tested until either failure occurs or until the time of test reaches a specified time w_1. In this stress level n_1 failure times observed. Under stress x_2, units will be tested until the time of test reaches a specified time w_2 and n_2 failure times observed.

Let n_i denote the number of units failed at stress level x_i in time interval $(w_{i-1}, w_i]$, $w_0 = 0$, and t_{ij} denote the j-th ordered failure time of n_i units at x_i, ($j = 1, 2, \ldots, n_i$ and $i = 1, 2$), while $r_i = N_i - n_i$ denotes the number of units censored at time w_i.

The log-likelihood function of t_{ij} under simple CSALT with Type I censoring at can be written as

$$\ell(\lambda_i, k; t_{ij}) \quad \propto \quad \log k \sum_{i=1}^{2} n_i - k \sum_{i=1}^{2} n_i \log \lambda_i + (k-1) \sum_{i=1}^{2} \sum_{j=1}^{n_i} \log t_{ij} - \sum_{i=1}^{2} \frac{A_i}{\lambda_i^k}.$$

where

$$A_i = \sum_{j=1}^{n_i} t_{ij}^k + r_i w_i^k \tag{C.1}$$

Now, using log-linear link given in (B.4), the log-likelihood function of (β_0, β_1) can be obtained as

$$\begin{aligned} \ell(\beta_0, \beta_1) \quad \propto \quad & \log k \sum_{i=1}^{2} n_i - k \sum_{i=1}^{2} n_i(\beta_0 + \beta_1 x_i) + (k-1) \sum_{i=1}^{2} \sum_{j=1}^{n_i} \log t_{ij} \\ & - \sum_{i=1}^{2} e^{-k(\beta_0 + \beta_1 x_i)} A_i. \end{aligned} \tag{C.2}$$

Upon differentiating (C.2) with respect to β_0 and β_1, the MLEs $\hat{\beta}_0$ and $\hat{\beta}_1$ are obtained as simultaneous solutions to the following two equations:

$$\Big[\sum_{i=1}^{2} n_i\Big]\Big[\sum_{i=1}^{2} x_i e^{-k\hat{\beta}_1 x_i} A_i\Big] \quad = \quad \Big[\sum_{i=1}^{2} x_i n_i\Big]\Big[\sum_{i=1}^{k} e^{-k\hat{\beta}_1 x_i} A_i\Big], \tag{C.3}$$

$$\hat{\beta}_0 \quad = \quad \frac{1}{k} \log\Big(\frac{\sum_{i=1}^{2} e^{-k\hat{\beta}_1 x_i} A_i}{\sum_{i=1}^{2} n_i}\Big). \tag{C.4}$$

As shown above, $\hat{\beta}_0$ and $\hat{\beta}_1$ are nonlinear functions of random quantities and thus, statistical inference with these MLEs can be based on the asymptotic distributional result that the vector $(\hat{\beta}_0, \hat{\beta}_1)$ is approximately distributed as a bivariate normal with mean vector (β_0, β_1) and variance-covariance matrix $\mathbf{I}_n^{-1}(\beta_0, \beta_1)$, where $\mathbf{I}_n(\beta_0, \beta_1)$ is the Fisher information matrix. The Fisher information matrix is obtained through taking expectation on the negative of the second partial derivatives of (C.2) with respect to β_0 and β_1.

The second partial derivatives of the maximum likelihood function are given as the following:

$$\frac{\partial^2 \ell(\beta_0, \beta_1)}{\partial \beta_0^2} = -k^2 \sum_{i=1}^{2} e^{-k(\beta_0 - 2\beta_1 x_i)} A_i, \tag{C.5}$$

$$\frac{\partial^2 \ell(\beta_0, \beta_1)}{\partial \beta_1^2} = -k^2 \sum_{i=1}^{2} x_i^2 e^{-k(\beta_0 - 2\beta_1 x_i)} A_i, \tag{C.6}$$

$$\frac{\partial^2 \ell(\beta_0, \beta_1)}{\partial \beta_1 \partial \beta_0} = -k^2 \sum_{i=1}^{2} x_i e^{-k(\beta_0 - 2\beta_1 x_i)} A_i. \tag{C.7}$$

D Step-Stress Tests

Consider N identical units that are subjected to simple SSALT with initial stress level x_1. At prefixed time period w_1, stress level is changed to x_2 and the test is continued until the censoring time w_2. When all units fail before w_2, it would result in complete data. Total n_i failures are observed at time t_{ij} , $j = 1, 2, \ldots, n_i$, while testing at stress level x_i, $i = 1, 2$ and $N - n_1 - n_2$ products remain unfailed and censored at time w_2. With step stress loading, an assumption is required to represent the effect of increased stress levels on the lifetime distribution of a test unit. The Khamis-Higgins (KH) model is appropriate. The KH model is based on a time transformation of the cumulative exposure model. Khamis and Higins (3) have proposed this model for multiple step-stress testing. Their c.d.f. under step-stress testing can be written as

$$G(t) = \begin{cases} 1 - \exp\left\{-\left(\frac{t^k}{\lambda_1^k}\right)\right\}, & 0 \le t < w_1, \\ 1 - \exp\left\{\left(-\frac{w_1^k}{\lambda_1^k} - \frac{t^k - w_1^k}{\lambda_2^k}\right)\right\}, & w_1 \le t < w_2. \end{cases} \tag{D.1}$$

The corresponding p.d.f. is

$$g(t) = \begin{cases} \frac{k}{\lambda_1^k} t^{k-1} \exp\left\{-\left(\frac{t^k}{\lambda_1^k}\right)\right\}, & 0 \le t < w_1, \\ \frac{k}{\lambda_2^k} t^{k-1} \exp\left\{\left(-\frac{w_1^k}{\lambda_1^k} - \frac{t^k - w_1^k}{\lambda_2^k}\right)\right\}, & w_1 \le t < w_2. \end{cases} \tag{D.2}$$

Then, using (D.1), (D.2) and the log-linear link in (B.4), the likelihood function of t_{ij} under simple SSALT with Type I censoring is obtained as in (C.2) where

$$A_i = \sum_{j=1}^{n_i} (t_{ij}^k - w_{i-1}^k) + \big(N - \sum_{l=1}^{i} n_l\big)\big(w_i^k - w_{i-1}^k\big). \tag{D.3}$$

As a result, we obtain the MLEs $\hat{\beta}_0$ and $\hat{\beta}_1$ as simultaneous solutions to (C.3) and (C.4) with A_i given in (D.3).

Just like in the case of constant-stress testing, $\hat{\beta}_0$ and $\hat{\beta}_1$ are nonlinear functions of random quantities and hence, inference using these MLEs are based on the asymptotic distributional result that the vector $(\hat{\beta}_0, \hat{\beta}_1)$ is approximately distributed as a bivariate normal with mean (β_0, β_1) and variance-covariance matrix $\mathbf{I}_n^{-1}(\beta_0, \beta_1)$.

The second partial derivatives of the maximum likelihood function are as in (C.5)-(C.7) where A_i given in (D.3).

E Optimality Criteria

In this section, we define different optimality criteria for determining the optimal design points, which then can be used to compare between the simple constant-stress test and step-stress test. For the simple constant-stress testing, the focus is to determine the w_1 and w_2. We assumed that the duration of each steps are all equal for simplicity of discussion; i.e. , $w_1 = w_2 = \Delta$. The equi-length assumption is also convenient for practitioners. For the simple step-stress testing, the duration of each steps are $\Delta_i = w_i - w_{i-1}$ for $i = 1, 2$. With the equi-length assumption $\Delta_1 = \Delta_2 = \Delta$. These objective functions are purely based on the Fisher information matrix. In this paper, two optimality criteria are considered.

E.1 D-optimality

Another optimality criterion often used in planning ALT is based on the determinant of the Fisher information matrix, which equals to the reciprocal of the determinant of the asymptotic variance-covariance matrix. Note that the overall volume of the Wald-type joint confidence region of (β_0, β_1) is proportional to $|\mathbf{I}_n^{-1}(\beta_0, \beta_1)|^{1/2}$ at a fixed level of confidence. Consequently, a smaller asymptotic joint confidence ellipsoid of (β_0, β_1) would correspond to a higher joint precision of the estimators of β_0 and β_1. For this purpose, the D-optimal design points are obtained by minimizing $|\mathbf{I}_n^{-1}(\beta_0, \beta_1)|$ for the maximal joint precision of $(\hat{\beta}_0, \hat{\beta}_1)$.

E.2 T-optimality

Another optimality criterion considered in this study is based on the trace of the first-order approximation of the variance-covariance matrix of the MLEs. It is identical to the sum of the diagonal elements of $\mathbf{I}_n^{-1}(\beta_0, \beta_1)$. The T-optimality criterion provides an overall measure of the average variance of the parameter estimates and gives the sum of the eigenvalues of the inverse of the Fisher information matrix. The T-optimal design points minimize the trace of $\mathbf{I}_n^{-1}(\beta_0, \beta_1)$.

F Constrained Optimization and Cost Function

In order to conduct an ALT experiment efficiently with constrained resources in practice, stress durations should be determined carefully at the design stage. It is because these decision variables affect the experimental cost as well as the precision of the parameter estimates of interest. Under the constraint that the total experimental cost does not exceed a pre-specified budget, a typical decision problem of interest can be formulated as to optimize an objective function of choice subject to $C_T \leq C_B$, where C_B is the total pre-specified budget and C_T is the total cost for running an ALT. In general, the total cost of test can be expressed in a simplified form as

- The cost of setting up a life experiment, which includes the costs of facility and testing chambers, say C_s.
- The cost of test units is $N \times C_u$, where C_u denotes the cost of each test unit, including the costs of manufacturing, purchasing, and/or installation.
- The cost of operating an experiment is $2\Delta \times C_o$, where C_o is the operation cost in the each step for per unit.

Therefore, the total cost of experiment is $C_T = C_s + NC_u + 2\Delta C_o$.

Since the objective functions nonlinear functions of Δ, Matlab software can be used to find the optimal solution. $\Delta^{\star}$ is the optimal step duration.

G Numerical Results

The numerical study was conducted in order to determine the optimal design points under the cost and time constraints.

Tables 1 and 2 present the values of the optimal step durations along with the corresponding optimal objective functions described in the section E and total experimental cost without and with the cost constraint, respectively. In this study proposed a simple SSALT plan with $N = 150$, $x = (0.4, 0.7)$, $\beta_0 = 0.3$ and $\beta_1 = 0.5$. It is also assumed that at an appropriate cost measurement unit $C_s = 10\$$, $C_u = 0.1\$$, $C_o = 3\$$. The cost constraint there is that the maximum total experimental cost does not exceed the pre-specified budget $C_B = 30\$$.

In Table 1, it is observed that $\Delta^*_T < \Delta^*_D$ in the unconstrained. This order, however, was found to be a consequence of the specific setting chosen here and did not necessarily hold for another study. For this case study, the D-optimality criterion not only take more time to complete the test, but also more expensive than the T-optimality. The T-optimality is the most optimal design in terms of cost and duration of the test. On the other hand, both of the criteria, have cost more than 30\$; for this reason, the cost constraint 30\$ used. The results are presented in Table 2. In this table, it is observed that the cost constraint, for the both of the criteria achieves considerable decrease the step duration. In addition, it is observed that $\Delta_T^* < \Delta_D^*$. This is the same result with the unconstrained mode. The T-optimality is still the optimal criterion, because it has the lowest cost and the minimum duration of the test.

Tables 3 and 4 present the values of the optimal durations, along with the corresponding optima of each objective function and the total cost for the both of the criteria without and with the cost constraint, for the simple SSALT. In Tables 4 and 4, it is observed that T-optimality is an optimal plan, because it has the lowest cost and test time. Also, D-optimality lead to the more cost and time of the test. Similar to the simple CSALT, we have $\Delta_T^* < \Delta_D^*$. In the simple SSALT compared to the simple CSALT, the cost and duration of the test are reduced. In addition, under the cost constraint, time and cost of the tests are reduced.

Overall, the CSALT is empirically shown to be more expensive compared to the corresponding SSALT one under the unconstrained and constrained optimal situations.

H Conclusions

In this article, the optimal simple CSALT and simple SSALT were compared for the weibull failure data under Type-I censoring. One of the objectives of this article was to quantify the advantage of using the SSALT relative to the CSALT. A log-linear relationship was assumed between the scale parameter and stress level, and the KH model was assumed for the effect of changing stress levels in the SSALT. The MLEs of the regression parameters and the associated Fisher information were derived. After obtaining the explicit cost functions, the optimal design points were determined according to the D-optimality and T-optimality criteria based on the information matrix under a pre-specified budget constraint. Regardless of the stress loadings, the D-optimal design was generally found to cost the most for test completion, while the T-optimal design was found optimal criterion. The results of the numerical study also quantified the advantage of using SSALT compared to CSALT. It was demonstrated that the SSALT is overall more affordable than the corresponding CSALT under the unconstrained/constrained situations.

Table 1: Optimal step durations and total experimental cost without the cost constraint for simple CSALT.

	Unconstrained ($C_B = \infty$)	
	T-Optimality	**D-Optimality**
Step Duration (Δ^*)	1.1313	2.4223
Total Experimental Cost (C_T)	31.7878	39.5338

Table 2: Optimal step durations and total experimental cost with the cost constraint for simple CSALT.

	Constrained ($C_B = 30\$$)	
	T-Optimality	**D-Optimality**
Step Duration (Δ^*)	0.7823	0.8253
Total Experimental Cost (C_T)	29.6938	29.9518

References

[1] Han, D. and Ng, H.T. (2014), Asymptotic comparison between constant-stress testing and step-stress testing for Type-I censored data from exponential distribution, *Communications in Statistics-Theory and Methods*, **43**, 10-12. 2384–2394.

[2] Hu, C.H., Plante, R.D. and Tang, J. (2013), Statistical equivalency and optimality of simple step-stress accelerated test plans for the exponential distribution, *Naval Research Logistics*, **60**, 19-30.

[3] Khamis, I.H. and Higgins, J.J. (1998), A new model for step-stress testing, *IEEE Transactions on Reliability*, **47**, 131–134.

[4] Sharon, V.A. and Vaidyanathan, V.S. (2016), Analysis of simple step-stress accelerated life test data from Lindley distribution under type-I censoring, *Statistica*, **76**, 233-248.

Table 3: Optimal step durations and total experimental cost without the cost constraint for simple SSALT.

[h] c c c height

Unconstrained ($C_B = \infty$)
T-Optimality
D-Optimality
Step Duration (Δ^*)
0.9832
1.3321
Total Experimental Cost (C_T)
30.8992
32.9926

Table 4: Optimal step durations and total experimental cost with the cost constraint for simple SSALT.

	Constrained ($C_B = 30\$$)	
	T-Optimality	**D-Optimality**
Step Duration (Δ^*)	0.3628	0.5372
Total Experimental Cost (C_T)	27.1768	28.2232

[5] Nelson, W. B. (1990), *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*, John Wiley and Sons, New York.

Some Maintenance Policies for Coherent Systems

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Abstract: The optimal strategies to maintain the technical systems in good working condition are of important goals in reliability engineering. In this talk, some optimal maintenance policies for n-component coherent systems under some partial information on the component failures of the system are proposed. We introduce two criteria to compute the probability of the number of failed components in the system. Also, by imposing some cost functions, we introduce two new approaches to the optimal *corrective and preventive* maintenance of a coherent system based on the proposed criteria.

Keywords Preventive maintenance, Corrective maintenance, Minimal repair, Signature, Survival signature.
Mathematics Subject Classification (2010) : 90B25, 60K10.

A Introduction

Nowadays, coherent systems are appearing in different aspects of human life such as industrial manufacturing lines, telecommunication systems etc. for various goals. In the literature of reliability engineering, a system consisting of n components is known as a coherent system if the structure function is nondecreasing in every component and it has no irrelevant component (see, Barlow and Proschan, 1975).

In recent years a large number of research works have been reported in the literature to assess the reliability and stochastic properties of coherent systems using the concept of *signature*. Let $X_1, \dots, X_n$ denote the lifetimes of an n-component coherent system and let T be the system lifetime. Under the assumption that the component lifetimes are independent and identically distributed (i.i.d.), Samaniego (1985) defined the concept of signature to express the reliability function of the system lifetime as a mixture of the reliability functions of the ordered lifetimes of its components. Let $X_{1:n}, \dots, X_{n:n}$ denote the corresponding order statistics of X_i's. Then

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the reliability function of the system lifetime can be represented as

$$P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t), \tag{A.1}$$

where $s_i = P(T = X_{i:n})$, $i = 1, \ldots, n$. The probability vector $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ is called the signature vector of the system.

In the study of the reliability and stochastic characteristics of the systems, a problem of interest, for engineers and system designers, is to maintain the system in good working conditions and to determine the number of spares that should be available in the depot for this purpose. The justification for the importance of this problem arises from the fact that the failure of the system and its unavailability may cause high costs for the users. In many complex coherent systems, the design of the structure of the system is such that the system operates, even though a number of components have already failed. Hence, the computation of the probability of the number of failed components in the system, under different conditions, is important for the system operators. These probabilities provide crucial information for preventing and maintaining the system in optimal operating conditions. The aim of maintenance schedules is mainly to diminish the occurrence of system failure or to change the status of a failed system to the working state.

Maintenance actions can generally be divided into two types: corrective maintenance (CM) and preventive maintenance (PM). For a deteriorating repairable system, the CM action is conducted upon failure to recover the system from a failure, whereas the PM action is performed at the planned time to improve the system reliability performance.

The aim of the present research is to give some maintenance policies for a coherent system under some partial information on the number of failures in the operating system. The system starts to work at time $t = 0$ and each component may fail over the time. We consider two conditional probabilities for an operating coherent system at time t. First we assume that the system is working at time t and at least k components have been failed before t. Under these assumptions, in Section 2, we compute the probability of the number of failed components in the system. Let N_t denote the number of failed components until time t. Then, we define the conditional probabilities

$$P(N_t = i \mid X_{k:n} \leq t < T), \quad i = k, ..., n-1.$$

In the second conditional probability, we consider the condition that exactly k components fail at the time t_1, and at time t_2 ($t_2 > t_1$) the system is still operating. Under this condition the probability of the number of failed components, N_{t_2}, is as follows:

$$P(N_{t_2} = i \mid X_{k:n} \leq t_1 < X_{k+1:n}, T > t_2), \quad i = k, ..., n-1.$$

As applications of two conditional probabilities mentioned above, two optimal maintenance polices for coherent systems are presented in Section 3.

B The number of failed components in the system

Consider a coherent system with lifetime T, as described in Introduction Section. The system starts to work at time $t = 0$ and each component may fail over the time. Suppose the system has been inspected at time t and we observed that at least k components have been failed before t, but the system is still working. Under this situation, the number of failed components on $[0, t]$, denoted by N_t, may be of interest. That is $(N_t \mid X_{k:n} \leq t < T)$, $k = 0, 1, ..., n-1$. Here, by the convention, $X_{0:n} \equiv 0$. Asadi and Berred (2012) have studied several properties of the above conditional random variable in the case that $k = 0$.

The probability mass function of $(N_t \mid X_{k:n} \leq t < T)$ can be computed as (see, Hashemi and Asadi, 2019)

$$P(N_t = i \mid X_{k:n} \leq t < T) = \frac{\bar{S}_i \binom{n}{i} \phi^i(t)}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \phi^j(t)}, \quad i = k, ..., n-1, \tag{B.1}$$

where $\bar{S}_i = \sum_{j=i+1}^{n} s_j$ and $\phi(t) = F(t)/\bar{F}(t)$. Another interesting quantity is the expected number of spares that are needed at time t to replace all failed components of the system:

$$E(N_t \mid X_{k:n} \leq t < T) = \frac{n \sum_{i=k}^{n-1} \bar{S}_i \binom{n-1}{i-1} \phi^i(t)}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \phi^j(t)}, 0 \leq k \leq n-1. \tag{B.2}$$

Now, it is assumed that the system is monitored in two time instances t_1 and t_2, $(t_1 < t_2)$ by the operator. Suppose that the total number of component failures at time t_1 is k, and at time t_2 the system is still alive. Under these circumstances, we are interested in the study of the number of failed components in the system at time t_2; that is we are interested in conditional random variable

$$(N_{t_2} \mid X_{k:n} \leq t_1 < X_{k+1:n}, T > t_2), \quad k = 0, 1, ..., n-1.$$

The probability mass function of this conditional random variable, can be computed, for $i = k, ..., n-1$, as

$$P(N_{t_2} = i \mid X_{k:n} \leq t_1 < X_{k+1:n}, T > t_2) = \frac{\bar{S}_i \binom{n}{i} \binom{i}{k} (\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1)^{i-k}}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \binom{j}{k} (\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1)^{j-k}}.$$

Hence, the mean number of failed components up to time t_2 can be expressed as

$$E(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2) = \frac{n \sum_{i=k}^{n-1} \bar{S}_i \binom{n-1}{i-1} \binom{i}{k} (\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1)^i}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \binom{j}{k} (\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1)^j}. \tag{B.3}$$

C Optimal corrective and preventive maintenance models

In this section, we develop two maintenance strategies for n-component coherent systems based on the conditional probabilities introduced in the previous section.

Strategy I

Assume that a coherent system begins to operate at time 0. A minimal repair has been performed on each component of the system that fails in the interval $(0, \tau)$ at a cost c_{min}. Thus, we can suppose that the system, consisting of n unfailed components with age τ, is alive at τ. Here, τ is a predetermined constant. The system has been inspected at t $(t > \tau)$. The operator decides to perform CM on the whole system at a cost c_{cms} once the system fails in the interval (τ, t), or to perform CM at t on the failed components of the system together with PM of all unfailed but deteriorating ones at a cost c_{cm} for CM and a cost c_{pm} for PM, whichever occurs first.

The justification of this policy in the interval $(0, \tau)$ is that each component is young and there is no need for major repair. Thus, before τ, only minimal repairs, which may not take much time and money, are carried out.

The expected cost of minimal repairs for the whole system in a renewal cycle is

$$c^*_{min} = nc_{min}H(\tau),$$

where $H(\tau) = \int_0^\tau r(t)dt$. For a more general cost structure, see, for example, Pham and Wang (2000).

The average system maintenance cost per unit time is then defined as

$$\begin{aligned} \eta_I(t) &= \frac{nc_{min}H(\tau) + F_\tau(t-\tau)c_{cms}}{\tau + E(\min(t-\tau, T_\tau))} \\ &+ \frac{\bar{F}_\tau(t-\tau)\left[E(N_{t,\tau} \mid T_\tau > t)(c_{cm} - c_{pm}) + nc_{pm}\right]}{\tau + E(\min(t-\tau, T_\tau))}, \end{aligned}$$

where $F_\tau(\cdot)$ denotes the distribution function of the lifetime of a system consisting of n components of age τ and $E(N_{t,\tau} \mid T_\tau > t)$ is the expected number of failed components of alive system at t when all components are functioning at τ, $(\tau < t)$. It can be easily shown that

$$\bar{F}_\tau(t-\tau) = \sum_{j=0}^{n-1} \bar{S}_j \binom{n}{j} \left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^j \left(\frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{n-j},$$

and

$$E(\min(t-\tau, T_\tau)) \quad = \quad \int_0^{t-\tau} [1 - F_\tau(x)]dx.$$

Also, we can obtain

$$E(N_t \mid T > t) = \frac{n \sum_{i=2}^{n} \bar{S}_{i-1} \binom{n-1}{i-2} \phi^i(t)}{\sum_{j=1}^{n} \bar{S}_{j-1} \binom{n}{j-1} \phi^j(t)}.$$

By substituting $\phi(t)$ with $\left(\frac{\bar{F}(\tau)}{\bar{F}(t)} - 1\right)$, we may obtain the corresponding formula for $E(N_{t,\tau} \mid T_\tau > t)$.

Now, let us assume that minimal repair takes negligible time, CM combined with PM takes w_1 time units and CM on the whole system at time t takes w_2 time units. The stationary availability for Strategy I is given by

$$A_I(t) = \frac{\tau + E(\min(t-\tau, T_\tau))}{\tau + E(\min(t-\tau, T_\tau)) + w_1 \bar{F}_\tau(t-\tau) + w_2 F_\tau(t-\tau)}.$$

Consider the bridge system whose components lifetimes are i.i.d. having Weibull distribution with reliability function $\bar{F}(t) = \exp\{-t^2\}$, $t \geq 0$. It is known that the system signature is $(0, 0.2, 0.6, 0.2, 0)$. In Table 1, the optimal times t^* that minimize the expected cost per unit of time and $\eta_I(t^*)$ are presented for several time instants τ. We observed that the optimal value for the pair (τ, t^*) is $(1.596, 1.60735)$, which results in the minimum maintenance cost 8.80022. Figure 1 shows the two-dimensional plot of the cost function in terms of (τ, t) for $c_{min} = 0.5$, $c_{cms} = 25$, $c_{cm} = 2$ and $c_{pm} = 1$. Also, in Figure 2(a), the graph of $\eta_I(t)$ is presented for different values of τ, and the above mentioned costs. Figure 2(b) depicts the plots of $A_I(t)$ for $w_1 = 0.08$ and $w_2 = 0.2$ and for different values of τ. As the plots show, the system availability first increases to achieve its maximum and then decreases.

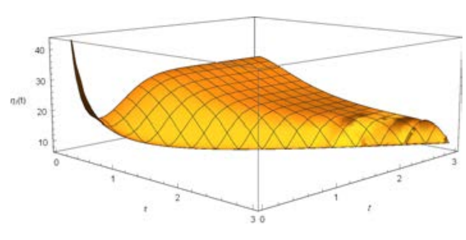

Figure 1: The average system maintenance cost per unit time in Example C.

Strategy II

Assume that a new coherent system starts to work at time 0. Suppose that the system has been inspected at two time instants t_1 and t_2, $t_1 < t_2$. If the system has failed before t_1, then the

Table 1: Optimal maintenance time for $c_{min} = 0.5$, $c_{cms} = 3$, $c_{cm} = 2$ and $c_{pm} = 1$.

τ	t^*	$\eta_I(t^*)$
0.100	0.46754	15.98490
0.500	0.60056	12.07660
1.000	1.02629	9.38755
1.500	1.51255	8.81008
1.595	**1.60636**	**8.80021**
1.596	1.60735	8.80022
1.598	1.60932	8.80023
1.700	1.71026	8.81158
2.000	2.00795	8.94730

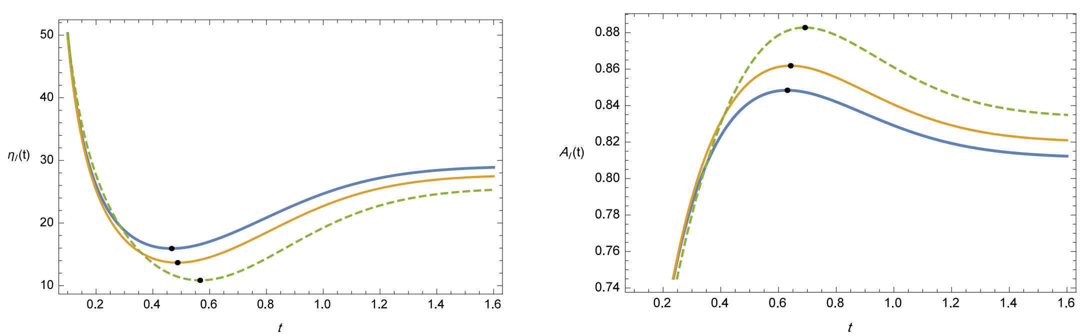

Figure 2: (a) The average system maintenance cost per unit time in Example C: $\tau = 0.1, 0.3, 0.5$ from up to down, (b) The stationary availability in Example C: $\tau = 0.1, 0.3, 0.5$ from up to down.

operator decides to perform a CM on the whole system at a cost c_{cms} as soon as the system fails. He/she performs the same action if the system has failed during the time interval (t_1, t_2). On the other hand, if the system is functioning at t_2, the operator decides to perform three different actions: (a) If the number of failed components N_{t_1} at t_1 is at most $(k_1 - 1)$, the operator performs a PM on the whole system at a cost c_{pms}; (b) If $k_1 \leq N_{t_1} \leq k_2$, then he/she decides to perform CM on the failed components of the system together with PM of all unfailed but deteriorating ones at a cost c_{cm} for CM and a cost c_{pm} for PM, respetively; (c) If N_{t_1} is at least $(k_2 + 1)$, then the operator decides to perform a more rigid PM on the system (than Case (a)) at a cost c^*_{pms}. In this strategy, we assume that t_2 is the decision variable, while t_1, k_1 and k_2 are some fixed constants.

The average system maintenance cost per unit of time is

$$\eta(t_2) = \frac{D(t_2)}{E(\min(t_2, T))},$$

where

$$\begin{aligned} D(t_2) &= c_{cms}P(T \le t_2) + c_{pms}P(T > t_2, N_{t_1} \le k_1 - 1) \\ &+ [(c_{cm} - c_{pm})E(N_{t_2} \mid k_1 \le N_{t_1} \le k_2, T > t_2) + nc_{pm}] \\ &\times P(T > t_2, k_1 \le N_{t_1} \le k_2) + c^*_{pms}P(T > t_2, N_{t_1} \ge k_2 + 1). \end{aligned}$$

In a special case where $k_1 = k_2 = k$, $D(t_2)$ may be reduced to

$$\begin{aligned} D(t_2) &= c_{cms}P(T \le t_2) + c_{pms}P(T > t_2, N_{t_1} \le k - 1) \\ &+ [(c_{cm} - c_{pm})\varphi(t_1, t_2) + nc_{pm}]P(T > t_2, N_{t_1} = k) \\ &+ c^*_{pms}P(T > t_2, N_{t_1} \ge k + 1), \end{aligned}$$

where, from (B.1),

$$\begin{aligned} \varphi(t_1, t_2) &= E(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2) \\ &= \frac{n\sum_{i=k}^{n-1} \bar{S}_i \binom{n-1}{i-1}\binom{i}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^i}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j}\binom{j}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^j}. \end{aligned}$$

Also

$$\begin{aligned} P(T > t_2, N_{t_1} < k - 1) &= \sum_{i=1}^{k} s_i \sum_{j=0}^{i-1} \binom{n}{j} F^j(t_2)\bar{F}^{n-j}(t_2) \\ &+ \sum_{i=k+1}^{n} s_i \sum_{m=n-i+1}^{n} \sum_{l=\max(m,n-k+1)}^{n} \binom{n}{l}\binom{l}{m} F^{n-l}(t_1) \\ &\qquad \times (\bar{F}(t_1) - \bar{F}(t_2))^{l-m}\bar{F}^m(t_2) \end{aligned}$$

and

$$\begin{aligned} P(T > t_2, N_{t_1} \ge k + 1) &= \sum_{i=k+2}^{n} s_i \sum_{j=k+1}^{i-1} \sum_{m=n-i+1}^{n-k-1} \binom{n}{m}\binom{n-m}{j} F^j(t_1) \\ &\times (\bar{F}(t_1) - \bar{F}(t_2))^{n-j-m}\bar{F}^m(t_2). \end{aligned}$$

On the other hand,

$$P(T > t_2, N_{t_1} = k) = \sum_{i=k+1}^{n} s_i \sum_{j=k}^{i-1} \binom{n}{j}\binom{n}{k} F^k(t_1)\bar{F}^{n-j}(t_2)(\bar{F}(t_1) - \bar{F}(t_2))^{j-k}.$$

Also, we may obtain

$$P(T < t_2) = 1 - \sum_{j=0}^{n-1} \bar{S}_j \binom{n}{j} F^j(t_2) \bar{F}^{n-j}(t_2).$$

Consider again the bridge system, where the component lifetimes are i.i.d. having Weibull distribution with reliability function $\bar{F}(t) = \exp\{-t^2\}$, $t \geq 0$. Figure 3 shows the plot of $\eta(t_2)$ for $t_1 = 0.5$, $c^*_{pms} = 20$, $c_{cms} = 20$, $c_{pms} = 5$, $c_{cm} = 2$ and $c_{pm} = 1$ and different values $k = 1, 2, 3$. In Table 2, the optimal times t^*_2 and $\eta(t^*_2)$ are presented for different values of c_{cms} and c_{pm}. As expected, when c_{cms} increases (c_{pm} decreases) then t^*_2 decreases.

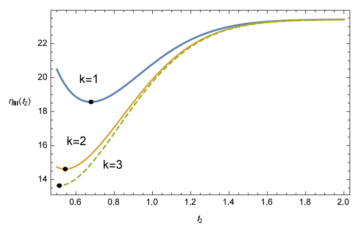

Figure 3: The average maintenance cost per unit time in Example C.

Table 2: Optimal maintenance time for Strategy II with $t_1 = 0.5$, $k = 1$.

$c_{cm} = 2$, $c_{pms} = 5$, $c^*_{pms} = 20$					
$c_{pm} = 1$			$c_{cms} = 20$		
c_{cms}	t^*_2	$\eta_{III}(t^*_2)$	c_{pm}	t^*_2	$\eta_{III}(t^*_2)$
10	1.3000	11.6949	0.6	0.6465	17.8146
15	0.8010	15.8536	1.0	0.6774	18.5673
20	0.6774	18.5673	1.5	0.7159	19.3944
25	0.6101	20.6165	2.0	0.7545	20.1078

References

[1] Asadi, M. and Berred, A. (2012), On the number of failed components in a coherent operating system, *Statistics and Probability Letters*, **82**, 2156-2163.

[2] Barlow, R.E. and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, New York.

[3] Hashemi, M. and Asadi, M. (2019), On components failure in coherent systems with applications to maintenance strategies, *Submitted*.

[4] Pham, H. and Wang, H. (2000), Optimal (τ, T) opportunistic maintenance of a k-out-of-$n : G$ system with imperfect PM and partial failure, *Naval Research Logistics*, **47**, 223-239.

[5] Samaniego, F.J. (1985), On closure of the IFR class under formation of coherent systems, *IEEE Transactions on Reliability*, **34**, 69-72.

Condition-based Maintenance Strategy Based on the Inverse Gaussian Degradation Process

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Abstract: In this study a condition based maintenance (CBM) considering for a single unit system with two competing causes, degradation-based failure and shock-based failure. Inverse Gaussian process (IG) utilized to describe the degradation behavior of this system and External shocks arriving at random times according Non-homogeneous poisson processes (NHPP). To increasing the life time of the system, an imperfect maintenance performed. The main objective of this study is to minimize the expected cost per unit time by consider a relationship between the degradation level after imperfect maintenance and the cost of this action. This relationship can be linear and non-linear. Finally a numerical example introduced to describe the proposed maintenance policy.

Keywords Inverse Gaussian process, External shocks, Condition-based maintenance, Imperfect maintenance.

Mathematics Subject Classification (2010) : 90B25**.**

A Introduction

All industrial systems suffer from inevitable failures over the time. This failures can be very costly for a company. So, Nowadays maintenance has played an important role in industry because an effective maintenance programme could minimize the maintenance cost. Generally maintenance divided into two tasks, Preventive maintenance and Corrective maintenance. System can failed by two competing causes degradation and random shocks. In these systems, a failure occur when the degradation levels exceed a critical threshold. Wiener (7), Gamma (4)and Inverse Gaussian process ((9),(5),(1)) are most popular process to describe the degradation behavior of a system. In addition systems suffer from fatal shocks. These shocks arriving at random times according non-homogeneous poisson process (NHPP) and lead system to fail

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immediately. Degradation-Threshold-Shock (DTS) models have widely use for systems with two causes for failed. For example (6) proposed a system subject generalized mixed shock.

Imperfect maintenance action widely use to reduced the expected cost per unit time by considering stochastic process as like as wiener process ((10),(3)), gamma process ((2),(4)) and Inverse gaussian process (1).

In this study a single unit system is considering with two competing causes of failures, i.e., degradation and shock. Inverse gaussian process (IG) used to describe the degradation behavior of this system and external shocks arriving at random times according Non-homogeneous poisson processes. This shocks are fatal so they can cause the system to fail immediately. Both perfect and imperfect maintenance are considering for this system. We try to find an optimal degradation levels after imperfect maintenance by determined a relationship between the expected degradation reduction and the cost of the imperfect maintenance.

The rest of the paper is organized as follows. Section B discribes the failure behavior of the system. Section C introduced the proposed maintenance policy and section D a numerical example describe the proposed maintenance policy.

B System description

B.1 Degradation behaviors

The degradation behavior is modeled based on Inverse gaussian process with shape and scale parameters given by μ and λ, $(\mu, \lambda > 0)$. Inverse gaussian process has the following properties.

1. X(0)=0, whit probability one.

2. Inverse gaussian process has independent increment.

3. Each increment follow Inverse gaussian distribution $X(t+h) - X(t) \sim IG(\mu h, \lambda h^2)$, for $h \geq 0$. Probability density function (PDF) of $IG(\mu h, \lambda h^2)$ is defined by:

$$f_{X(t)}(x) = \sqrt{\frac{\lambda h^2}{2\pi x^3}} \exp\left\{ - \frac{\lambda h^2 (x - \mu h)^2}{2(\mu h)^2 x} \right\}. \tag{B.1}$$

So, Inverse gaussian process has mean μh and variance $\frac{\mu^3 h}{\lambda}$.

B.2 External shocks

In addition internal-based degradation, systems suffer from external shocks. This shocks are fatal and the system can fail immediately. External shocks arriving at random times according

Non-homogeneous Poisson Processes. $\nu(t)$ denote the occurrence rate of a shock and intensity function is:

$$\Lambda(t) = \int_0^t \lambda(x)x. \tag{B.2}$$

If T_s denote the random time to an external shock the cumulative distribution function (CDF) of T_s can be written as

$$F_{T_s}(t) = 1 - \exp\left\{ - \int_0^t \nu(y)y \right\}. \tag{B.3}$$

C Maintenance policy

We consider a single unit system that can fail by two competing causes, degradation and random shock. The basic assumptions of the system are following:

a) System subject periodic inspections every T time to determine degradation levels and the state of a system. These Inspections are assumed to instantaneous and perfect.

b) System is non-self announced so, If a failure occurred the system stopped until next inspection.

c) At an inspection if degradation level exceeded a critical threshold L or a fatal shock is detected, then a corrective replacement performed and system replaced by a new one.

e) We assume y_{i_k} is the degradation level after k^{th} imperfect maintenance, Δ is imperfect maintenance threshold and M is preventive maintenance threshold. At an inspection if degradation level is between (M, L),then If $M - y_{i_k} \leq \Delta$ a perfect preventive replacement performed and system replaced by a new one. If $M - y_{i_k} > \Delta$ an imperfect preventive replacement performed and degradation level reduce to y_{i_k} by cost C_p^k.

f) At an inspection if degradation level is less than M then no preventive actions is performed.

g) The cost of maintenance actions for this system are: The corrective replacement action cost C_c; the perfect preventive replacement action cost C_p; the k^{th} imperfect preventive replacement action cost C_p^k; the expected downtime cost C_{down}; the inspection cost C_{ins}. It is assumed that

$$C_c > C_p \geq C_p^k > C_{down} > C_{ins}$$

.

C.1 Imperfect maintenance action

According imperfect maintenance actions system can divide into three parts such as initial part, middle part and final part. The initial part started at time 0 with degradation level 0 and ended with first imperfect maintenance. The length of this part could be 0 if the system failed before first imperfect maintenance. The middle part started at an imperfect maintenance until next imperfect maintenance. The length of this part can be 0 to infinity. At the end, final part started from last imperfect maintenance until a corrective maintenance or a preventive maintenance occurred. There are three possible scenarios for replacement system at the final part, preventive replacement, corrective replacement due to degradation and corrective replacement due to external shock.

In this way k^{th} imperfect maintenance performed at inspection i_k and degradation level reduce to a level y_{i_k} $(y_{i_{(k-1)}} < y_{i_k} < M)$, clearly $i_0 = 0$ and $y_{i_0} = 0$.

Figure 1 has shown this reduction for a system that replaced by a preventive maintenance.

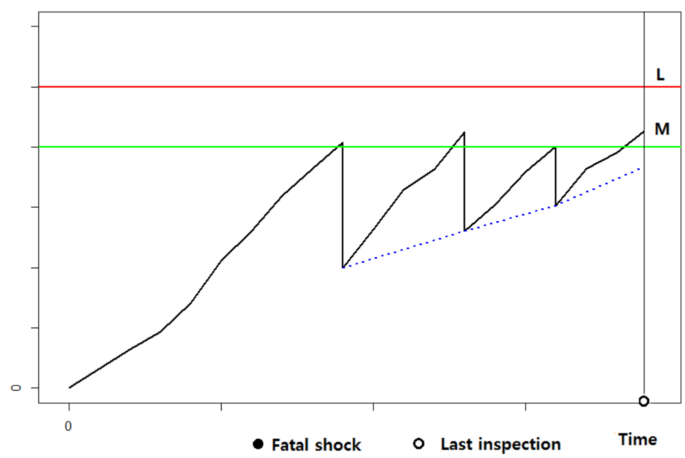

Figure 1: Impact of imperfect maintenance on degradation level

The number of imperfect maintenance is not known in this study, it's depend on system situation, in fact an imperfect maintenance performed if and only if the degradation level is between (M, L) and $M - y_{i_k} > \Delta$. The degradation level after k^{th} imperfect maintenance can be written as:

$$y_{i_k} = \gamma + (k-1)\left(\frac{X_{i_k}}{L}\right)^{\alpha}. \tag{C.1}$$

In this equation X_{i_k} is degradation level before imperfect maintenance. This reduction has performed based on a relationship between degradation reduction and the cost of imperfect maintenance. This relation can be linear so:

$$C_p^k = C_p^0 U(i_k) + \beta(k-1), \tag{C.2}$$

or non-linear that can be written as:

$$C_p^k = C_p^0 (U(i_k))^\eta + \beta(k-1). \tag{C.3}$$

In above equations $U(i_k)$ is the degradation improvement factor and is given by:

$$U(i_k) = \left(\frac{X_{i_k} - y_{i_k}}{X_{i_k}}\right) \tag{C.4}$$

Also C_p^0 is a constant cost for imperfect maintenance action when the degradation level of the system is reduced to 0 at the first time that imperfect maintenance action occured. Therefore, when $y_{i_1} = 0$ the imperfect maintenance cost is constant $C_p^1 = C_p^0 = C_p$.

If K is the number of last imperfect maintenance action, So the following relationshipes exists for y_{i_k}:

$$y_{i_0} \le y_{i_1} \le \cdots \le y_{i_K},$$

and for imperfect maintenance cost:

$$C_p^0 \ge C_p^1 \ge \cdots \ge C_p^K.$$

Figure 2 has shown the impact of imperfect maintenance cost function on the degradation improvement factor that can have three different shapes.

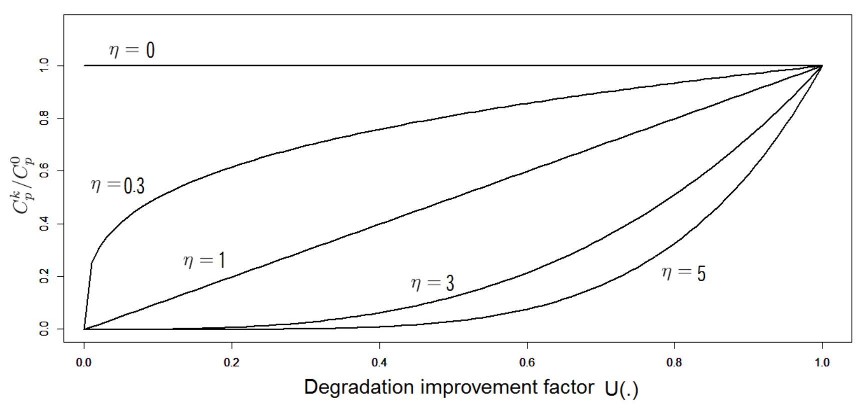

Figure 2: Impact of imperfect maintenance cost function

According above assumptions the decision process has shown by algorithm C.1 [h!] Decision process [1] Giving degradation and shock parameters

$L, M, T, \gamma, \Delta, \eta$

$k = 0$; $i_k = 0$;$X(0) = 0$; $y_{i_k} = 0$ Monitor the system at $i_k * T$ $X(i_k) \geq L \quad OR \quad i_k * T \geq T_s$ Corrective replacement $k = 0$ and $X(i_0) = 0$ $M \leq X(i_k) < L$ $M - y_{i_k} \leq \Delta$ Preventive replacement $k = 0$ and $X(i_0) = 0$ $M - y_{i_k} > \Delta$ imperfect maintenance $k = k + 1$ $y_{i_k} = \gamma + (k-1)\left(\frac{X_{i_k}}{L}\right)^{\alpha}$ $C_p^k = C_p^0\left(\frac{X_{i_k} - y_{i_k}}{X_{i_k}}\right)^{\eta} + \beta(k-1)$ $X(i_k) = y_{i_k}$ $X(i_k) < M$ $i_k = i_k + 1$ Go to line 2

C.2 Cost optimization

According the proposed maintenance policy the expected cost per unit time of the system is given by:

$$\overline{C}(T, M, \gamma, \alpha) = \frac{C_c P_c + C_p P_P + C_{ins} E[N_{ins}] + C_{down} E[T_{down}] + E[\sum_{k=0}^{\infty} C_p^k]}{E[N_{ins}]T} \tag{C.5}$$

In this equation P_c is the probability of corrective replacement, P_p probability of preventive replacement, $E[N_{ins}]$ the expected number of inspections, $E[T_{down}]$ the expected downtime and $E[\sum_{k=0}^{\infty} C_p^k]$ the expected imperfect maintenance cost.

So to minimize the expected cost per unit time we try to determine the optimal value of the decision variation. In fact we try to find

$$\overline{C^*}(T, M, \gamma, \alpha) = \inf\{\overline{C}(T, M, \gamma, \alpha); T \geq 0, 0 < \gamma < M < L, \alpha \geq 0\}. \tag{C.6}$$

The artificial bee colony (ABC) algorithm is used to detemine the optimal variables.

D Simulation study

The proposed maintenance policy is simulated. We consider Inverse gaussian process to describe degradation process by scale parameter $\mu = 0.5$ and shape parameter $\lambda = 5$ so each increment follows

$$X(t) \sim IG(0.5t, 5t^2)$$

Shock process occurred based on Non-homogeneous poisson processes and the occurrence rate of a shock is given by:

$$\nu(t) = 0.019t^{0.72}$$

Table 1 has shown the cost values for maintenance actions.

Table 1: Maintenance actions cost

C_c	C_p	C_P^0	C_{down}	C_{ins}	β
1000	800	800	400	100	3

And critical threshold for a failure due to initial degradation is $L = 20$, imperfect maintenance threshold is $\Delta = 0.5$. Table 1 has shown the expected cost per unit time based on the proposed maintenance policy and optimal variables for different η.

Table 2: Optimal maintenance policy with different η.

η	T^*	M^*	γ^*	α^*	Expected cost
1.5	3.22	17.56	1.29	4	148.0494
1	3.06	17.92	2.27	5	148.2259
0.3	2.95	18.14	8.22	3	148.2823

References

[1] Chen, Nan, et al. (2015), Condition-based maintenance using the inverse Gaussian degradation model, *European Journal of Operational Research*, **243**(1), 190-199.

[2] Do, P., Voisin, A., Levrat, E. and Iung, B. (2015), A proactive condition-based maintenance strategy with both perfect and imperfect maintenance actions, *Reliability Engineering and System Safety*, **133**, 22-32.

[3] Bin, Liu., Min, Xie., Zhengguo, Xu. and Way, Kuoa. (2016), An imperfect maintenance policy for mission-oriented systems subject to degradation and external shocks, *Computers and Industrial Engineering*, **102**, 21-32.

[4] Mercier, S. and Castro, I.T. (2019), Stochastic comparisons of imperfect maintenance models for a gamma deteriorating system, *European Journal of Operational Research*, **273**, 237-248.

[5] Li, R., Liu,X., Song, Y., Cai, Z. and Li, J. (2017), An Optimal Condition Based Maintenance Strategy Using Inverse-Gaussian Degradation Process, *IEEE Transactions on Reliability.*

[6] Rafiee, K., Feng, Q. and Coit, D.W. (2017), Reliability assessment of competing risks with generalized mixed shock models, *Reliability Engineering and System Safety,***159**,1-11.

[7] Yang, L., Ma, X., Peng, R., Zhai, Q. and Zhao, Y. (2017), A preventive maintenance model based on dependent two-stage deterioration and external shocks, *Reliability Engineering and System Safety*, **160**, 201-211.

[8] Yang, L., Ma, X. and Zhao, Y. (2017), A condition-based maintenance model for a three-state system subject to degradation and environmental shocks, *Computers and Industrial Engineering*, **105**, 210-222.

[9] Yang, L., Zhao, Y., Peng, R. and Ma, X. (2018), Opportunistic maintenance of production systems subject to random wait time and multiple control limits, *Journal of Manufacturing Systems*, **47**,12-34.

[10] Zhang, M., Gaudoin, O. and Xie, M. (2015), Degradation-based maintenance decision using stochastic ltering for systems under imperfect maintenance, *European Journal of Operational Research*, **245**, 531-541.

On Estimation of Stress-Strength Parameter for Discrete Weibull Distribution

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Abstract: In this paper, statistical inference for the reliability of stress-strength models when stress and strength are independent discrete Weibull random variables is discussed. The so called proportion method estimator of model is studied and is compared with maximum likelihood (ML) estimator. Also, based on simulation, the root mean square error (RMSE) and the relative bias (RB) of the estimator of $R = P(X < Y)$ and its variance are computed and compared. Furthermore, we have provied a confidence interval for R as well as its coverage rate. **Keywords** Discrete Weibull distribution, Stress-Strength model, Maximum likelihood estimator.

Mathematics Subject Classification (2010) : 62F10**,** 62F12**,** 62F15**,** 62N01**,** 62N05**.**

A Introduction

Stress-Strength models are one of the key issues in reliability and many other sciences. If Y represents the strength of a certain system and X the stress on it, $R = P(X < Y)$ represents the probability that the strength overcomes the stress, and then the system works (R is then referred to as the reliability parameter). Stress-Strength model in the last decades has attracted much interest from various fields (8; 12), ranging from engineering to biostatistics. Most research in this area is carried out for continuous distribution and less work is done for discrete case.

Among the recent works on continuous case, one can refer to the works of Kzlaslan (14), Chaudhary and Tomer (6), Bai et al. (2; 3), Eryilmaz (7), Wang et al. (21), Yadav and Singh (22) and Cetinkaya and Gen (5).

The majority of papers that study estimation of R deal with continuous probability distributions. However, in some real-life situations stress or strength can have the discrete distribution. For example, this is the case when the stress is the number of the products that customers want to buy and the strength is the number of the products that factory produces. The number of cars that passes crossroads for a specified period of time has the Poisson distribution and

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the number of interviews an employer needs to conduct until he finds a suitable candidate for a vacant position follows the geometric distribution. These are also examples of the discrete stress.

Ahmad et al. (1) and Maiti (15) considered the case when X and Y were from the geometric distribution. The negative binomial distribution was studied by Ivshin and Lumelskii (9) and Sathe and Dixit (19). Recently, the case when X is from the geometric distribution, and Y is from the Poisson distribution was investigated by Obradovic et al. (17). Also Jovanovic (10) has study the estimation of $P\{X < Y\}$ for geometric-exponential model based on complete and censored samples.

In reliability and lifetime analysis, the bathtub shaped hazard rate function is widely used in many applications and lifetime analysis and Weibull distribution is of those distributions that has bathub shaped hazard rate function. In discrete lifetime distributions, Nakagawa and Osaki (16), Stein and Dattero (20), and Padgett and Spurrier (18) proposed three different discrete versions of the Weibull distribution which were further studied by Khan et al. (11) and Kulasekera (13).

The probability mass function (pmf) and cumulative distribution function (CDF) of type I discrete Weibull distribution (denoted with $DW(q,\beta)$) which was first introduced by Nakagawa and Osaki (16) are:

$$f(x) = q^{(x-1)^{\beta}} - q^{x^{\beta}} \quad x = 1, 2, \cdots; \quad 0 < q < 1, \beta > 0, \tag{A.1}$$

$$F(x) = 1 - q^{x^{\beta}}, \tag{A.2}$$

respectively.

B Point Estimators of R

Let X and Y be two independent random variable with CDF, $F(x)$ and $G(y)$ which denoting stress and strength respectively and $X \sim DW(q_1, \beta_1)$ and $Y \sim DW(q_2, \beta_2)$. Then, the reliability $R = P(X < Y)$ of the stress-strength model is given by

$$\begin{aligned} R = P(X < Y) &= \sum_{t=1}^{\infty}(1 - G(t))f(t) \\ &= \sum_{t=1}^{\infty} q_2^{t^{\beta_2}}(q_1^{(t-1)^{\beta_1}} - q_1^{t^{\beta_1}}) \\ &= \lim_{k\to\infty} \sum_{t=1}^{k} q_2^{t^{\beta_2}}(q_1^{(t-1)^{\beta_1}} - q_1^{t^{\beta_1}}). \end{aligned} \tag{B.1}$$

Reliability can be actually computed taking into account only its first terms. As an example, we compute the reliability R when $\beta_1 = \beta_2 = \beta$ and q_1, q_2, β and k take different values. Some parts of the results are shown in Table 1. As it is seen, the value of R is already stable at the 6th decimal digit when $\beta > 1$ and $k = 4$. So, for $\beta > 1$, we can use the closed approximation form of reliability parameter as follow,

$$R \approx \sum_{x=1}^{4} {q_2}^{x^{\beta_2}} ({q_1}^{(x-1)^{\beta_1}} - {q_1}^{x^{\beta_1}}). \tag{B.2}$$

Also, for estimating the parameters with method of moments (MM), we have

$$\begin{aligned} E(X) &= \sum_{x=1}^{\infty} q^{x^{\beta}}, \\ E(X^2) &= 2\sum_{x=1}^{\infty} x q^{x^{\beta}} + E(X). \end{aligned}$$

So, we have to equate the population moments to the corresponding sample moments and then solve the two equations simultaneously for q and β. But since $E(X)$ and $E(X^2)$ have not closed form the equations cannot be solved by ordinary techniques. Furthermore, for method of maximum likelihood (ML) the log-likelihood function is,

$$\log L = \sum_{i=1}^{n} \log\{q^{(x_i-1)^{\beta}} - q^{{x_i}^{\beta}}\}.$$

Equating the partial derivatives with respect to q and β to 0 yields equations which again cannot be solved easily. Barbiero (4) has presented an R software package named "*DiscreteWeibull*" which can estimate the parameters of discrete Weibull distribution with MM and ML methods.

Khan et al. (11) have proposed a simple method of estimating the parameters and call it the method of proportions. If y denotes the number of 1's in the sample of size n from discrete Weibull distribution, then using the fact that $f(1) = P(X = 1) = 1 - q$, we have

$$\tilde{q} = 1 - \frac{y}{n}, \tag{B.3}$$

and similary if z is the number of 2's in the sample, then by using $f(2) = P(X = 2) = q - q^{2^{\beta}}$ and (B.3), we have,

$$\tilde{\beta} = \log\{\log(1 - y/n - z/n)/\log(1 - y/n)\}/\log 2. \tag{B.4}$$

It is well-known that an empirical CDF is an unbiased and consistent estimator of the actual CDF. Here $\tilde{q} = 1 - \frac{y}{n}$ is an empirical estimate of $q = P(X > 1)$ and consistent estimator of q and we have similar results for $\tilde{\beta}$.

To compare the above estimates, we first generate two random samples of sizes n_1 and n_2 from discrete Weibull distribution, then based on (B.2), the MLE of R ($\hat{R}$) and the estimate of R based on (B.3) ($\tilde{R}$) are computed for different sample sizes. Table 2 shows the consequence. As the results show, the difference between two estimators are tolerable.

C Variance of the Estimators and Confidence Interval

Whereas the exact value of the variance or the mean square error of either estimator introduced, is almost impracticable to derive, an approximate value can be easily supplied recalling the delta method. Considering the use of the delta method, we have

$$Var(\tilde{R}) \approx \left[\partial\tilde{R}/\partial\tilde{q_1}, \partial\tilde{R}/\partial\tilde{q_2}\right] \begin{bmatrix} Var(\tilde{q_1}) & Cov(\tilde{q_1},\tilde{q_2}) \\ Cov(\tilde{q_1},\tilde{q_2}) & Var(\tilde{q_2}) \end{bmatrix} \begin{bmatrix} \partial\tilde{R}/\partial\tilde{q_1} \\ \partial\tilde{R}/\partial\tilde{q_2} \end{bmatrix},$$

where

$$\begin{aligned} Var(\tilde{q_1}) =& var(1 - y/n_1) \\ =& \tilde{q_1}(1-\tilde{q_1}), \\ Var(\tilde{q_2}) =& var(1 - y/n_2) \\ =& \tilde{q_2}(1-\tilde{q_2}). \end{aligned}$$

Since $\underline{X}$ and $\underline{Y}$ are independent $Cov(\tilde{q_1},\tilde{q_2}) = 0$, then,

$$Var(\tilde{R}) \approx \left(\partial\tilde{R}/\partial\tilde{q_1}\right)^2 Var(\tilde{q_1}) + \left(\partial\tilde{R}/\partial\tilde{q_2}\right)^2 Var(\tilde{q_2})$$

Once one has computed $Var(\tilde{R})$, an approximate $(1-\alpha)$ 100 % confidence interval for R can be built, recalling the asymptotic normality of R,

$$\left(\tilde{R} + z_{\alpha/2}\sqrt{Var(\tilde{R})}, \tilde{R} + z_{1-\alpha/2}\sqrt{Var(\tilde{R})}\right)$$

Since R is bounded in $[0, 1]$, the corrected lower and upper bounds are,

$$\left(\max\left(0, \tilde{R} + z_{\alpha/2}\sqrt{Var(\tilde{R})}\right), \min\left(1, \tilde{R} + z_{1-\alpha/2}\sqrt{Var(\tilde{R})}\right)\right)$$

D Simulation study

For each couple (q_1, q_2) and fixed $\beta = 2$, a huge number ($B = 1000$) of samples $\underline{X}$ of size n_1 and $\underline{Y}$ of size n_2 are drawn from $DW(q_1, 2)$ and $DW(q_2, 2)$ independently. Different and unequal sample sizes are here considered. The empirical estimators are computed on each sample, their approximate variances are calculated, and the corresponding 95 % confidence intervals for R are built. In more detail, the root mean square error (RMSE) and the percentage relative bias (RB) of the estimator are provided by,

$$RMSE(\tilde{R}) = \sqrt{\frac{1}{B}\sum_{s=1}^{B}(\tilde{R}(s) - R)^2},$$
$$RB(\tilde{R}) = \frac{\left((1/B)\sum_{s=1}^{B}\tilde{R}(s) - R\right)}{R} \cdot 100,$$

where $\tilde{R}(s)$ denotes the value of $\tilde{R}$ for the sth sample. Also, the approximation of the variance of the estimator is computed via,

$$\widehat{Var}(\tilde{R}) = E\left[\left(\tilde{R} - \bar{\tilde{R}}\right)^2\right],$$

where $\bar{\tilde{R}} = \sum_{s=1}^{B}\tilde{R}(s)/B$. Table 3 shows the true variance of $\tilde{R}$, $(Var(\tilde{R}))$ and its approxiamtion for different values of parameters and sample sizes.

The results show the accuracy of the estimators. Table 4 presented the root mean square error (RMSE) and the relative bias (RB) of the estimators of R and its variance.

Also, Figure 1 shows the behaviour of the RMSE based on the different values of R and sample sizes. According to the results with increasing sample size value of RMSE decreases.

The coverage rate of the confidence interval is simply defined as follows,

$$CR = \frac{1}{B}\sum_{s=1}^{B} I\left[\tilde{R}(s) + z_{\alpha/2}\sqrt{Var(\tilde{R}(s))} \leq R \leq \tilde{R}(s) + z_{1-\alpha/2}\sqrt{Var(\tilde{R}(s))}\right],$$

where $I(E)$ is the indicator function, taking value 1 if E is true and 0 otherwise. The length of the confidence interval is then equal to $2z_{1-\alpha/2}\sqrt{Var(\tilde{R}(s))}$.

Table 5 shows the coverage rate of the confidence interval for different values of parameters and sample sizes. It can be easily understood that increasing sample size cause increases real percentage. The simulation result are presented in the below table.

References

[1] Ahmad, K.E., Fakhry, M.E. and Jaheen, Z.F. (1995), Bayes estimation of $P(Y > X)$ in the geometric case, *Microelectronic Reliability*, **35**(5), 817-820.

[2] Bai, X., Shi, Y., Liu, Y. and Liu, B. (2018), Reliability estimation of multicomponent stress-strength model based on copula function under progressively hybrid censoring, *Journal of Computational and Applied Mathematics*, **344**, 100-114.

[3] Bai, X., Shi, Y., Liu, Y. and Liu, B. (2019), Reliability inference of stressstrength model for the truncated proportional hazard rate distribution under progressively Type-II censored samples, *Applied Mathematical Modelling*, **65**, 377-389.

[4] Barbiero, A. (2015). An *R* Package "DiscreteWeibull". https://CRAN.R-project.org/ package=DiscreteWeibull

[5] Cetinkaya, C. and Gen, A.I. (2019). Stressstrength reliability estimation under the standard two-sided power distribution, *Applied Mathematical Modelling*, **65**, 72-88.

[6] Chaudhary, S. and Tomer, S.K. (2018), Estimation of stressstrength reliability for Maxwell distribution under progressive Type-II censoring scheme, *International Journal of System Assurance Engineering and Management*, **11**, 1-13.

[7] Eryilmaz, S. (2018), Phase type stress=strength models with reliability applications, *Communications in Statistics-Simulation and Computation*, **47**(4), 954=963.

[8] Gnedenko, B.V. and Ushakov, I.A. (1995), *Probabilistic Reliability Engineering*, John Wiley and Sons, New York, NY, USA.

[9] Ivshin, V.V., Lumelskii, Ya. P. (1995), *Statistical Estimation Problems in StressStrength Models*, Perm, Russia: Perm University Press.

[10] Jovanovic, M. (2017), Estimation of $P\{X < Y\}$ for geometric-exponential model based on complete and censored samples, *Communications in Statistics - Simulation and Computation*, **46**(4), 3050-3066.

[11] Khan, M.S.A., Khalique, A. and Abouammoh, A.M. (1989), On estimating parameters in a discrete Weibull distribution, *IEEE Transactions on Reliability*, **38**, 348-350.

[12] Kotz, S., Lumelskii, L. and Pensky, M. (2003), *The Stress-strength Model and Its Generalizations: Theory and Applications*, World Scientific, New York, NY, USA.

[13] Kulasekera, K.B. (1994), Approximate MLEs of the parameters of a discrete Weibull distribution with type I censored data, *Microelectronics Reliability*, **34**, 1185-1188.

[14] Kzlaslan, F. (2017), Classical and Bayesian estimation of reliability in a multicomponent stress-strength model based on the proportional reversed hazard rate mode, *Mathematics and Computers in Simulation*, **136**, 36-62.

[15] Maiti, S.S. (1995), Estimation of $P(XY)$ in the geometric case, *Journal of Indian Statistical Associatin*, **33**, 87-91.

[16] Nakagawa, T. and Osaki, S. (1975), The discrete Weibull distribution, *IEEE Transactions on Reliability*, **24**, 300-301.

[17] Obradovic, M., Jovanovic, M., Miloevic, B., Jevremovic, V. (2015), Estimation of $P\{XY\}$ for Geometric-Poisson Model, *Hacettepe Journal of Mathematics and Statistics*, **44**(4), 949-964.

[18] Padgett, W.J. and Spurrier, J.D. (1985), Discrete failure models, *IEEE Transactions on Reliability*, **34**, 253-256.

[19] Sathe, Y.S. and Dixit, U.J. (2001), Estimation of $P(XY)$ in the negative binomial distributionm *Journal of Statistical Planning and Inference*, **93**, 83-92.

[20] Stein, W.E. and Dattero, R. (1984), A new discrete Weibull distribution, *IEEE Transactions on Reliability*, **33**, 196-197.

[21] Wang, B., Geng, Y. and Zhou, J. (2018), Inference for the generalized exponential stress-strength model, *Applied Mathematical Modelling*, **53**, 267-275.

[22] Yadav, A.S. and Singh, S.K.S.U. (2018), Estimation of stress-strength reliability for inverse Weibull distribution under progressive Type-II censoring scheme, *Journal of Industrial and Production Engineering*, **35**(1), 48-55 .

Table 1: The convergence of series (B.1) for different values of parameters and k

			R				
β	q_1	q_2	$k=2$	$k=3$	$k=4$	$k=20$	$k=50$
0.5	0.2	0.6	0.527254	0.5442277	0.5519919	0.562267	0.5623231
2	0.5	0.8	0.5792	0.5873265	0.587381	0.5873811	0.5873811
3	0.7	0.9	0.5465115	0.5498599	0.5498599	0.5498599	0.5498599
3	0.3	0.3	0.2100197	0.2100197	0.2100197	0.2100197	0.2100197
4	0.7	0.8	0.2596097	0.2596097	0.2596097	0.2596097	0.2596097

Table 2: Comparing the ML estimator ($\hat{R}$) and estimator based on method of proportions ($\tilde{R}$)

n_2	n_1	$\tilde{R}$	$\hat{R}$	$diff$
10	10	0.587381	0.5184652	0.06891
10	50	0.9987037	0.9215955	0.077
10	100	0.5198944	0.5137716	0.0061
10	1000	0.8037962	0.8055314	-0.0017
100	50	0.5458164	0.518862	0.027
100	100	0.6655324	0.6740383	-0.0085
100	1000	0.516161	0.5392271	-0.023

Table 3: The true variance of estimator of R ($Var(\tilde{R})$) and its bootstrap approximation ($\widehat{Var}(\tilde{R})$)

(n_1, n_2, q_1, q_2)	$Var(\tilde{R})$	$\widehat{Var}(\tilde{R})$
$(10, 10, 0.8, 0.6)$	0.02043607	0.02534769
$(10, 20, 0.8, 0.6)$	0.01305601	0.01381681
$(10, 50, 0.8, 0.6)$	0.009513626	0.01041398
$(20, 50, 0.6, 0.3)$	0.001997601	0.001921655
$(50, 50, 0.6, 0.3)$	0.001322801	0.001171524

Table 4: The root mean square error (RMSE) and the relative bias (RB) of the estimators of R and its variance

	$R = 0.8$	$R = 0.7$	$R = 0.6$	$R = 0.5$
	$(n_1, n_2) = (10, 10)$			
$RB(\tilde{R})$	1.395774	2.195337	7.773502	5.166667
$RB(V(\tilde{R}))$	-1.601882e-15	-7.10597e-15	-5.813333e-15	-7.220954e-15
$RMSE(\tilde{R})$	0.1717109	0.1894999	0.2125658	0.2343813
$RMSE(V(\tilde{R}))$	0.01931855	0.01756745	0.01618998	0.02455306
	$(n_1, n_2) = (10, 50)$			
$RB(\tilde{R})$	-1.211659	0.2836987	1.495738	-5.449599
$RB(v(\tilde{R}))$	1.219214e-14	-1.503061e-15	5.004339e-15	6.370609e-15
$RMSE(\tilde{R})$	0.09928318	0.1060792	0.1203305	0.1489042
$RMSE(v(\tilde{R}))$	0.005752886	0.006123595	0.004949186	0.008004615
	$(n_1, n_2) = (50, 50)$			
$RB(\tilde{R})$	-0.5644108	0.7567041	3.351207	-0.258137
$RB(v(\tilde{R}))$	2.921723e-15	8.362234e-16	-2.476251e-15	-3.423953e-15
$RMSE(\tilde{R})$	0.07974157	0.08697079	0.09353506	0.09945494
$RMSE(v(\tilde{R}))$	0.00172438	0.001253519	0.001019633	0.001823376

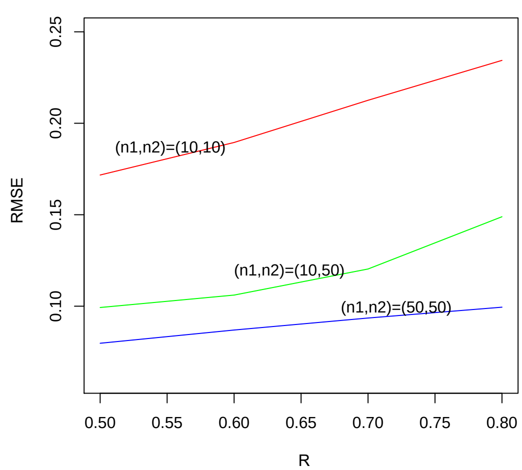

Figure 1: The RMSE of estimmator of R

Table 5: The coverage rate (CR) of confidence interval for R

(n_1, n_2, q_1, q_2)	CR
$(10, 10, 0.8, 0.6)$	0.795
$(10, 20, 0.8, 0.6)$	0.832
$(10, 50, 0.8, 0.6)$	0.872
$(20, 50, 0.6, 0.3)$	0.909
$(50, 50, 0.6, 0.3)$	0.924

A Confidence Interval for Stress-Strength Reliability in Gamma Model

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Abstract: In this paper we consider to construct confidence interval for the stress-strength reliability parameter under the gamma distribution. A generalized pivotal quantity is proposed for this parameter and a Monte Carlo simulation approach is given to obtain a generalized confidence interval. This approach is illustrated using a real data set.

Keywords Cornish-Fisher expansion, Gamma distribution, generalized confidence interval.

Mathematics Subject Classification (2010) : 62F40**,** 62F25**.**

A Introduction

For two independent random variables X and Y, the stress-strength reliability model is defined as $R = P(X > Y)$. In engineering, this model has very application and X is considered as strength of a structure and Y is considered as the stress imposed on it. The system fails if the stress exceeds the strength. In medicine, let X represents the response for a control group and Y represents the response for a treatment group. In biology, this probability is useful in estimating heritability of a generic trait. For other applications, see (10).

The term stress-strength was first introduced by (4) and various distributions of X and Y are considered in the literature, for example exponential, normal and Weibull distributions. When both X and Y have gamma distributions, (7) paid to estimate of the stress-strength reliability parameter. Bootstrap estimators and confidence intervals are proposed by (5; 6). When the shape parameters are known, the uniformly minimum variance unbiased estimator (UMVUE) of R is obtained by (9). Some normal-based approaches for inference on R is proposed by (12).

In this paper, using the concept of generalized pivotal quantity (GPQ) introduced by(14), we propose a generalized confidence interval (GCI) for the parameter R when both X and Y have gamma distributions, and the shape and scale parameters are unknown.

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The paper is organized as follows. In Section 2, some properties of gamma distribution are reviewed. In Section 3, a GCI is presented for the stress-strength reliability parameter in gamma distribution. In Section 4, the proposed approach is illustrated using a real example.

B Properties of gamma distribution

The gamma distribution with shape parameter α and scale parameter λ has the probability density function

$$f_X(x) = \frac{x^{\alpha-1}\lambda^{\alpha}e^{-\lambda\ x}}{\Gamma(\alpha)}, \quad x > 0,\ \alpha > 0,\ \lambda > 0,$$

where $\Gamma(.)$ is the gamma function. It is denoted by $Ga(\alpha,\ \lambda)$.

Let $X_1, \ldots, X_n$ be a random sample of size n from $Ga(\alpha,\ \lambda)$. The likelihood function can be written as

$$L(\alpha,\ \lambda) = \frac{\lambda^{n\alpha}}{[\Gamma(\alpha)]^n}\tilde{x}^{n(\alpha-1)}e^{-\lambda\ n\bar{x}},$$

where $\bar{x}$ and $\tilde{x}$ are the observed values of $\bar{X}$ and $\tilde{X}$, respectively, which denote the arithmetic and geometric means of the random samples, i.e. $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $\tilde{X} = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$.

When the both parameters are unknown, the maximum likelihood estimator (MLE) of λ is $\hat{\lambda} = \frac{\hat{\alpha}}{\bar{X}}$, but the MLE of α does not have closed form and can be obtained by using numerical methods (3). An approximation to $\hat{\alpha}$ is given by

$$\hat{\alpha} \approx \frac{3 + T + \sqrt{(T+2)^2 - 24T}}{-12T},$$

Where $T = \log(W)$ and $W = \frac{\tilde{X}}{\bar{X}}$ (11).

In the following Lemma some properties of statistics in gamma distribution. For more details see (1; 13).

Lemma1:

1. $\bar{X}$ and W are jointly sufficient and complete statistics for the vector parameters (α, λ).

2. The distribution of W does not depend on λ.

3. The statistics $\bar{X}$ and W are independent random variables.

4. $n\bar{X} \sim Ga(n\alpha, \lambda)$ and $2n\lambda\bar{X} \sim \chi^2_{(2n\alpha)}$.

5. the ith cumulant of T is given by

$$\kappa_1(\alpha) = \log(n) + \psi(\alpha) - \psi(n\alpha),$$
$$\kappa_i(\alpha) = \frac{1}{n^{i-1}}\psi^{(i-1)}(\alpha) - \psi^{(i-1)}(n\alpha), \qquad i = 2, 3, \ldots,$$

where $\psi(\alpha)$ is the digamma function and $\psi^{(k)}(\alpha)$ is the kth derivative of $\psi(\alpha)$.

C Inference on stress-strength reliability

Let X and Y are independent continuous random variables. Then, the stress-strength reliability R is calculated as

$$R = P(Y < X) = \int_0^{+\infty} F_Y(x) f_X(x)\, dx,$$

where f_X is probability density function (pdf) of X and F_Y is cumulative distribution function (cdf) of Y. Now, consider X and Y are the independent random variables such that $X \sim Ga(\alpha_1, \lambda_1)$ and $Y \sim Ga(\alpha_2, \lambda_2)$. Therefore, the parameter R can be expressed as

$$R = \frac{\lambda_1^{\alpha_1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{+\infty} \gamma(\alpha_2, \lambda_2 x)\; x^{\alpha_1 - 1} e^{-\lambda_1 x} dx := R(\alpha_1, \lambda_1, \alpha_2, \lambda_2), \tag{C.1}$$

where $\gamma(.,.)$ is lower incomplete gamma function.

In especial case, if $\alpha_2 = 1$, the stress-strength reliability parameter in (C.1) becomes to

$$R = \frac{\lambda_1^{\alpha_1}}{\Gamma(\alpha_1)} \int_0^{+\infty} \left[1 - e^{-\lambda_2 x}\right] x^{\alpha_1 - 1} e^{-\lambda_1 x} dx = 1 - \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{\alpha_1}.$$

If $\alpha_i = \frac{n_i}{2}$, $i = 1, 2$, where n_i's are integer values, then, the reliability parameter R is given by

$$R = 1 - F_{2\alpha_1, 2\alpha_2}\left(\alpha_2 \lambda_1 / (\alpha_1 \lambda_2)\right),$$

where $F_{2\alpha_1, 2\alpha_2}$ denotes the cdf of F distribution with $2\alpha_1$ and $2\alpha_2$ degrees of freedom. See (5). If α_1 is an integer, then R can also be expressed as

$$R = \sum_{j=0}^{\alpha_1 - 1} \frac{\Gamma(j + \alpha_2)}{\Gamma(j+1)\Gamma(\alpha_2)} \left(\frac{\rho}{1+\rho}\right)^{\alpha_2} \left(\frac{1}{1+\rho}\right)^{j},$$

where $\rho = \frac{\lambda_2}{\lambda_1}$ (9).

The concepts of GPQ and GCI are introduced by (14). These are explained as follow. For details see (15).

Let $\boldsymbol{X}$ be a random vector whose distribution depends on the vector of parameters $(\theta, \boldsymbol{\eta})$ where θ is the parameters of interest and $\boldsymbol{\eta}$ is the vector of nuisance parameters. Furthermore, let $\boldsymbol{x}$ be the observed value of $\boldsymbol{X}$.

Definition. A GPQ for θ, to be denoted by $T(\boldsymbol{X};\ \boldsymbol{x}; \theta, \eta)$, is a function of $\boldsymbol{X}$, $\boldsymbol{x}$, θ and $\boldsymbol{\eta}$, and satisfies the following conditions:

(i) The distribution of $T(\boldsymbol{X};\ \boldsymbol{x}; \theta, \boldsymbol{\eta})$ is free of all unknown parameters.

(ii) The observed value of $T(\boldsymbol{X};\ \boldsymbol{x}; \theta, \boldsymbol{\eta})$, i.e., t_{obs}=$T(\boldsymbol{x};\ \boldsymbol{x}; \theta, \boldsymbol{\eta})$ is free of the nuisance parameters $\boldsymbol{\eta}$.

If t_{obs} equals to the parameter of interest θ, then the GPQ is called the fiducial GPQ. In this case, a two-sided equally tailed $100(1-\gamma)\%$ GCI for θ is given by $(T_{\gamma/2}, T_{1-\gamma/2})$ where T_τ is τth percentile of the distribution of T. It is proved that the GCIs based fiducial GPQs have asymptotically correct frequentist coverage probability (8).

Using Cornish–Fisher expansion, (13) obtained the approximate GPQs for the parameters of gamma distribution. Consider

$$h(\alpha) = \kappa_1(\alpha) + [\kappa_2(\alpha)]^{1/2} Q(\alpha,\ U) - t,$$

where $U \sim U(0,1)$, $\kappa_i(\theta)$ is the ith cumulant of T, t is observed value of T and

$$\begin{aligned} Q(\alpha,\gamma) \quad = \quad & z_\gamma + \frac{1}{6}\kappa_3'(\alpha)\left(z_\gamma^2 - 1\right) + \frac{1}{24}\kappa_4'(\alpha)\left(z_\gamma^3 - 3z_\gamma\right) \\ & -\frac{1}{36}[\kappa_3'(\alpha)]^2\left(2z_\gamma^3 - 5z_\gamma\right) + \frac{1}{120}\kappa_5'(\alpha)\left(z_\gamma^4 - 6z_\gamma^2 + 3\right) \\ & -\frac{1}{24}\kappa_3'(\alpha)\kappa_4'(\alpha)\left(z_\gamma^4 - 5z_\gamma^2 + 2\right) \\ & +\frac{1}{324}[\kappa_3'(\alpha)]^3\left(12z_\gamma^4 - 53z_\gamma^2 + 17\right), \end{aligned}$$

where $\kappa_i'(\alpha) = \kappa_i(\alpha)/[\kappa_2(\alpha)]^{i/2}$, $i = 2, 3, 4, 5$, and z_γ is the γ quantile of the standard normal distribution $N(0, 1)$.

Let T_α be the solution of $h(\alpha) = 0$. When $n \geq 5$ and $z_\gamma \leq 4$, (13) showed that $h(\alpha)$ is a strictly increasing function of α. Therefore, the solution of $h(\alpha) = 0$ is unique and can be obtained by the bisection method. Therefore, T_α is an approximate GPQ for α. Define $T_\lambda = \frac{V}{2n\bar{x}}$ where $V \sim \chi^2_{(2nT_\alpha)}$, and $\bar{x}$ is observed value of $\bar{X}$. Therefore, based on Lemma 1, T_λ is GPQ for λ.

Suppose $X_1, \ldots, X_n$ is a random sample from the strength population and $Y, \ldots, Y_m$ is a random sample from the stress population such that $X_i \sim Ga(\alpha_1, \lambda_1)$ and $Y_j \sim Ga(\alpha_2, \lambda_2)$.

Consider

$$T_1 = \log\left(\frac{\tilde{X}}{\bar{X}}\right), \qquad T_2 = \log\left(\frac{\tilde{Y}}{\bar{Y}}\right),$$

where $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$, $\tilde{X} = \left(\prod_{I=1}^{n} X_i\right)^{1/n}$, $\bar{Y} = \frac{\sum_{j=1}^{n} Y_i}{m}$, $\tilde{Y} = \left(\prod_{j=1}^{m} X_j\right)^{1/m}$.

Also, consider that the observed values of $\bar{X}$, $\bar{Y}$, T_1, T_2 are $\bar{x}$, $\bar{y}$, t_1, t_2. A GPQ for the parameter stress-strength reliability can be obtained by using the GPQs for the parameters $\alpha_1, \lambda_1, \alpha_2, \lambda_2$ as

$$T_R = R(T_{\alpha_1},\ T_{\lambda_1}, T_{\alpha_2},\ T_{\lambda_2}),$$

where $T_{\alpha_1},\ T_{\lambda_1}, T_{\alpha_2},\ T_{\lambda_2}$ are GPQ for $\alpha_1, \lambda_1, \alpha_2, \lambda_2$. It can be used to construct a GCI for R. It can be obtained using Monte Carlo simulation based on the following algorithm:

Algorithm 1: For given $\bar{x}$, $\bar{y}$, t_1, t_2,

1. Generate U_j from $U(0,1)$, $j = 1, 2$.

2. Obtain the solution of $h(\alpha) = 0$, and call it T_{α_i} where

$$h_j(\alpha) = \ \kappa_1(\alpha\ ) + [\kappa_2(\alpha\ )]^{1/2}\, Q(\alpha,\ U_j) - t_j.$$

3. Generate V_1 from $\chi^2_{(2nT_{\alpha_1})}$ and V_2 from $\chi^2_{(2mT_{\alpha_2})}$.

4. Compute $T_{\lambda_1} = V_1/(2n\bar{x}\ )$ and $T_{\lambda_2} = V_2/(2m\bar{y}\ )$.

5. Compute $T_R = R(T_{\alpha_1},\ T_{\lambda_1}, T_{\alpha_2},\ T_{\lambda_2})$

6. Repeat Steps 1 and 5, M times. Then, there are M values of T_R.

7. A GCI for R is $(T_R^{\gamma/2}, T_R^{1-\gamma/2})$ where T_R^{γ} is the γth quantile of T_R.

D A real example

Here, we illustrate the proposed generalized approach for the stress-strength reliability using a real data set. It is given by (2) to study comparing lifetimes two kind of drills that a company uses in cutting machines. After a certain period of usage, it is of interest to know which brand is more reliable so that the factory can make the subsequent purchase decision. Table 1 presents the lifetimes of the drill of size 1.88 mm from the two suppliers.

The probability that the drill lifetime by the first supplier is larger than that by the second supplier can be computed by $R = P(X >\ Y)$. (2) showed that the gamma distribution can be

Table 1: Lifetimes (in Minutes) of 1.88-mm Drill from Two Suppliers

X	135 98 114 137 138 144 99 93 115 106 132 122 94 98 127 122
	102 133 114 120 93 126 119 104 119 114 125 107 98 117 111 106
	108 127 126 135 112 94 127 99 120 120 121 122 96 109 123 105
Y	105 105 95 87 112 80 95 97 77 103 78 87 107 96 79 91
	108 97 80 76 92 85 76 96 77 80 100 94 82 104 91 95
	93 99 99 94 84 99 91 85 86 79 89 89 100

good fitted for each these drill lifetimes. Here,

$$\bar{x} = 115.125, \quad \bar{y} = 91.422, \qquad t_1 = -0.0069, \qquad t_2 = -0.0055.$$

Based on Algorithm 1 by $M = 10000$, a 95% lower GCI for R is 0.855. Therefore, we can concluded that drills from the first supplier have a higher quality.

References

[1] Bain, L.J. and Engelhardt, M. (1987), *Introduction to Probability and Mathematical Statistics*, Brooks/Cole.

[2] Chen, P. and Ye, Z.S. (2017), Approximate statistical limits for a gamma distribution, *Journal of Quality Technology*, **49**(1), 64-77.

[3] Choi, S.C. and Wette, R. (1969), Maximum likelihood estimation of the parameters of the gamma distribution and their bias, *Technometrics*, **11**(4), 683-690.

[4] Church, J.D. and Harris, B. (1970), The estimation of reliability from stress-strength relationships, *Technometrics*, **12**(1), 49-54.

[5] Constantine, K., Karson, M. and Tse, S.K. (1989), Bootstrapping estimators of in the gamma case, *Journal of statistical computation and simulation*, **33**(4), 217-231.

[6] Constantine, K., Karson, M.J. and Tse, S.K. (1990), Confidence interval estimation of in the gamma case, *Communications in Statistics-Simulation and Computation*, **19**(1), 225-244.

[7] Constantine, K., Tse, S.K. and Karson, M. (1986), Estimation of in the gamma case, *Communications in Statistics-Simulation and Computation*, **15**(2), 365-388.

[8] Hannig, J., Iyer, H. and Patterson, P. (2006), Fiducial generalized confidence intervals, *Journal of the American Statistical Association*, **101** (473), 254-269.

[9] Huang, K., Mi, J. and Wang, Z. (2012), Inference about reliability parameter with gamma strength and stress, *Journal of Statistical Planning and Inference*, **142** (4), 848-854.

[10] Kotz, S. and Pensky, M. (2003), *The Stress-Strength Model and Its Generalizations: Theory and Applications*, World Scientific.

[11] Krishnamoorthy, K., Lee, M. and Xiao, W. (2015), Likelihood ratio tests for comparing several gamma distributions, *Environmetrics*, **26** (8), 571-583.

[12] Krishnamoorthy, K., Mathew, T. and Mukherjee, S. (2008), Normal-based methods for a gamma distribution: Prediction and tolerance intervals and stress-strength reliability, *Technometrics*, **50** (1), 69-78.

[13] Wang, B.X. and Wu, F. (2018), Inference on the gamma distribution, *Technometrics*, **60** (2), 235-244.

[14] Weerahandi, S. (1993), Generalized confidence intervals, *Journal of the American Statistical Association*, **88** (423), 899-905

[15] Weerahandi, S. (1995), *Exact Statistical Methods for Data Analysis*, Springer, New York, NY.

Allocating Two Redundancies in Series Systems with Dependent Component Lifetimes

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Abstract: This article deals with the problem of allocating redundancies for improving engineering system performances where component lifetimes are dependent and heterogeneous and an active policy is possible. Stochastic orders are implemented for comparing allocation policies. Allocations are derived under general conditions and hold for arbitrary dependency structures among lifetimes. Various examples are also given.

Keywords Dependence, Redundancy, Reliability, Stochastic orders.

Mathematics Subject Classification (2010) : 90B25, 60E15.

A Introduction

Spares are usually used to attain high reliable systems. Finding optimal allocations of spares is then essential in practice. Since system lifetimes are stochastic phenomena, partial (stochastic) orders are implemented for comparison purposes in literature; See, e.g. Boland et al. (4), Romera et al. (14), Valdes and Zequeira (17; 18), Belzunce et al. (2; 3), Jeddi and Doostparast (8), Zhang (19) and references therein. Boland et al. (4) considered two-component series and parallel systems with a spare and then compared system lifetimes under various policies for allocating the spare. They assumed that component lifetimes and spare lifetimes are statistically independent. The assumption of independence for components and spares are restrictive for practical purposes and occur rarely in engineering systems (Barlow and Proschan, 1975, P.29 (1)). Jeddi and Doostparast (8) relaxed the assumption of independence and provided optimal allocations for arbitrary dependence structure among component lifetimes; See also Kotz et al. (10), da Costa Bueno and Martins do Carmo (6), Belzunce et al. (2; 3) for more information. The redundancy allocation problem (RAP) may be extended for $k(\geq 2)$ spares. The RAP with $k = 2$ has been studied in the literature under the assumption of independence; See, e.g. Romera et al. (14), Valdes and

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Zequeira (17; 18), Hu and Wang (7), Brito et al. (5). This paper considers RAP with $k = 2$ in general and then removes the assumption of independence among component and spare lifetimes. Specifically, there is a n-component series system with lifetimes $X_1, \cdots, X_n$. Also, there exist two spares with lifetimes S_1 and S_2 and one wishes to find optimal design for allocating the spares to the original components. Following Boland et al. (4), two possible schemes are considered: (I) Allocating S_1 to X_1 and S_2 to X_2; (II) Allocating S_2 to X_1 and S_1 to X_2. If spares work jointly with original components, then the RAP is called "active policy". Therefore, for the n-component series system and under active policy, we face with two possible systems which should be compared. The system lifetimes are $T_{[1]}^{[AC]} = \wedge\{\vee\{X_1, S_1\}, \vee\{X_2, S_2\}, X_3, \cdots, X_n\}$ and $T_{[2]}^{[AC]} = \wedge\{\vee\{X_1, S_2\}, \vee\{X_2, S_1\}, X_3, \cdots, X_n\}$. Here, "$\wedge\{a_1, \cdots, a_m\}$" and "$\vee\{a_1, \cdots, a_m\}$" call for the minimum and the maximum of real numbers $a_i (i = 1, \cdots, m)$, respectively.

The rest of this article is organized as follows. In Section 2, we review some notions of stochastic orderings that will be used in sequel. Comparison of allocations of two active spars for series systems with dependent component lifetimes is considered in Section 3. Also in Section 3, the RAP is considered for two-component series in greater details. In Section 4, a guidance to obtain optimal allocation is given. Conclusions and further works are given in Section 5.

Notice that the active policy for a n-component parallel system with two spares under Schemes I and II are identical and hence this case is not discussed.

If spares wait to fail the respective original components and then joint to system, the RAP is called "standby policy". The standby policies are not studied in the paper.

B Stochastic orders

In this section, we review some (marginal and jointly) stochastic orders which are used in the next section. [Shaked and Shantikumar, (16)] Let $F_X(t) = P(X \le t)$ and $F_Y(t) = P(Y \le t)$, for $t > 0$, be distribution functions (DFs) of lifetimes X and Y, respectively. Then X is said to be smaller than Y in the "usual stochastic order", denoted by $X \le_{st} Y$, if $\overline{F}_X(t) \le \overline{F}_Y(t)$ for all $t > 0$, where $\overline{F}_X(t) = 1 - F_X(t)$ and $\overline{F}_Y(t) = 1 - F_Y(t)$ are the survival functions (SFs) of X and Y, respectively. Let X and Y be two random variables. X is smaller than Y in the "joint likelihood ratio order", denoted by $X \le_{lr:j} Y$, if

$$E(g(X, Y)) \le E(g(Y, X)), \tag{B.1}$$

for all $g \in G_{lr}$, where $G_{lr} = \{g \in D|g(y,x) \leq g(x,y),\ \forall\ y \leq x\}$ and D is the set of all functions from $\mathbb{R}^2$ to $\mathbb{R}$. If Inequality (B.1) holds for all $g \in G_{st} = \{g \in D|g(x,y) - g(y,x)$ is increasing in x, $\forall y \in \mathbb{R}\}$, then X is smaller than Y in "jointly stochastic order", denoted by $X \leq_{st:j} Y$. [Shaked and Shantikumar (16), p. 266] Let $\mathbf{U} = (U_1, \cdots, U_n)$ and $\mathbf{V} = (V_1, \cdots, V_n)$ be two random vectors. If $E[\phi(\mathbf{U})] \leq E[\phi(\mathbf{V})]$ for all increasing function ϕ from $\mathbb{R}^n$ to $\mathbb{R}$, then $\mathbf{U}$ is smaller than $\mathbf{V}$ in stochastic order and denoted by $\mathbf{U} \leq_{st} \mathbf{V}$. [Lehmann, (11)] The random variables X and Y are said "positive (negative) quadrant dependent" (PQD/NQD) if for every $(y_1, y_2) \in \mathbb{R}^2$,

$$F_{Y_1,Y_2}(y_1, y_2) \geq (\leq) F_{Y_1}(y_1) F_{Y_2}(y_2). \tag{B.2}$$

[Shaked and Shantikumar, (16), p. 388] Let (U_1, U_2) and (V_1, V_2) be two random vectors with the joint DFs $F_{U_1,U_2}(.,.)$ and $F_{V_1,V_2}(.,.)$, respectively. Furthermore, F_{U_1,U_2} and F_{V_1,V_2} have identical marginals. If

$$F_{U_1,U_2}(a,b) \leq F_{V_1,V_2}(a,b),\ \forall (a,b) \in \mathbb{R}^2, \tag{B.3}$$

then (U_1, U_2) is called smaller than (V_1, V_2) in the PQD order, and denoted by $(U_1, U_2) \leq_{PQD} (V_1, V_2)$. [Shaked and Shantikumar, (16), p. 392]

Let $\mathbf{U} = (U_1, \cdots, U_n)$ and $\mathbf{V} = (V_1, \cdots, V_n)$ be two random vectors with joint DFs F and G and joint SFs $\bar{F}$ and $\bar{G}$, respectively. Then, $\mathbf{U}$ is smaller than $\mathbf{V}$ in the PQD order if $F(\mathbf{x}) \leq G(\mathbf{x})$ and $\bar{F}(\mathbf{x}) \leq \bar{G}(\mathbf{x})$ for all $\mathbf{x} = (x_1, \cdots, x_n)$.

C Active-based RAPs for series systems

We study series systems including n dependent components under the active redundancies in Subsection 3.1. The 2-component series systems are investigated with a greater detail in Subsection 3.2.

C.1 n-component series system

Let X_i $(i = 1, 2, \cdots, n)$ and S_i $(i = 1, 2)$ denote the i-th component lifetime and the i-th spare lifetimes, respectively. The component lifetimes $X_1, \cdots, X_n$ and S_1, S_2 are also dependent. In sequel, "$U \overset{D}{\equiv} V$" means that the distribution functions of U and V are identical. Let $T^{AC}_{[1]} = \wedge\{\vee\{X_1, S_1\}, \vee\{X_2, S_2\}, X_3, \cdots, X_n\}$, $T^{AC}_{[2]} = \wedge\{\vee\{X_1, S_2\}, \vee\{X_2, S_1\}, X_3, \cdots,$

$X_n\}$. Then $T^{AC}_{[1]} \geq_{st} T^{AC}_{[2]}$, if and only if

$$P(B_{(-\{2,n+1\})}) + P(B_{(-\{1,n+2\})}) \geq P(B_{(-\{1,n+1\})}) + P(B_{(-\{2,n+2\})}), \tag{C.1}$$

where $B_{(-\{j,k\})} = \bigcap_{i=1, i\notin\{j,k\}}^{n+2} [X_i > a]$ for $1 \leq j \leq n$ and $1 \leq k \leq n$, by convention $X_{n+1} \overset{D}{\equiv} S_1$ and $X_{n+2} \overset{D}{\equiv} S_2$.

. For all $a > 0$, one can see that

$$\begin{aligned}
\overline{F}_{T^{AC}_{[1]}}(a) &= P(X_1 > a, \vee\{X_2, S_2\} > a, \cdots, X_n > a) \\
&+ P(X_1 < a, S_1 > a, \vee\{X_2, S_2\} > a, \cdots, X_n > a) \\
&= \overline{F}_{X_1,\cdots,X_n}(a,\cdots,a) + P(\{X_2 < a\} \cap B_{(-\{2,n+1\})}) + P(\{X_1 < a\} \cap B_{(-\{1,n+2\})}) \\
&+ P(\{X_1 < a\} \cap \{X_2 < a\} \cap B_{(-\{1,2\})}).
\end{aligned} \tag{C.2}$$

Similarly

$$\begin{aligned}
\overline{F}_{T^{AC}_{[2]}}(a) &= \overline{F}_{X_1,\cdots,X_n}(a,\cdots,a) + P(\{X_2 < a\} \cap B_{(-\{2,n+2\})}) + P(\{X_1 < a\} \cap B_{(-\{1,n+1\})}) \\
&+ P(\{X_1 < a\} \cap \{X_2 < a\} \cap B_{(-\{1,2\})}).
\end{aligned} \tag{C.3}$$

Then

$$\begin{aligned}
T^{AC}_{[1]} \geq_{st} T^{AC}_{[2]} &\iff P(\{X_2 < a\} \cap B_{(-\{2,n+1\})}) + P(\{X_1 < a\} \cap B_{(-\{1,n+2\})}) \\
&\geq P(\{X_2 < a\} \cap B_{(-\{2,n+2\})}) + P(\{X_1 < a\} \cap B_{(-\{1,n+1\})}) \\
&\iff P(B_{(-\{2,n+1\})}) - P(B_{(-\{n+1\})}) + P(B_{(-\{1,n+2\})}) - P(B_{(-\{n+2\})}) \\
&\geq P(B_{(-\{2,n+2\})}) - P(B_{(-\{n+2\})}) + P(B_{(-\{1,n+1\})}) - P(B_{(-\{n+1\})}).
\end{aligned} \tag{C.4}$$

So

$$\begin{aligned}
T^{AC}_{[1]} \geq_{st} T^{AC}_{[2]} &\iff P(B_{(-\{2,n+1\})}) + P(B_{(-\{1,n+2\})}) \geq P(B_{(-\{2,n+2\})}) + P(B_{(-\{1,n+1\})}) \\
&\iff P\Big(\bigcap_{i=1, i\notin\{2,n+1\}}^{n+2} [X_i > a]\Big) + P\Big(\bigcap_{i=1, i\notin\{1,n+2\}}^{n+2} [X_i > a]\Big) \\
&\geq P\Big(\bigcap_{i=1, i\notin\{2,n+2\}}^{n+2} [X_i > a]\Big) + P\Big(\bigcap_{i=1, i\notin\{1,n+1\}}^{n+2} [X_i > a]\Big),
\end{aligned}$$

as required. $\square$

In Theorem C.1, there exist a perspective of reliability engineering. The quantity $P(B_{-\{2,n+1\}})$ denotes the reliability function of the original series system in which X_2 is replaced by S_2. Denote this by $R_{S_2}^{[-2]}$. Similarly, let $R_{S_2}^{[-1]} = P(B_{-\{1,n+1\}})$, $R_{S_1}^{[-2]} = P(B_{-\{2,n+2\}})$ and $R_{S_1}^{[-1]} = P(B_{-\{1,n+2\}})$. Condition (C.1) is equivalent by

$$R_{S_2}^{[-2]} - R_{S_1}^{[-2]} \geq R_{S_2}^{[-1]} - R_{S_1}^{[-1]}. \tag{C.5}$$

Therefore, Inequality (C.5) holds if increment in original system reliability by replacing Component 2 with either S_2 or S_1 is greater than increment in original system reliability by replacing Component 1 with either S_2 or S_1. That is, $T_{[1]}^{AC} \geq_{st} T_{[2]}^{AC}$ if and only if change of system reliability caused by replacing spares S_1 and S_2 instead of Component 2 is greater than change of system reliability caused by replacing spares S_1 and S_2 instead of Component 1.

A special case of Theorem C.1 is given in the next corollary which extends result obtained by Romera et al. (14).

Suppose that (X_1, X_2), $(X_3, \cdots, X_n)$ and (S_1, S_2) are independent. Then, $T_1^{[AC]} \geq_{st} T_2^{[AC]}$ if $X_1 \geq_{st} X_2$ and $S_1 \leq_{st} S_2$ or $X_1 \leq_{st} X_2$ and $S_1 \geq_{st} S_2$. Notice that in Corollary C.1, $X_3, \cdots, X_n$ may be dependent. It is only necessary that the first two original components X_1 and X_2 be independent of the rest (original) components $X_3, \cdots, X_n$ as well as independent of the spars S_1 and S_2. Corollary C.1 says that the stronger spare should be allocated to the weaker component.

[Jeddi and Doostparast (9)] If $(X_1, X_3, \cdots, X_n | S_2 = a) \geq_{st} (X_2, X_3, \cdots, X_n | S_2 = a)$ and $(X_2, X_3, \cdots, X_n | S_1 = a) \geq_{st} (X_1, X_3, \cdots, X_n | S_1 = a)$ for all $a > 0$, then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$. [Jeddi and Doostparast (9)] If $(X_1, X_3, \cdots, X_n, S_2) \geq_{st} (X_2, X_3, \cdots, X_n, S_2)$ and $(X_2, X_3, \cdots, X_n, S_1) \geq_{st} (X_1, X_3, \cdots, X_n, S_1)$ then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$.

Now, we provide sufficient conditions for Theorem C.1 in terms of the quadratic dependence structure in the next proposition. The proof is immediately derived from Theorem C.1 and Definition B. $T_1^{[AC]} \geq_{st} T_2^{[AC]}$ provided that one of the following conditions holds:

(I) $(X_1, X_3, \cdots, X_n, S_2) \geq_{PQD} (X_2, X_3, \cdots, X_n, S_2)$ and
$(X_2, X_3, \cdots, X_n, S_1) \geq_{PQD} (X_1, X_3, \cdots, X_n, S_1)$;

(II) $(X_1, X_3, \cdots, X_n, S_2) \geq_{PQD} (X_1, X_3, \cdots, X_n, S_1)$ and
$(X_2, X_3, \cdots, X_n, S_1) \geq_{PQD} (X_2, X_3, \cdots, X_n, S_2)$.

As mentioned by Shaked and Shantikumar (16) (p. 392), the multivariate PQD order in Definition B implies the random vectors must have same univariate marginals. Therefore, in Proposition B, we require that $X_1 \overset{D}{\equiv} X_2$ in Case I and $S_1 \overset{D}{\equiv} S_2$ in Case II.

Assume that components and spares are homogeneous. Then Proposition B concludes that $T_1^{[AC]} \geq_{st} T_2^{[AC]}$ if strength of positive dependency in $(X_2, X_3, \cdots, X_n, S_2)$ is smaller than $(X_1, X_3, \cdots, X_n, S_2)$ and $(X_1, X_3, \cdots, X_n, S_1)$ is smaller than $(X_2, X_3, \cdots, X_n, S_1)$, then one should redundant the spare to the former in series systems. In summarize, one should allocate spares to components with weaker positive dependency. The assumptions of $X_1 \overset{D}{\equiv} X_2$ and $S_1 \overset{D}{\equiv} S_2$ in Remark B are not restrictive specially if the units are coming from the same production company/line. Albeit, there exist statistical tests for verifying this assumption on the basis of reliability component data sets. For more information, see Meeker and Escobar (12) and references therein. Meanwhile, there are systems in which component lifetimes do not follow the same marginal distributions.

C.2 Two-component series systems

Let $T_1^{[AC]} = \wedge\{\vee\{(X_1, S_1)\}, \vee(X_2, S_2)\}$ and $T_2^{[AC]} = \wedge\{\vee(X_1, S_2), \vee\{(X_2, S_1\}\}$. For $n = 2$, Theorem C.1 concludes:

$T_1^{[AC]} \geq_{st} T_2^{[AC]}$ if and only if

$$\bar{F}_{X_1,S_2}(a,a) - \bar{F}_{X_2,S_2}(a,a) \geq \bar{F}_{X_1,S_1}(a,a) - \bar{F}_{X_2,S_1}(a,a), \quad \forall a > 0. \tag{C.6}$$

Similarly to Section 3, Condition (C.6) is defined on the basis of the pairwise dependences in random vectors $(X_1, S_1), (X_1, S_2), (X_2, S_1)$ and (X_2, S_2) while the dependence structures in (X_1, X_2) and (S_1, S_2) play no role. Moreover, if $(X_1, S_2) \geq_{st} (X_2, S_2)$ and $(X_2, S_1) \geq_{st} (X_1, S_1)$ then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$.

Now for two-component series systems, we provide a sufficient condition for $T_1^{[AC]} \geq_{st} T_2^{[AC]}$ on the basis of the jointly likelihood order.

[Jeddi and Doostparast (9)] If $(X_1|S_2 = a) \geq_{lr:j} (X_2|S_2 = a)$ and $(X_2|S_1 = a) \geq_{lr:j} (X_1|S_1 = a)$ then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$.

Suppose that the pairs (X_1, S_2) and (X_2, S_1) are PQD and the pairs (X_1, S_1) and (X_2, S_2) are NQD. Then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$ provided that either $X_1 \overset{st}{=} X_2$ or $S_1 \overset{st}{=} S_2$. . The proof is immediately concluded by Proposition C.2 and Definition B. □

Let $(X_1, S_2) \geq_{PQD} (X_2, S_2)$ and $(X_2, S_1) \geq_{PQD} (X_1, S_1)$ or $(X_1, S_2) \geq_{PQD} (X_1, S_1)$ and $(X_2, S_1) \geq_{PQD} (X_2, S_2)$ then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$.

(FGM copula) The Farlie-Gumbel-Morgenstern (FGM) n-copulas is defined by

$$C(u_1, \cdots, u_n) = u_1 \cdots u_n \left(1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \theta_{j_1 \cdots j_k}(1-u_{j_1}) \cdots (1-u_{j_k})\right), \tag{C.7}$$

for $0 \leq u_i \leq 1$, $i = 1, \cdots, n$, and $-1 < \theta_{j_1 \cdots j_k} < 1$. Notice that $\theta_{j_1 \cdots j_k}$ are the parameters of the FGM copulas in Equation (C.7) to capture the dependency structures among the corresponding random variables. One can easily verify that these parameters are the correlation coefficients among the respective random variables. For a greater detail, see Nelsen (13). Now, assume a joint DF with the FGM 4-copulas for the lifetime vector (X_1, X_2, S_1, S_2). From Equations (C.6) and (C.7), $T_1^{[AC]} \geq_{st} T_2^{[AC]}$ if and only if

$$\begin{aligned}(\overline{F}_{X_1}(a) - \overline{F}_{X_2}(a))(\overline{F}_{S_2}(a) - \overline{F}_{S_1}(a)) \quad &+ \quad \overline{F}_{S_2}(a) F_{S_2}(a)\big(\theta_{14}\overline{F}_{X_1}(a)F_{X_1}(a) - \theta_{24}\overline{F}_{X_2}(a)F_{X_2}(a)\big) \\ &+ \quad \overline{F}_{S_1}(a) F_{S_1}(a)\big(\theta_{23}\overline{F}_{X_2}(a)F_{X_2}(a) - \theta_{13}\overline{F}_{X_1}(a)F_{X_1}(a)\big)\end{aligned} \tag{C.8}$$

for all $a > 0$. Note that Inequality (C.8) is free of θ_{12}, and θ_{34}, the dependency parameters within (X_1, X_2) and (S_1, S_2), respectively. From Equation (C.8), one can see that: (1) If $X_1 \overset{D}{\equiv} X_2$ or $S_1 \overset{D}{\equiv} S_2$ and $\theta_{14} \geq \theta_{24}$ and $\theta_{23} \geq \theta_{13}$ then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$. Note that this sufficient condition for the FGM distribution is also derived directly by Proposition B; (2) For $-1 < \theta_{12}, \theta_{34} < 1$, and $\theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = 0$, we have $X_1 \geq_{st} X_2$ and $S_2 \geq_{st} S_1$ if and only if $T_1 \geq_{st} T_2$. For $\theta_{12}, \theta_{34} = 0$ the claim of Romera et al. (14) is obtained. This result can also be proved by Corollary C.1; (3) If $X_1 \geq_{st} X_2$ and $S_2 \geq_{st} S_1$ and $\theta_{14}, \theta_{23} \geq 0$ and $\theta_{24}, \theta_{13} \leq 0$ then $T_1^{[AC]} \geq_{st} T_2^{[AC]}$. To see this, notice that the FGM copula in Equation (C.8) for $\theta_{ij} \in [0,1]$ is PQD and for $\theta_{ij} \in [-1,0]$ is NQD (Nelsen (13), p. 188). Then, Proposition C.2 implies $T_1^{[AC]} \geq_{st} T_2^{[AC]}$. □

D A guidance to obtain optimal allocation

In practice, one should consider all $\binom{n}{2}$ possible combinations and apply Theorem C.1 to derive the optimal allocation. To see this, the next example is given. Let $X_i \sim Exp(i), 1 \leq i \leq 3$ and $S_j \sim Exp(1/j), j = 1, 2$, where $Exp(\theta)$ stands for the exponential distribution with mean $\theta \geq 0$. Suppose that the lifetime vectors (X_1, X_2, X_3) and (S_1, S_2) are independent. There exist

$\binom{3}{2} = 6$ possible schemes to allocate S_1 and S_2 as follow:

$$\begin{aligned}
T_1^{[AC]} &= \wedge\{\vee\{(X_1, S_1)\}, \vee(X_2, S_2), X_3\}, \\
T_2^{[AC]} &= \wedge\{\vee\{(X_1, S_2)\}, \vee(X_2, S_1), X_3\}, \\
T_3^{[AC]} &= \wedge\{\vee\{(X_1, S_1)\}, X_2, \vee(X_3, S_2)\}, \\
T_4^{[AC]} &= \wedge\{\vee\{(X_1, S_2)\}, X_2, \vee(X_2, S_1)\}, \\
T_5^{[AC]} &= \wedge\{X_1, \vee\{(X_2, S_1)\}, \vee(X_3, S_2)\}, \\
T_6^{[AC]} &= \wedge\{X_1, \vee\{(X_2, S_2)\}, \vee(X_3, S_1)\}.
\end{aligned}$$

By Corollary C.1, we see that $T_1^{[AC]} \geq_{st} T_2^{[AC]}$, $T_3^{[AC]} \geq_{st} T_4^{[AC]}$ and $T_5^{[AC]} \geq_{st} T_6^{[AC]}$. Moreover, Theorem 3.4 in Jeddi and Doostparast (8) implies that $T_1^{[AC]} \geq_{st} T_3^{[AC]} \geq T_5^{[AC]}$. Hence the optimal allocation assigns S_1 to X_1 and S_2 to X_2. □ Suppose that there exist m spares. For implementing the above-mentioned guideline, one should first select two spares among $\binom{m}{2}$ possible ways, and then find the best optimal allocation. Now, the new system consists $n+2$ components. In fact, it includes n components in which two of them have been strengthened by two selected spares. Hence, we can assume that the new system is still a series system with size n. Secondly, she/he must select the next two spares from $\binom{m-2}{2}$ possible ways and proceed similarly until all the spares been allocated. For illustration purpose, let $n = 3$ and $m = 4$. Denote by X_i, $1 \leq i \leq 3$ and S_j, $1 \leq j \leq 2$, the original and spare component, respectively. Therefore, there exist $\binom{4}{2} = 6$ ways to select two spares. Suppose that we selected S_2 and S_4. Thus there are $\binom{3}{2} = 3$ possible allocations as follows:

$$\begin{aligned}
T_1^{[AC]} &= \wedge\{\vee\{(X_1, S_2)\}, \vee(X_2, S_4), X_3\}, \\
T_2^{[AC]} &= \wedge\{\vee\{(X_1, S_4)\}, \vee(X_2, S_2), X_3\}, \\
T_3^{[AC]} &= \wedge\{\vee\{(X_1, S_2)\}, X_2, \vee(X_3, S_4)\}, \\
T_4^{[AC]} &= \wedge\{\vee\{(X_1, S_4)\}, X_2, \vee(X_2, S_2)\}, \\
T_5^{[AC]} &= \wedge\{X_1, \vee\{(X_2, S_2)\}, \vee(X_3, S_4)\}, \\
T_6^{[AC]} &= \wedge\{X_1, \vee\{(X_2, S_4)\}, \vee(X_3, S_2)\}. \qquad \text{(D.1)}
\end{aligned}$$

Similarly, to Example D, we can compare the pair lifetimes $(T_1, T_2), (T_3, T_4)$ and (T_5, T_6) by Theorem C.1. For example assume that $X_1 \leq_{st} X_2 \leq_{st} X_3$ and $S_2 \geq_{st} S_4$. Then, $T_1 \geq_{st} T_2, T_3 \geq_{st} T_4$ and $T_5 \geq_{st} T_6$. Moreover by Theorem 3.4 in Jeddi and Doostparast (8), $T_3 \geq_{st} T_5$ since $X_1 \leq_{st} X_2$. Similarly $T_1 \geq_{st} T_3$ since $X_2 \leq_{st} X_3$. In summarize, T_1 is the best allocation based on two spares S_2 and S_4. Now, let $X_1^* = \vee(X_1, S_2)$ and $X_2^* = \vee(X_2, S_4)$ and $X_3^* = X_3$. Then, we have a series system of size 3 with component lifetimes X_1^*, X_2^* and X_3^*, the system lifetime is $T^* = \wedge(X_1^*, X_2^*, X_3^*)$. For allocating S_1 and S_3, similarly there exist possible ways as given by (D.1), replacing X_i by X_i^*, $(1 \leq 3)$ and (S_1, S_3) by (S_2, S_4). Now, one must verify conditions (C.1) in order to find the optimal configuration for allocating S_1 and S_3. To end this, we usually need to specify the SFs of the component and spare lifetimes. Let m be odd. Then, there is only one spare for allocating to the recently improved system, in the last step. Therefore, we have a series system with dependent components and an additional spare which must be allocated to the system. Hence one may use Theorem 3.4 in Jeddi and Doostparast (8).

E Conclusions

This article considered RAP with two spares for series engineering systems when component lifetimes are dependent. The findings do not rely on any specific form for structural dependency among component lifetimes. This paper is based on the first author's PhD thesis and an extended version of this paper is presented in Jeddi and Doostparast (9).

References

[1] Barlow, R.E. and Proschan, F. (1975), *Statistical theory of reliability and life testing*, Holt, Rinehart and Winston, Inc., New York.

[2] Belzunce, F., Martinez-Puertas, H. and Ruiz, J.M. (2011), On optimal allocation of redundant components for series and parallel systems of two dependent components, *Journal of Statistical Planning and Inference,* **141**, 3094-3104.

[3] Belzunce, F., Martinez-Puertas, H. and Ruiz, J.M. (2013), On allocation of redundant components for systems with dependent components, *European Journal of Operational Research,* **230**, 573-580.

[4] Boland, P.J., EI-Neweihi, E. and Proschan, F. (1992), Stochastic order for redundancy allocations in series and parallel systems, *Advances in Applied Probability,* **24**, 161-171.

[5] Brito, G, Valdes, J.E. and Zequeira, R.I. (2011), On the hazard rate and reversed hazard

rate orderings in two-component series systems with active redundancies, *Statistics and Probability Letters,* **81**, 201-206.

[6] da Costa Bueno, V., Martins do Carmo, I. (2007). Active redundancy allocation for a k-out-of-n:F system of dependent components. *European Journal of Operational Research,* **176**, 1041-1051.

[7] Hu, T. and Wang, Y. (2009), Optimal allocation of active redundancies in r-out-of-n systems, *Journal of Statistical Planning and Inference,* **139**, 3733-3737.

[8] Jeddi, H. and Doostparast, M. (2016), Optimal redundancy allocation problems in engineering systems with dependent component lifetime, *Applied Stochastic Models in Business and industry,* **32**, 199-208.

[9] Jeddi, H. and Doostparast, M. (2019), Allocation of redundancies in systems: A general dependency-base framework, *submitted*.

[10] Kotz, S., Lai, C.D. and Xie, M. (2003), On the effect of redundancy for systems with dependent components, *IIE Transactions,* **35**, 1103-1110.

[11] Lehmann, E.L. (1966), Some concepts of dependence, *The Annals of Mathematical Statistics,* **37**, 1137-1153.

[12] Meeker, W.Q. and Escobar, L.A. (1998), *Statistical Methods for Reliability Data*, John Wiley & Sons, New York.

[13] Nelsen, R.B. (2006), *An introduction to copulas*, Springer, New York.

[14] Romera, R., Valdes, J.E. and Zequeira, R.I. (2004), Active-redundancy allocation in systems, *IEEE Transactions on Reliability,* **53**, 313-318.

[15] Shanthikumar, J.G. and Yao, D.D. (1991), Bivariate Characterization of some stochastic order relations, *Advances in Applied Probability,* **23**, 642-659.

[16] Shaked, M. and Shanthikumar, J.G. (2007), *Stochastic Orders*, Springer-Verlag, New York.

[17] Valdes, J.E. and Zequeira, R.I. (2003), On the optimal allocation of an active redundancy in a two-component series system, *Statistics and Probability Letters,* **63**, 325-332.

[18] Valdes, J.E. and Zequeira, R.I. (2006), On the optimal allocation of two active redundancies in a two-component series system, *Operations Research Letters,* **34**, 49-52.

[19] Zhang. Y. (2018), Optimal allocation of active redundancies in weighted k-out-of-n systems, *Statistics and Probability Letters,* **135**, 110-117.

Bayesian Conditional Estimation of Weibull Distributions Under Type-II Censored Order Statistics

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Abstract: Two methods of estimation of parameters in Weibull distribution under type-II censored order statistics have been considered. It is because of the complex behavior in the calculation of the likelihood function of the presented scheme in this situation without loss of generality, this problem fixed with Gumbel model. The one to one transformation between these models and its satisfying in their parameters able us for the use of this alternative model. Moreover, some statistical inferences of a new strategy of estimating based on The Bayesian conditional method have provided and the numerical results of these strategies are compared.

Keywords Bayes, Conditional, Gumbel, Maximum Likelihood Estimators, Weibull.

Mathematics Subject Classification (2010) : 62E17, 62H12**.**

A Introduction

Weibull distribution is commonly used in reliability studies. It is used for modeling observed failures of many different types of components and phenomena. In some situations, statistical inference about its parameters in quality control and some engineering problems is of our interest. Depending on a kind of sampling plan, and based on maximum likelihood estimation, the performance of estimating in this model can differ. One of the most applicable sampling strategies in engineering problems is type-II censored order statistics scheme. For a comprehensive, perfect and complete discussion on censored sampling scheme see (2). The main problem of estimating parameters under this scheme is a major fault occurrence, especially for large scale parameters in the confrontation with Weibull models. A routine solution of this end is utilizing Gumbel or extreme value distribution, which have the capacity of one to one transformation between corresponding random variables and related parameters. Yet, the problem of bad performance of maximum likelihood estimators (MLEs) is still holding on but a little better than

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previous estimating.

However, in the situations of having a location scale family of distributions for sampling, some authors proposed a conditional method of estimating. This method was introduced by (10), and applied in some inferences problem by (11) and (9). A generalize conditional strategy of estimating in location scale family of distribution done by (16). In continuing, the conditional Bayes estimators have been discussed in some quality control disciplines by (5). In addition, in this concept the conditional strategy has been used alone in (7) and (6).

The combined of Bayes and conditional method have used effectively in (5) for quality control disciplines. In the Bayes method, choosing a good prior distribution is so important but the presented method has focused on improper Jeffreys prior or proportion to this prior. The extension of this prior has been considered in this study and the best of these priors has been identified.

For text shortening and page boundaries, only formulas and relationships are presented in the Gumbel distribution. The derivation of the results for the Weibull distribution is accomplished with a simple transformation of e^x, and conversions are kept one by one in the parameters (see (5)). Therefore, without diminishing loss of generality, in section 2, the required relationships in the Gumbel distribution are shown, and in Section 3, the numerical comparison of the two methods presented in deducing the parameters of the location and the scale for $\mu = -5, 0, 5, 10$ and $\sigma = 1, 5, 10$.

B Notations and Models

Suppose that X_1, X_2, …, X_n are some random variables following to the Gumbel distribution with parameters μ and σ. The probability density function of this distribution is

$$f_{X_i}(x) = \frac{1}{\sigma} e^{\frac{x-\mu}{\sigma}} e^{-e^{\frac{x-\mu}{\sigma}}}, \; i = 1, 2, \ldots, n, \; (x, \mu) \in \mathbb{R}, \; \sigma \in \mathbb{R}^+. \tag{B.1}$$

For simplicity the notation of these representation is considered as $X \sim G(\mu, \sigma)$.

The joint probability density function of first r ordinary order statistics $X_{1:n}$, $X_{2:n}$, …, $X_{r:n}$ arising from independent and identical random variables X_1, X_2, …, X_n is (see (1))

$$f_{X_{1:n}, X_{2:n}, \ldots, X_{r:n}}(x_1, x_2, \ldots, x_r) = \frac{1}{\sigma^r} exp(\sum_{j=1}^{r} \frac{x_j - \mu}{\sigma}) exp(-((n-r) e^{\frac{x_r - \mu}{\sigma}} + \sum_{j=1}^{r} e^{\frac{x_j - \mu}{\sigma}})). \tag{B.2}$$

It is noteworthy that this sampling method can be considered as a special case of progressively type-II censored order statistics which named as type-II censored order statistics sampling scheme. However, the maximum likelihood estimators (MLEs) based on maximizing likelihood function B.2, called by $\hat{\mu}$ and $\hat{\sigma}$ respectively referred as MLE of μ and σ. In continuing another method of estimating these parameters according to the Bayesian conditional strategy are introduced. The performance of this manner of estimating is constructed as follows

Let $\pi(\mu,\sigma) = \dfrac{1}{\sigma^m},\ m \in \mathbb{R}^+$ denote the improper prior distribution for two location and scale parameters and immediately from B.2, the posterior distribution $\pi(\mu,\sigma \mid \underline{x})$ satisfies

$$\pi(\mu,\sigma \mid \underline{x}) = \frac{1}{\sigma^{r+m}} exp(\sum_{j=1}^{r} \frac{x_j-\mu}{\sigma}) exp(-((n-r)e^{\frac{x_r-\mu}{\sigma}} + \sum_{j=1}^{r} e^{\frac{x_j-\mu}{\sigma}}))$$

Now, consider two ancilary statistics $Z_1 = \dfrac{\hat{\mu}-\mu}{\hat{\sigma}}$ and $Z_2 = \dfrac{\hat{\sigma}}{\sigma}$ and moreover pivotal quantity $a_i = \dfrac{x_i-\hat{\mu}}{\hat{\sigma}},\ i = 1,2,\ldots,r$. According to these notations

$$f_{(Z_1,Z_2)|(a_1,a_2,\ldots,a_r)}(z_1,z_2) \propto z_2^{r+m-2} exp(\sum_{j=1}^{r}(z_1+a_j)z_2) exp(-((n-r)e^{(z_1+a_r)z_2} + \sum_{j=1}^{r} e^{(z_1+a_j)z_2}))$$

or equivalently

$$f_{(Z_1,Z_2)|(a_1,a_2,\ldots,a_r)}(z_1,z_2) = C z_2^{r+m-2} exp(\sum_{j=1}^{r}(z_1+a_j)z_2) exp(-((n-r)e^{(z_1+a_r)z_2} + \sum_{j=1}^{r} e^{(z_1+a_j)z_2}))$$

where C is normalizing constant such that

$$\frac{1}{C} = \int_{\mathbb{R}} \int_{\mathbb{R}+} z_2^{r+m-2} exp(\sum_{j=1}^{r}(z_1+a_j)z_2) exp(-((n-r)e^{(z_1+a_r)z_2} + \sum_{j=1}^{r} e^{(z_1+a_j)z_2})) dz_2 dz_1$$

In order to calculate Bayesian estiomator of Z_1 and Z_1, the full conditional probability density function of these parameters are be needed which be derived in the following

$$f_{Z_1}(z_1) \propto e^{rz_1z_2 - e^{z_1z_2}[(n-r)e^{a_rz_2} + \sum_{j=1}^{r} e^{a_jz_2}]},\ z_1 \in \mathbb{R}$$

and

$$f_{Z_2}(z_2) \propto z_2^{r+m-2} e^{z_2(\sum_{j=1}^{r}(z_1+a_j)) - [(n-r)e^{z_2[z_1+a_r]} - \sum_{j=1}^{r} e^{z_2[z_1+a_j]}]},\ z_2 \in \mathbb{R}^+$$

It is clearly that finding a closed normalizing constant for these distribution do not be manipualting esay and we shoud apply the Gibbs sampling method for generating random variables from these two probability density functions. Moreover, it is because of Z_1 and Z_2 are the linear combination of parameters μ and σ, the Bayes estimator of these new parameters can keep their Bayesian features under the first and last of these loss functions. However, under the second loss function, some challenges have appeared. It is clearly understood with some slight mathematical calculations that the Bayes estimator of a linear combination $\delta_{2B}^*(a\theta+b)$ under the LinEx loss function is $a\delta_{2B}^*(a\theta)+b$ (see (12)).

Finally, it is worth to mention that the similar distribution of Z_1 is

$$u(x) = \frac{\beta\lambda^{\frac{\alpha}{\beta}}}{\Gamma(\frac{\alpha}{\beta})} e^{\alpha x - \lambda e^{\beta x}},\ x \in \mathbb{R},\ (\alpha, \beta, \lambda) \in \mathbb{R}^+.$$

where it's cumulative distribution function is

$$U(x) = \int_0^{e^{\beta x}} \frac{\beta\lambda^{\frac{\alpha}{\beta}}}{\Gamma(\frac{\alpha}{\beta})} t^{\frac{\alpha}{\beta}-1} e^{-\lambda t} dt \tag{B.3}$$

Hence, the generating random samples from $f_{Z_1}(z_1)$ can be easily done but for $f_{Z_2}(z_2)$ some complicated problems has been appeared. For calculation conditional Bayes estimator of Z_1 and Z_2 , three loss functions have considered and these estimators have compared with their MLE's.

- Absolute Error Loss Functions (AEL)

$$L(\delta, \theta) = \mid \delta - \theta \mid$$

$$\delta^*_{1B}(\theta) = median\ of\ \theta\ in\ posterior\ density\ function$$

- Linear Exponential Loss Functions (LinEx)

$$L(\delta, \theta) = e^{c(\delta-\theta)} - c(\delta - \theta) - 1,\ c \in \mathbb{R}^+$$

$$\delta^*_{2B}(\theta) = \frac{-log(E[e^{-c\theta}])}{c}$$

  In present study $c = 1$ has been considered.

- Squared Error Loss Functions (SE)

$$L(\delta, \theta) = (\delta - \theta)^2$$

$$\delta^*_{3B}(\theta) = mean\ of\ \theta\ in\ posterior\ density\ function$$

In each case, $\delta^*_B(\theta)$, represent the corresponding bayesian estimator to each of loss functions (see (12)).

C Numerical comparison of two presented methods

In this section, we provide some comparison results for the performance of two presented methods of estimating parameters in Gumbel distribution. The MLE estimators of these parameters can be easily derived by maximizing the likelihood function B.2, but the calculation of Bayes estimators have some conflicts.

Metropolis et al. (13), introduced the Metropolis-Hastings (M-H) algorithmic rule in program as a general Mont Carlo Markov Chain (MCMC) technique and afterward Hastings (8), expanded the M-H algorithm. One can apply the M-H algorithm to get random sample from any subjectively complicated target distribution of any dimension that is known up to a normalizing constant. Gibbs sampling method is a particular instance for the MCMC method. It can be utilized to generate a sample from the full conditional probability distributions of two or more random variables. Gibbs sampling requires decomposing the joint posterior distribution into full conditional distributions for each parameter and then sampling from them. We propose using the Gibbs sampling plan to generate a sample from the posterior density functions $f_{Z_1}(z_1)$ and $f_{Z_2}(z_2)$ in turn compute the Bayesian estimates under given loss functions (see (14) and (15)).

Generating random samples such that distributed as similar as Z_1 is expressed previously. For such a same task in Z_2 we propose the following steps.

I: fix values $n = 15,\ r = 8,\ R_1 = 0, R_1 = R_2 = \cdots = R_7 = 0,\ \text{ and } R_8 = 7.$

II: Utilize the given algorithm in (3), to generating type-II censored order statistics arising from independent and identical Gumbel random variables under the scheme which introduced in previous step.

III: Calculate MLEs of parameters μ and σ based on samples that generated in previous step, say $\hat{\mu}$ and $\hat{\sigma}$ respectively.

IV: Construct $a_i = \dfrac{x_i - \hat{\mu}}{\hat{\sigma}},\ i = 1, 2, \ldots, 8.$

V: Assume that a fix value of z_2 say $z_2 = 1$.

VI: Generate a random sample z_1 based on a cdf B.3, with parameters $\alpha = 8z_2$, $\beta = z_2$ and $\lambda = 7e^{a_8 z_2} + \sum_{j=1}^{8} e^{a_j z_2}$.

VI: Generate a new random sample t based on a cdf B.3, with parameters $\alpha = \sum_{j=1}^{8}(z_1 + a_j)$, $\beta = z_1 + a_8$ and $\lambda = 7$.

VII: For $M = 20000$ and $N = 100000$ calculate $CC = \dfrac{\sum_{j=20001}^{100000} t^{m+6} e^{-\sum_{j=1}^{8} e^{t(z_1+a_j)}}}{N - M}$. Hence,

$$f_{z_2}(z_2) = \frac{1}{CC} z_2^{r+m-2} e^{z_2(\sum_{j=1}^{r}(z_1+a_j)) - [(n-r)e^{z_2[z_1+a_r]} - \sum_{j=1}^{r} e^{z_2[z_1+a_j]}]},\ z_2 \in \mathbb{R}^+$$

.

VIII: It is clearly that z_2 belong to the log concave family of distribution. Therefore, using the given algorithm of this family of distributions in (4), and generate one random variable from this pdf.

IX: Call the generated random variable z_{1s} and z_{2s}. Moreover repeat this step for $s = 1, 2, \ldots, 100000$ and afterward delete the first 20000th and construct new vectors z_{1s} and z_{2s} with length 80000. It is straightforeward to see that $\delta^*_{1B}(z_i) = \dfrac{z_{(i(60000))} + z_{(i(60001))}}{2}$, $\delta^*_{2B}(z_i) = -log[\sum_{s=20001}^{100000} \dfrac{e^{-z_{is}}}{80000}]$, and $\delta^*_{3B}(z_i) = \sum_{s=20001}^{100000} \dfrac{z_{is}}{80000}$, $i = 1, 2$

Now, based on these samples calculate the bayes estimators of Z_1 and Z_2, and sequently calculate the bayes estimators of μ and σ. The number of repeated in each tables is 1000000.

True Values of Parameters	Bias	MSE	MAE
$\mu = -5$	0.68680050	0.63839000	0.70501050
$\sigma = 1$	-0.04166361	0.21870780	0.38150950
$\mu = -5$	-0.93424835	4.32319800	1.62188400
$\sigma = 5$	0.05166332	4.49697100	1.62527700
$\mu = -5$	-2.88157680	22.1155700	3.71012200
$\sigma = 10$	0.10755720	18.0756800	3.25669600
$\mu = 0$	0.68716310	0.63873880	0.70513560
$\sigma = 1$	-0.04246840	0.21780420	0.38138020
$\mu = 0$	-0.93734297	4.32370200	1.62178800
$\sigma = 5$	0.05174331	4.50535600	1.62830300
$\mu = 0$	-2.87358300	22.0749800	3.70312600
$\sigma = 10$	0.10011000	17.9499900	3.24967700
$\mu = 5$	0.68562740	0.63654430	0.70387780
$\sigma = 1$	-0.04257320	0.21866210	0.38165130
$\mu = 5$	-0.93823280	4.33715900	1.62496700
$\sigma = 5$	0.05137040	4.52099100	1.62886800
$\mu = 5$	-2.87603700	22.0890500	3.70735800
$\sigma = 10$	0.10698600	18.0351600	3.25654700
$\mu = 10$	0.65110020	0.56575350	0.67037940
$\sigma = 1$	1.13773740	1.44024000	1.13773740
$\mu = 10$	-0.93614990	4.32951100	1.62322800
$\sigma = 5$	1.05268570	5.60223000	1.71455800
$\mu = 10$	-2.87460300	22.0702600	3.70402900
$\sigma = 10$	1.11072600	19.2965000	3.24181500

Conditional Bayes Estimators True Values of Parameters	Bias	MSE	MAE
$\mu = -5$	0.8722585	0.791762	0.8722585
$\sigma = 1$	-0.0479138	0.04196371	0.1637934
$\mu = -5$	-0.89537	0.8310668	0.89537
$\sigma = 5$	-0.04946586	0.04220974	0.1640008
$\mu = -5$	-1.176859	1.385527	1.176859
$\sigma = 10$	0.09995236	0.04999095	0.1791974
$\mu = 0$	0.6835907	0.5058638	.6836182
$\sigma = 1$	-0.04048524	0.04148003	0.1622532
$\mu = 0$	-0.9004636	0.839695	0.9004636
$\sigma = 5$	0.04918261	0.04233654	0.1642744
$\mu = 0$	-1.17571	1.382866	1.17571
$\sigma = 10$	0.09990332	0.04973746	0.1786718
$\mu = 5$	0.6806113	0.5016124	0.6806389
$\sigma = 1$	-0.04001441	0.04170052	0.163145
$\mu = 5$	-0.900369	0.8393726	0.900369
$\sigma = 5$	0.05015226	0.04250714	0.1644665
$\mu = 5$	-1.176583	1.384879	1.176583
$\sigma = 10$	0.01089497	0.4598741	0.5843977
$\mu = 10$	0.0729195	0.4612334	0.5853499
$\sigma = 1$	0.125118	0.4633722	0.5870599
$\mu = 10$	-0.1040864	0.4626862	0.586616
$\sigma = 5$	0.1218691	0.4643177	0.5874308
$\mu = 10$	-0.3145018	0.4995298	0.6156897
$\sigma = 10$	0.1265605	0.4652878	0.5883557

The last table denotes the behavior of Conditional Bayes estimators of corresponding parameters under LinEx loss function. The similar tables for another loss functions are omitted. Finally, as you can see the new strategy of estimating has so good performance in comparison with MLEs.

References

[1] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992), *A first course in order statistics*, Volume 54, Siam.

[2] Balakrishnan, N. and Cramer, E. (2014), *The art of progressive censoring*, Springer New York.

[3] Balakrishnan, N. and Sandhu, R. A. (1995), A simple simulational algorithm for generating progressive Type-II censored samples, *The American Statistician*, **49**(2), 229-230.

[4] Devroye, L. (1984), A simple algorithm for generating random variates with a log-concave density, *Computing*, **33**(3-4), 247-257.

[5] Haghighi, F. (2017), Bayes conditional control charts for Weibull quantiles under type II censoring, *Quality and Reliability Engineering International*, **33**(5), 959-968.

[6] Haghighi, F. and Castagliola, P. (2015), Conditional control charts for monitoring the Weibull shape parameter under progressively type II censored data, *Quality and Reliability Engineering International*, **31**(6), 1013-1022.

[7] Haghighi, F., Pascual, F. and Castagliola, P. (2015), Conditional control charts for Weibull quantiles under type II censoring, *Quality and Reliability Engineering International*, **31**(8), 1649-1664.

[8] Hastings, W.K. (1970), Monte Carlo sampling methods using Markov chains and their applications, *Biometrika*, **57**, 97-109.

[9] Lawless, J.F. (1978), Confidence interval estimation for the Weibull and extreme value distributions, *Technometrics*, **20**(4), 355-364.

[10] Lawless, J.F. (1973), Conditional versus unconditional confidence intervals for the parameters of the Weibull distribution, *Journal of the American Statistical Association*, **68**(343), 665-669.

[11] Lawless, J.F. (1975), Construction of tolerance bounds for the extreme value and Weibull distributions, *Technometrics*, **17(2)**, 255-261.

[12] Lehmann, E.L. and Casella, G. (2006), *Theory of point estimation*, Springer Science & Business Media.

[13] Metropolis, N., Rosenbluth, A.W., Rosenbluth, M.N., Teller, A.H. and Teller, E. (1953), Equation of state calculations by fast computing machines, *The journal of chemical physics*, **21**(6), 1087-1092.

[14] Soliman, A.A., Abd-Ellah, A.H., Abou-Elheggag, N.A. and Ahmed, E. A. (2012), Modified Weibull model: A Bayes study using MCMC approach based on progressive censoring data, *Reliability Engineering & System Safety*, **100**, 48-57.

[15] Soliman, A.A., Abd-Ellah, A.H., Abou-Elheggag, N.A. and Ahmed, E. A. (2013), Reliability estimation in stressstrength models: an MCMC approach, *Statistics*, **47**(4), 715-728.

[16] Viveros, R. and Balakrishnan, N. (1994), Interval estimation of parameters of life from progressively censored data, *Technometrics*, **36**(1), 84-91.

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Allocation Policy of Redundancies in Two-Parallel-Series System with Randomized Components

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Abstract: In this paper, we consider two-parallel-series system consisting two types of components chosen from two batches of n independent components. Suppose that the number of components from the first batch (say K) is chosen randomly according to a probability distribution. We purpose to compare the systems when K is distributed by two probability distributions such that they stochastically ordered.

Keywords Coherent system, Redundancy, Stochastic orders, Randomized components.

Mathematics Subject Classification (2010): 60E15, 62N05.

A Introduction

In the theory of coherent systems, the study of the stochastic properties of coherent systems which composed of different types of components is an important topic. Consider a situation that there are two different batches of components to built the system. The random lifetimes of the components in the first and the second batches are denoted by $\{X_i, i = 1, \cdots, n\}$ and $\{Y_i, i = 1, \cdots, n\}$, respectively. Assume that the k components from the first batch and the $n-k$ components from the second batch are selected to construct the system.

One way to improve the reliability of systems is to allocate redundancies, at component level or at system level. The former means that some spares connect in parallel to each components and in the latter case, a duplicated system consisting of spares connects to the original system in parallel.

In this paper, we consider a series structure and suppose that to improve the reliability of the system, the redundancy at system level is employed as it can shown in Figure 1. Note that in up-series subsystem there are k components from the first batch and the remaining components from the second batch while in the down-series subsystem, the components combination is vice

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versa. This system which is called "two-parallel-series" system, studied by Laniado and Lillo

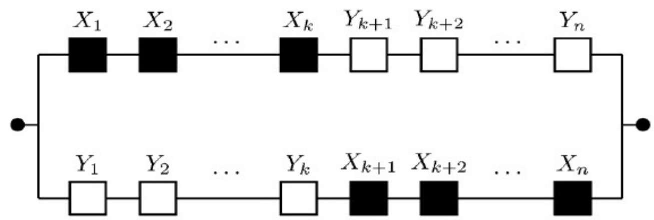

Figure 1: Two-parallel-series system

(3), for which they stochastically compared the allocation policies by some common stochastic orders.

In this paper, we suppose that K is random variable with support in $\{0, \cdots, n\}$. This assumption is considered by Di Crescenzo and Pellery (1) for series and parallel systems where, they compared the systems when random variables K_1 and K_2 are stochastically ordered. Their results are strengthened by Hazra and Nanda (2) using some other stochastic orders. Navarro et al. (5) extended the results from series and parallel systems to general coherent systems with possibly dependent components.

Here, we suppose that the random number of components (K) is chosen with two probability distributions, say K_1 and K_2 where they are stochastically ordered and compare the corresponding two-parallel-series systems.

A.1 Some useful definitions and lemmas

Let X and Y be random variables with corresponding distribution functions F and G, reliability functions $\bar{F}$ and $\bar{G}$, density functions f and g, hazard rate functions r_X and r_Y and reverse hazard rate functions $\tilde{r}_X$ and $\tilde{r}_Y$. Then X is said to be smaller than Y in the

- usual stochastic order (denoted by $X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all x,
- hazard rate order (denoted by $X \leq_{hr} Y$) if $\frac{\overline{G}(x)}{\overline{F}(x)}$ is increasing in x, or, equivalently, $r_X(x) \geq r_Y(x)$ for all x,
- reversed hazard rate order (denoted by $X \leq_{rhr} Y$) if $\frac{G(x)}{F(x)}$ is increasing in x, or, equivalently, $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$ for all x,
- likelihood ratio order (denoted by $X \leq_{lr} Y$) if $\frac{g(x)}{f(x)}$ is increasing in x, for all x.

It is known that the following implications hold:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr[rhr]} Y \Rightarrow X \leq_{st} Y,$$

(Karlin (1968). A function $h(x, y)$ is said to be Sign-Regular of order 2 (SR_2) if $\epsilon_1 h(x, y) \geq 0$ and

$$\epsilon_2 \begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geq 0$$

whenever $x_1 < x_2, y_1 < y_2$ for ϵ_1 and ϵ_2 equal to +1 or -1. If the above relations hold with $\epsilon_1 = +1$ and $\epsilon_2 = +1$, then h is said to be Totally Positive of order 2 (TP_2); and if they hold with $\epsilon_1 = +1$ and $\epsilon_2 = -1$ then h is said to be Reverse Regular of order 2 (RR_2).

It should be pointed out that TP_2 (RR_2) property of $h(t, x)$ is equivalent to that $\frac{h(t,x_2)}{h(t,x_1)}$ is increasing (decreasing) in t whenever $x_1 \leq x_2$.

Let $\psi_i : [0, \infty) \times [0, \infty) \to \mathbb{R},\ i = 1, 2$, be a function and let $g_i(\theta)$ be the Lebesgue probability distribution function of a random variable $T_i,\ i = 1, 2$. Let

$$\psi(x) = \frac{\int_0^\infty \psi_2(x, \theta) g_2(\theta) d\theta}{\int_0^\infty \psi_1(x, \theta) g_1(\theta) d\theta}, \quad x > 0. \tag{A.1}$$

(Misra and Naqvi (2018) (4)) Suppose that $\frac{\psi_2(x,\theta)}{\psi_1(x,\theta)}$ increases (decreases) in $x \in (0, \infty)$ and increases in $\theta \in (0, \infty)$. Further suppose that any of the following three conditions hold:

(i) $T_1 \leq_{lr} T_2$ and $\psi_1(x, \theta)$ or $\psi_2(x, \theta)$ is TP_2 (RR_2) in $(x, \theta) \in (0, \infty) \times (0, \infty)$;

(ii) $T_1 \leq_{hr} T_2$ and

$\psi_1(x, \theta)$ is TP_2 (RR_2) in $(x, \theta) \in (0, \infty) \times (0, \infty)$ and is increasing in $\theta \in (0, \infty)$ or;

$\psi_2(x, \theta)$ is TP_2 (RR_2) in $(x, \theta) \in (0, \infty) \times (0, \infty)$ and is increasing in $\theta \in (0, \infty)$,

(iii) $T_1 \leq_{rhr} T_2$ and

$\psi_1(x, \theta)$ is TP_2 (RR_2) in $(x, \theta) \in (0, \infty) \times (0, \infty)$ and is decreasing in $\theta \in (0, \infty)$ or;

$\psi_2(x, \theta)$ is TP_2 (RR_2) in $(x, \theta) \in (0, \infty) \times (0, \infty)$ and is decreasing in $\theta \in (0, \infty)$.

Then the function $\psi(x)$ as defined in (A.1), is increasing (decreasing) in $x \in (0, \infty)$.

B Main results

Let S_k denote the lifetime of the system shown in Figure 1, in which the components are assumed to be independent and hence the corresponding distribution function is obtained as

$$F_{S_k}(t) = [1 - \bar{F}^k(t)\bar{G}^{n-k}(t)][1 - \bar{F}^{n-k}(t)\bar{G}^k(t)]. \tag{B.1}$$

Laniado and Lillo (3) supposed that the $\bar{F}$ and $\bar{G}$ follow the proportional hazards (PH) model as $\bar{G}(t) = \bar{F}^{\alpha}(t)$ for some positive $\alpha > 0$ and for all $t \geq 0$ and proved the following result. (3) For the system with different combinations of components, we have for every $k_1 \leq k_2 \leq \frac{n}{2}$

(i) $S_{k_1} \geq_{rhr} S_{k_2}$

(ii) $S_{k_1} \geq_{hr} S_{k_2}$.

Afterwards, Wang and Laniado (6) extended the stochastic ordering to likelihood ratio order. (6) For the system with different combinations of components, we have

$$S_{k_1} \geq_{lr} S_{k_2}$$

for every $k_1 \leq k_2 \leq \frac{n}{2}$. Their results state that, if the allocation of components and redundancies is unbalanced as much as possible (in other words, the heterogeneity is maximized) then one can get the higher reliable system.

In this paper, we want to suppose that K is random variable which take the value in $\{0, 1, \cdots, n\}$. In this case, we denote the lifetime of the system by S_K. The distribution function of S_K is represented as follows

$$F_{S_K}(t) = \sum_{k=0}^{n} [1 - \bar{F}^k(t)\bar{G}^{n-k}(t)][1 - \bar{F}^{n-k}(t)\bar{G}^k(t)]P(K = k),$$

and under the PH model, we have

$$F_{S_K}(t) = \sum_{k=0}^{n} [1 - \bar{F}^{k+\alpha(n-k)}(t)][1 - \bar{F}^{n-k+\alpha k}(t)]P(K = k).$$

At the continue, we purpose to compare the systems when the random number K follow two probability distributions which are ordered stochastically. If $K_1 \leq_{st} K_2$ then $S_{K_1} \geq_{st} S_{K_2}$. .
From Theorem B we have the the $\bar{F}_{S_k}$ is a decreasing function of k. We can write

$$\begin{aligned}
\bar{F}_{S_{K_1}}(t) &= \sum_{k=0}^{n} \left[1 - (1 - \bar{F}^{k+\alpha(n-k)}(t))(1 - \bar{F}^{n-k+\alpha k}(t))\right] P(K_1 = k) \\
&= E(\phi_{\bar{F}(t)}(K_1)) \\
&\geq E(\phi_{\bar{F}(t)}(K_2)) \\
&= \sum_{k=0}^{n} \left[1 - (1 - \bar{F}^{k+\alpha(n-k)}(t))(1 - \bar{F}^{n-k+\alpha k}(t))\right] P(K_2 = k) = \bar{F}_{S_{K_2}}(t)
\end{aligned}$$

where, $\phi_{\bar{F}(t)}(k) = 1 - F_{S_k}(t)$. The inequlaity follows from Shaked and shantikumar (2007). $\square$

This result show that the reliability of the system increases in ususal stochastic order sense, when the random number of components decreases in the usual stochastic order. In the simpler words, as the mean value of K increases, the reliability of the system decreases.

In the next of the paper, we extend the obtained result to the other stochastic orders. If $K_1 \leq_{hr} K_2$ then $S_{K_1} \geq_{rhr} S_{K_2}$. . The desired result is equivalent to show that

$$\frac{\sum_{k=0}^{n}[1-\bar{F}^{k+\alpha(n-k)}(t)][1-\bar{F}^{n-k+\alpha k}(t)]P(K_2=k)}{\sum_{k=0}^{n}[1-\bar{F}^{k+\alpha(n-k)}(t)][1-\bar{F}^{n-k+\alpha k}(t)]P(K_1=k)} \tag{B.2}$$

is decreasing function of t.

From Theorem B(i) and Definition A.1, we have that

$$\frac{[1-\bar{F}^{k_2+\alpha(n-k_2)}(t)][1-\bar{F}^{n-k_2+\alpha k_2}(t)]}{[1-\bar{F}^{k_1+\alpha(n-k_1)}(t)][1-\bar{F}^{n-k_1+\alpha k_1}(t)]}$$

is decreasing in t and hence $[1-\bar{F}^{k+\alpha(n-k)}(t)][1-\bar{F}^{n-k+\alpha k}(t)]$ is RR_2 in $(t,k) \in \mathbb{R}^+ \times \{0,1,\cdots,n\}$. Also, from Theorem B, it is clear that $F_{S_k}(t)$ is increasing function of k, then according to Lemma A.1 (ii) we obtain the result. $\square$

If $K_1 \leq_{rhr} K_2$ then $S_{K_1} \geq_{hr} S_{K_2}$. . The desired result is equivalent to show that

$$\frac{\sum_{k=0}^{n}\left(1-[1-\bar{F}^{k+\alpha(n-k)}(t)][1-\bar{F}^{n-k+\alpha k}(t)]\right)P(K_2=k)}{\sum_{k=0}^{n}\left(1-[1-\bar{F}^{k+\alpha(n-k)}(t)][1-\bar{F}^{n-k+\alpha k}(t)]\right)P(K_1=k)} \tag{B.3}$$

is decreasing function of t.

From Theorem B(ii) and Definition A.1, we have that

$$\frac{1-[1-\bar{F}^{k_2+\alpha(n-k_2)}(t)][1-\bar{F}^{n-k_2+\alpha k_2}(t)]}{1-[1-\bar{F}^{k_1+\alpha(n-k_1)}(t)][1-\bar{F}^{n-k_1+\alpha k_1}(t)]}$$

is decreasing in t and hence $1-[1-\bar{F}^{k+\alpha(n-k)}(t)][1-\bar{F}^{n-k+\alpha k}(t)]$ is RR_2 in $(t,k) \in \mathbb{R}^+ \times \{0,1,\cdots,n\}$. Also, it is evident that $\bar{F}_{S_k}(t)$ is decreasing function of k, then according to Lemma A.1 (iii) we get the result. $\square$

If $K_1 \leq_{lr} K_2$ then $S_{K_1} \geq_{lr} S_{K_2}$. . First, note that the density function of S_k is

$$f_{S_K}(t) = \frac{f(t)}{\bar{F}(t)}[(k+\alpha(n-k))\bar{F}^{k+\alpha(n-k)}(t) + (\alpha k + (n-k))\bar{F}^{\alpha k+(n-k)}(t) - (\alpha+1)n\bar{F}^{(\alpha+1)n}]$$

The desired result is equivalent to show that

$$\frac{\sum_{k=0}^{n} f_{S_k}(t)P(K_2=k)}{\sum_{k=0}^{n} f_{S_k}(t)P(K_1=k)}$$

is decreasing function of t. From Theorem B it follows that $\frac{f_{S_{k_2}}(t)}{f_{S_{k_1}}(t)}$ decreases in t and hence, $f_{s_k}(t)$ is RR_2 in $(t,k) \in \mathbb{R}^+ \times \{0,1,\cdots,n\}$. Now, using Lemma A.1 (i) we get the result. $\square$

References

[1] Di Crescenzo, A. and Pellerey, F. (2011), Stochastic comparisons of series and parallel systems with randomized independent components, *Operations Research Letters*, **39**(5), 380-384.

[2] Hazra, N.K. and Nanda, A.K. (2014), Some results on series and parallel systems of randomized components, *Operations Research Letters*, **42**(2), 132-136.

[3] Laniado, H. and Lillo, R.E. (2014), Allocation policies of redundancies in two-parallel-series and two-series-parallel systems, *IEEE Transactions on Reliability*, **63**(1), 223-229.

[4] Misra, N. and Naqvi, S. (2018), Some unified results on stochastic properties of residual lifetimes at random times, *Brazilian Journal of Probability and Statistics*, **32**(2), 422-436.

[5] Navarro, J., Pellerey, F. and Di Crescenzo, A. (2015), Orderings of coherent systems with randomized dependent components, *European Journal of Operational Research*, **240**(1), 127-139.

[6] Wang, J. and Laniado, H. (2015), A note on allocation policy in two-parallel-series and two-series-parallel systems with respect to likelihood ratio order, *Statistics and Probability Letters*, **102**, 17-21.

A Probabilistic Model for Structure Functions of Coherent Systems

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Abstract: Recently Coolen and Coolen-Maturi (2016) suggested the probabilistic idea for structure function of a coherent system. They did not give any specific model for their suggestion. In this paper we tried to make the preliminaries for this model. The author welcomes the interest readers and participants of the Seminar for their collaborations that are crucial for further progress of the present work.

Keywords Coherent systems, Probabilistic structure function, Survival signature.

Mathematics Subject Classification (2010) : 62N05, 60E15**.**

A Introduction

Mathematical and statistical theory of system reliability is based on the central concept "structure function", which is a binary function and describes deterministically the state of a system when the states of its components are given. In practical uses and real world applications it is somewhat restricted as for various reasons the functioning of system components does not always provide absolute certainty that the system will function. In other words, except the failures of system components there are other random factors that may cause to system failure. For example in a car system sometimes we have seen that the main and key components of the car are functioning but the car does not work. Therefore because of lack of our perfect knowledge and our uncertainty about the quality of the system functioning, a generalization of structure function from binary function to a probability may have substantial advantages for realistic system reliability quantification. This idea was recently suggested by Coolen and Coolen-Maturi (2016). They did not give any specific model for probabilistic structure function. In this paper, among different factors that may have effects on system performance, we only consider the quality of the links between system components that we think it is an important factor for functioning of the system. In the following section we explain our model in details.

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Illustrative examples are also given. Finally in Section 3, we show that the survival signature a concept defined by Coolen and Coolen-Maturi (2012) can be successfully used in our model for probability structure function.

B Probabilistic structure function

In this section, for the sake of completeness we first review the binary structure function and then present our probabilistic model for structure function.

2.1 Binary Structure Function

Consider a system consists of n components and assume that all components and the system are in functioning or failed state. In a fixed point of time let state vector $\mathbf{X} = (X_1, \dots, X_n) \in \{0,1\}^n$, with

$$X_i = \begin{cases} 1 & \text{if } i\text{th component is working} \\ 0 & \text{otherwise} \end{cases}$$

The structure function $\phi : \{0,1\}^n \to \{0,1\}$ is defined as

$$\phi(\mathbf{X}) = \phi(X_1, \dots, X_n) = \begin{cases} 1 & \text{if the system is working} \\ 0 & \text{otherwise} \end{cases}$$

The system is said to be coherent if $\phi(\mathbf{X})$ is not decreasing in any X_i and all components be relevant, that is

$$\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1$$

at least for one $\mathbf{X} \in \{0,1\}^{n-1}$.

Obviously $\phi(1, \dots, 1) = 1$ and $\phi(0, \dots, 0) = 0$.

Also $\phi(\mathbf{X}) = X_i \phi(1_i, \mathbf{X}) + (1 - X_i)\phi(0_i, \mathbf{X})$, $i = 1, \dots, n$ (pivotal decomposition).

If $\phi(\mathbf{X}) = 1(0)$ then $\mathbf{X}$ is said to be a path(cut) vector and the corresponding subset $P = \{1 \leq i \leq n | X_i = 1\}(C = \{1 \leq i \leq n | X_i = 0\})$ is called a path(cut) set of the system. Note that an arbitrary vector $\mathbf{X} \in \{0,1\}^n$ always is a path vector or a cut vector but not both. Whereas $P \subseteq \{1, \dots, n\}$ can be both a path and a cut set.

If $P(C) \subseteq \{1, \dots, n\}$ be a path(cut) set and $Q \subset P(C)$ is not a path(cut) set then $P(C)$ is a minimal path(cut) set of the system. It is known that if $P_1, \dots, P_r(C_1, \dots, C_s)$ be all minimal path(cut) sets of the system then

$$\phi(\mathbf{X}) = max_{1 \leq i \leq r} \;\; min_{j \in P_i} X_j = min_{1 \leq i \leq s} \;\; max_{j \in C_i} X_j \tag{B.1}$$

.

If $p_i = EX_i = P(X_i = 1)$ be the reliability of component i then $h(\mathbf{p}) = h(p_1, \ldots, p_n) = E\phi(\mathbf{X}) = P(\phi(\mathbf{X}) = 1)$ is the reliability function of the system. For more details on reliability of coherent systems with binary structure function, see Barlow and Proschan (1975).

2.2 Structure Function as Probability

Here we still assume that the binary state for the system components. We also take into account the quality of the links between the components as an effective factor on system performance. This factor may cause to failure of the system even if all its components are working. Therefore given the states of components, we consider the state of the system to be a conditional probability as follow

$$\phi_{\mathbf{a}}(\mathbf{x}) = P_{\mathbf{a}}(\text{system is functioning}|\mathbf{X} = \mathbf{x})$$

where $\mathbf{a} = (a_1, \ldots, a_m)$ with

$$a_l = P(\text{the } l^{th} \text{ minimal link is functioning}), l = 1, \ldots, m$$

and m is the number of minimal links between the system components. As usual we assume that the failure of the components leads to failure of the system definitely. Let

$$S = \begin{cases} 1 & \text{if system is functioning} \\ 0 & \text{otherwise} \end{cases}$$

then

$$\phi_{\mathbf{a}}(\mathbf{x}) = P_{\mathbf{a}}(S = 1|\mathbf{X} = \mathbf{x}).$$

ASSUMPTIONS

A1. $\phi_{\mathbf{a}}(\mathbf{x})$ is increasing in x_i, $i = 1, \ldots, n$.

A2. The component i is relevant, that is $\phi_{\mathbf{a}}(1_i, \mathbf{x}) > 0$ and $\phi_{\mathbf{a}}(0_i, \mathbf{x}) = 0$ at least for one $\mathbf{x} \in \{0, 1\}^{n-1}$, $i = 1, \ldots, n$. Under the above conditions we call the system as a coherent system. In a coherent system we have $\phi_{\mathbf{a}}(0, \ldots, 0) = 0$. Also $\phi_{\mathbf{a}}(1, \ldots, 1) > 0$, but it is not necessary equal to 1. It means that even if all components of the system are working, there exists a positive probability for system failure. But if $\mathbf{x}$ is a cut vector we have $\phi_{\mathbf{a}}(\mathbf{x}) = \phi(\mathbf{x}) = 0$ in which $\phi(\mathbf{x})$ is the binary structure function.

A3. We assume that the links between components are functioning independently and are independent of the states of components.

Minimal Path(Cut) Sets

If $\phi_{\mathbf{a}}(\mathbf{x}) > (=) 0$ we call $\mathbf{x}$ as a path(cut) vector. The path(cut) sets and the minimal path(cut) sets are defined as in previous subsection 2.1. Although $\phi_{\mathbf{a}}(\mathbf{x})$ is simply satisfied in pivotal decomposition but the Equation (2.1) does not hold true for $\phi_{\mathbf{a}}(\mathbf{x})$.

Lemma 2.1. For $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{a} = (a_1, \ldots, a_m)$ the system reliability is given by

$$h_{\mathbf{a}}(\mathbf{p}) = E_{\mathbf{a}} S = P_{\mathbf{a}}(S = 1) = \sum\nolimits_{\mathbf{x}} \phi_{\mathbf{a}}(\mathbf{x}) P(\mathbf{X} = \mathbf{x})$$

where summation is taken over all path vectors. Also

$$1 - h_{\mathbf{a}}(\mathbf{p}) = P_{\mathbf{a}}(S = 0) = \sum_{\mathbf{x}_p} (1 - \phi_{\mathbf{a}}(\mathbf{x}_p)) P(\mathbf{X} = \mathbf{x}_p) + \sum_{\mathbf{x}_c} P(\mathbf{X} = \mathbf{x}_c)$$

$\mathbf{x}_p(\mathbf{x}_c)$ is a path(cut) vector of the system.

Proof. We have

$$h_{\mathbf{a}}(\mathbf{p}) = \sum_{\mathbf{x}} P_{\mathbf{a}}(S = 1 | \mathbf{X} = \mathbf{x}) P(\mathbf{X} = \mathbf{x}) = \sum_{\mathbf{x}} \phi_{\mathbf{a}}(\mathbf{x}) P(\mathbf{X} = \mathbf{x}).$$

The second equality of the lemma given for unreliability of the system can similarly be proved. □

Remark 1. The above lemma shows clearly that the system failure is not only depended to components failures but also to the minimal links between components. The first sum in $1 - h_{\mathbf{a}}(\mathbf{p})$ gives in fact the contribution of the minimal links between the system components to failure of the system and the second sum gives the same for failure of system components.

Remark 2. Note that $\phi_{\mathbf{a}}(\mathbf{x})$ reduces to the binary structure function $\phi(\mathbf{x})$ when $\mathbf{a} = (1, \ldots, 1)$, that is all minimal links between components are in functioning state definitely. Therefore we can say that the coherent systems with binary structure functions is a subclass of the coherent systems with probabilistic structure functions. Obviously $\phi_{\mathbf{a}}(\mathbf{x}) \le \phi(\mathbf{x})$ and therefore $h_{\mathbf{a}}(\mathbf{p}) \le h(\mathbf{p})$.

In the following examples we also assume that the system components are independent. EXAMPLES

1(Series System).

In this system we have $m = n-1$, $\phi_{\mathbf{a}}(\mathbf{x}) = a_1 a_2 \cdots a_{n-1} x_1 x_2 \cdots x_n$ and $h_{\mathbf{a}}(\mathbf{p}) = a_1 a_2 \cdots a_{n-1} p_1 p_2 \cdots p_n$. Note that for $i = 1, \ldots, n-1$ we have

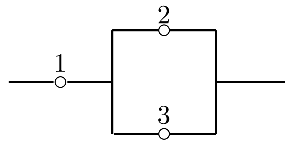
$$a_i = P(\text{link between componets } i \text{ and } i+1 \text{ is functioning}).$$

2(Parallel System).

In this system we have $m = n$, $\phi_{\mathbf{a}}(\mathbf{x}) = 1 - \prod_1^n (1 - a_i x_i)$ and $h_{\mathbf{a}}(\mathbf{p}) = 1 - \prod_1^n (1 - a_i p_i)$.

3(2-out-of-3 System). In this system we have $m = 3$, $\phi_{\mathbf{a}}(\mathbf{x}) = a_1 x_1 x_2 + a_2 x_1 x_3 + a_3 x_2 x_3 - x_1 x_2 x_3 (a_1 a_2 + a_1 a_3 + a_2 a_3 - a_1 a_2 a_3)$ and $h_{\mathbf{a}}(\mathbf{p}) = a_1 p_1 p_2 + a_2 p_1 p_3 + a_3 p_2 p_3 - p_1 p_2 p_3 (a_1 a_2 + a_1 a_3 + a_2 a_3 - a_1 a_2 a_3)$.

4(Series-Parallel System).



For this system we have $m = 2$, $\phi_{\mathbf{a}}(\mathbf{x}) = a_1x_1x_2 + a_2x_1x_3 - a_1a_2x_1x_2x_3$ and $h_{\mathbf{a}}(\mathbf{p}) = a_1p_1p_2 + a_2p_1p_3 - a_1a_2p_1p_2p_3$.

Remark 3. In a coherent system, the minimal links between system components can be obtained just by the collection of its minimal path sets. Suppose $P_{(r)} \subseteq \{1, \ldots, n\}$ be a minimal path set of the system consisting of $r \geq 2$ components. Obviously it contains $r-1$ minimal links between components. We also assume that $P_{(1)}$ has one minimal link(see the parallel system in Example 2). It is an interesting problem if one can obtain a closed form for $\phi_{\mathbf{a}}(\mathbf{x})$ in terms of its minimal path sets and $\mathbf{a}$, as like as the Equation (2.1) for $\phi(\mathbf{x})$.

C Survival Signature of Coherent Systems with Probability Structure Functions

The concept of system signature introduced by Samaniego (1985). It is a very useful tool and has a wide range of applications in study of reliability analysis of coherent systems. Let $T = \phi(T_1, \ldots, T_n)$ be the lifetime of a coherent system where T_i is the lifetime of component i. When T_i's are independent and identically distributed (i.i.d), Samaniego (1985) showed that

$$P(T > t) = \sum_{i=1}^{n} s_i P(T_{i:n} > t) \tag{C.1}$$

where $T_{i:n}$ is the i^{th} ordered component lifetime, $s_i = P(T = T_{i:n})$ and the probability vector $\mathbf{s} = (s_1, \ldots, s_n)$ is the signature of the system.

Although the system signature is an important tool and has many applications in reliability studies of coherent systems with binary structure function but it seems that is not case for the coherent systems with probabilistic structure functions as we have seen in previous section that the system failure is not determined by the failure of components deterministically. Note that the Equation (3.1) does not hold true for the systems with probability structure functions.

In this section we show that the "survival signature", a concept introduced by Coolen and Coolen-Maturi (2012), how plays the role of system signature in coherent systems with probabilistic structure functions.

Coolen and Coolen-Maturi (2012) introduced a new measure and called it the survival signature for a coherent system of order n with components of r different types. They assumed that the system has m_k components of type k, $k = 1, \ldots, r$ and also assumed that the components of the same type are exchangeable and the components of different types are independent. For $i_k = 0, \ldots, m_k$ and $k = 1, \ldots, r$ their measure is defined as follow

$$\bar{s}(i_1, \ldots, i_r) = P\left(\phi(\mathbf{X}) = 1 | \text{exactly } i_k \text{ components of type } k \text{ are working}\right)$$
$$= \left[\prod_{k=1}^{r} \binom{m_k}{i_k}\right]^{-1} \sum_{\mathbf{x} \in S_{i_1, \ldots, i_r}} \phi(\mathbf{x})$$

where $S_{i_1,\ldots,i_r} = \{\mathbf{x} | \sum_{j=1}^{m_k} x_j^k = i_k,\ k = 1, \ldots, r.\}$

They also obtained the system reliability as follow:

$$P(T > t) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} \bar{s}(i_1, \ldots, i_r) \prod_{k=1}^{r} P(C_t^k = i_k)$$

where $C_t^k \in \{0, 1, \ldots, m_k\}$ is the number of components of type k that function at time t.

Under the above stated assumptions they showed that the $\bar{s}$, as like as the system signature is not depended to the joint distribution of the components.

Definition 3.1. We define the survival signature of a coherent system with probability structure function as follow

$$\bar{s}_{\mathbf{a}}(i) = P_{\mathbf{a}}(\text{system is functioning}|\text{the number of working components is } i).$$

The following lemma gives an expression for $\bar{s}_{\mathbf{a}}(i)$ in a coherent system with i.i.d. components. It is useful for determining the reliability function of the system.

Lemma 3.1. In a coherent system with i.i.d. components and with probability structure function $\phi_{\mathbf{a}}(\mathbf{x})$ we have

$$\bar{s}_{\mathbf{a}}(i) = \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} \phi_{\mathbf{a}}(\mathbf{x})}{\binom{n}{i}}$$

where $|\mathbf{x}| = \sum x_j$.

Proof. We have

$$\bar{s}_{\mathbf{a}}(i) = P_{\bar{a}}(S = 1 | \sum_{j=1}^{n} X_j = i) = \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} P_{\mathbf{a}}(S = 1, \mathbf{X} = \mathbf{x})}{P(\sum X_j = i)}$$

$$= \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} P_{\mathbf{a}}(S = 1 | \mathbf{X} = \mathbf{x}) P(\mathbf{X} = \mathbf{x})}{P(\sum X_j = i)} = \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} \phi_{\mathbf{a}}(\mathbf{x}) p^i (1-p)^{n-i}}{\binom{n}{i} p^i (1-p)^{n-i}}$$

where p is the common reliability of components. $\square$

Lemma (3.1) shows that the survival signature $\bar{s}_{\mathbf{a}}(i)$ is not depended to component reliabilities. We now in the next theorem obtain the reliability function of the system.

Theorem 3.1. Under the assumptions of Lemma (3.1) we have

$$h_{\mathbf{a}}(p) = \sum_{i=1}^{n} \bar{s}_{\mathbf{a}}(i) \binom{n}{i} p^i (1-p)^{n-i}. \tag{C.2}$$

Proof. We have $h_{\mathbf{a}}(p) = P_{\mathbf{a}}(S=1)$

$$= \sum_{i=1}^{n} P_{\mathbf{a}}(S=1 | \sum_j X_j = i) \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=1}^{n} \bar{s}_{\mathbf{a}}(i) \binom{n}{i} p^i (1-p)^{n-i}.$$

$\square$

The Equation (3.2) is the similar version of the Equation (3.1).

References

[1] Barlow, R.E. and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston.

[2] Coolen, F.P.A. and Coolen-Maturi, T. (2012), On generalizing the signature to systems with multiple types of components. *Complex Systems and Dependability,* 115-130, Berlin, Springer.

[3] Coolen, F.P.A. and Coolen-Maturi, T. (2016), The structure function for system reliability as predictive (imprecise) probability, *Reliability Engineering and System Safety*, **154**, 180-187.

[4] Samaniego, F.J. (1985), on the closure of the IFR class under formation of coherent systems, *IEEE Transactions on Reliability*, **34**, 69-72.

A New Two-Sided Class of Lifetime Distributions

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Abstract: In the present paper, we introduce a new class of change point lifetime distributions. Some statistical and reliability characteristics are presented for the proposed model. In addition, we look at a real example of the data, and it can be seen that in practice the new model has a superiority over some of the statistical models.

Keywords Hazard rate function, Order statistics, Maximum likelihood estimator, Transmutation map, odd Transmuted two-sided distribution.

Mathematics Subject Classification (2010) : 62E10**,** 62F10**.**

A Introduction

Recently, several researchers study the statistical distribution theory and modeling. Since there are the various data in the real world, it is necessary to extend the classic statistical models and finally introduce the more accurate models. In recent year, the different methods have been introduced for constructing the statistical distributions by researchers.

The change point models are often important in the statistical distribution theory. The change point distributions are used in the different branch of sciences such as economic, engineering, agriculture and so on. A family of the change point distributions is introduced by Van Dorp and Kotz (2002a) so-called two-sided power distribution (TSP) with the *pdf*,

$$f(x;\alpha,\beta)=\begin{cases}\alpha\left(\frac{x}{\beta}\right)^{\alpha-1}, & 0<x\leq\beta,\\ \alpha\left(\frac{1-x}{1-\beta}\right)^{\alpha-1}, & \beta\leq x<1,\end{cases}\tag{A.1}$$

and with the cumulative distribution function (*cdf*),

$$F(x;\alpha,\beta)=\begin{cases}\beta\left(\frac{x}{\beta}\right)^{\alpha}, & 0<x\leq\beta,\\ 1-(1-\beta)\left(\frac{1-x}{1-\beta}\right)^{\alpha}, & \beta\leq x<1,\end{cases}\tag{A.2}$$

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where $0 \leq \beta \leq 1$ and $\alpha > 0$. The parameter β is the location parameter called "turning point" and α is the shape parameter that control the shape of distribution on the left and right of β.

Van Dorp and Kotz (2002b) introduced an extension of the three-parameter triangular distribution utilized in risk analysis. Their model includes the TSP distribution as a special case. Van Dorp and Kotz (2003) considered a family of continuous distributions on a bounded interval generated by convolutions of the TSP distributions.

Recently, some of researchers have proposed some new distributions by using Alzaatreh's technique (2013) that these distributions are one-sided and have not any change point. The Alzaatreh's technique, as an important method for developing statistical distributions, hasn't been used for the change point distributions, yet.

Alzaatreh et al. (2013) introduced a new technique to drive families of distributions by using any *pdf* as a generator. Suppose that $G(x)$ is a parent distribution and $r(x)$ is a probability distribution function (*pdf*). Then Alzaatreh et al. (2013) introduced a new lifetime distributions family by combining the parent G and the *pdf* $r(x)$ as follows.

$$F(x) = \int_{a}^{W(G(x))} r(t)dt, \tag{A.3}$$

where $T \in [a, b]$ and $-\infty \leq a < b \leq \infty$ and $W(G(x))$ satisfies the following conditions.

- $W(G(x)) \in [a, b]$
- $W(G(x))$ is differentiable and monotonically non-decreasing
- $W(G(x)) \to a$ as $x \to -\infty$ and $W(G(x)) \to b$ as $x \to \infty$.

A special case of $W(k)$ is generalized odd ratio function that have considered by scientific researchers widely in recent years. The generalized odd ratio is a well-known quantity in reliability, survival analyse and engineering systems. $W(k)$ should be state.

Kharazmi and Zargar (2018) introduced a new family of distribution to apply the transmutation technique in the two-sided distributions for increasing the flexibility and usefulness of the TSP distribution and generalized this class of distributions with *pdf*

$$f(x;\alpha,\beta,\lambda) = \begin{cases} \alpha\left((1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{x}{\beta}\right)^{2\alpha-1}\right), & 0 < x \leq \beta, \\ \alpha\left((1+\lambda)\left(\frac{1-x}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-x}{1-\beta}\right)^{2\alpha-1}\right), & \beta \leq x < 1, \end{cases} \tag{A.4}$$

and its *cdf* is given by

$$F(x;\alpha,\beta,\lambda) = \begin{cases} \beta\left((1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha} - \lambda\left(\frac{x}{\beta}\right)^{2\alpha}\right), & 0 < x \leq \beta, \\ 1-(1-\beta)\left((1+\lambda)\left(\frac{1-x}{1-\beta}\right)^{\alpha} - \lambda\left(\frac{1-x}{1-\beta}\right)^{2\alpha}\right), & \beta \leq x < 1. \end{cases} \tag{A.5}$$

The main motivation of the present paper is to apply the Alzaatreh's method for increasing the flexibility and usefulness of the two-sided distributions. For this purpose, a new distribution is proposed based on generalized odd ratio for transmuted two-sided model introduced by Kharazmi and Zargar (2018).

B Generalized odd transmuted two-sided-G distribution

Suppose that $G(x;\xi)$ be a parent *cdf* of a continuous random variable with *pdf* $g(x;\xi)$. Based on the relations (A.4) and (A.5) and parent $G(x;\xi)$, one can see the following distribution. A random variable X is said to be generalized odd transmuted two-sided-G distribution if its *pdf* is given by

$$f(x)=\begin{cases}\alpha\psi'(x;\eta,\xi)\left((1+\lambda)\left(\frac{\psi(x;\eta,\xi)}{\beta}\right)^{\alpha-1}-2\lambda\left(\frac{\psi(x;\eta,\xi)}{\beta}\right)^{2\alpha-1}\right), & -\infty<x\le\Omega_1,\\ \alpha\psi'(x;\eta,\xi)\left((1+\lambda)\left(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\right)^{\alpha-1}-2\lambda\left(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\right)^{2\alpha-1}\right), & \Omega_1\le x<\Omega_2,\end{cases} \tag{B.1}$$

and its *cdf* is given by

$$F(x)=\begin{cases}\beta\left((1+\lambda)\left(\frac{\psi(x;\eta,\xi)}{\beta}\right)^{\alpha}-\lambda\left(\frac{\psi(x;\eta,\xi)}{\beta}\right)^{2\alpha}\right), & -\infty<x\le\Omega_1,\\ 1-(1-\beta)\left((1+\lambda)\left(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\right)^{\alpha}-\lambda\left(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\right)^{2\alpha}\right), & \Omega_1\le x<\Omega_2,\end{cases} \tag{B.2}$$

where $\Omega_1=G^{-1}_{(x;\xi)}\left(\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\eta}}\right)$, $\Omega_2=G^{-1}_{(x;\xi)}\left(\left(\frac{1}{2}\right)^{\frac{1}{\eta}}\right)$, $\psi(x;\eta,\xi)=\frac{G^{\eta}_{(x;\xi)}}{1-G^{\eta}_{(x;\xi)}}$ and $\psi'(x;\eta,\xi)=\frac{\eta g_{(x;\xi)}G^{\eta-1}_{(x;\xi)}}{\left(1-G^{\eta}_{(x;\xi)}\right)^2}$ is derivative of $\psi(x;\eta,\xi)$ and ξ is a parameter vector in the *cdf* $G(x;\xi)$ and $G^{-1}_{(x;\xi)}(.)$ is its inverse. Also, $\psi(x;\eta,\xi)=W(G(x;\xi))$ satisfies the conditions in Alzaatreh 's method. We denote generalized odd transmuted two-sided-G family of distributions by $GOTTS{-}G(\alpha,\beta,\lambda,\eta,\xi)$. If $\lambda=0$, we get a new model with the *pdf*

$$f(x;\alpha,\beta,\eta,\xi)=\begin{cases}\alpha\psi'(x;\eta,\xi)\left(\frac{\psi(x;\eta,\xi)}{\beta}\right)^{\alpha-1}, & -\infty<x\le\Omega_1,\\ \alpha\psi'(x;\eta,\xi)\left(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\right)^{\alpha-1}, & \Omega_1\le x<\Omega_2,\end{cases} \tag{B.3}$$

and its *cdf* is given by

$$F(x;\alpha,\beta,\eta,\xi)=\begin{cases}\beta\left(\frac{\psi(x;\eta,\xi)}{\beta}\right)^{\alpha}, & -\infty<x\le\Omega_1,\\ 1-(1-\beta)\left(\frac{1-\psi(x;\eta,\xi)}{1-\beta}\right)^{\alpha}, & \Omega_1\le x<\Omega_2.\end{cases} \tag{B.4}$$

Notice that this new model is obtained by applying generalized odd quantity for two-sided power distribution and it called $GOTSP-G$.

If $\lambda = 0$ and $\alpha = \beta = 1$, we get have

$$f(x;\eta,\xi) = \psi'(x;\eta,\xi), \qquad -\infty < x \leq \Omega_2, \tag{B.5}$$

and its *cdf* is given by

$$F(x;\eta,\xi) = \psi(x;\eta,\xi), \qquad -\infty \leq x < \Omega_2. \tag{B.6}$$

B.1 Quantile function

For generating random variables from the $GOTTS-G$ distribution, one can use the inverse transformation method. The quantile of order q of the $GOTTS-G$ distribution is

$$x_q = F^{-1}(q;\alpha,\beta,\lambda,\eta,\xi) \quad = \quad \begin{cases} G^{-1}_{(x;\xi)}\left[\left(\frac{A_1}{1+A_1}\right)^{\frac{1}{\eta}}\right], & 0 < q \leq \beta, \\ \\ G^{-1}_{(x;\xi)}\left[\left(\frac{A_2}{1+A_2}\right)^{\frac{1}{\eta}}\right], & \beta \leq q < 1, \end{cases}$$

where, $A_1 = \beta\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-\frac{4\lambda q}{\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}$ and $A_2 = 1-(1-\beta)\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-\frac{4\lambda(1-q)}{1-\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}$.

C Estimation of the parameters of $GOTTS-G$ distribution

In this section, we obtain the estimation of parameters $GOTTS-G$ distribution by using maximum likelihood estimation (MLE).

C.1 Maximum likelihood estimation

We consider the estimation of the parameters of the new family from sample by maximum likelihood method. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the $GOTTS-G$ distribution and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the corresponding order statistics. The log-likelihood function is given by

$$\begin{aligned} \ell \quad = \quad & n\log\alpha + \sum_{i=1}^{n}\log(\psi'(x;\eta,\xi)) \\ & + \log\left\{\prod_{i=1}^{r}\left((1+\lambda)\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{2\alpha-1}\right)\right. \\ & \left.\times\prod_{i=r+1}^{n}\left((1+\lambda)\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{2\alpha-1}\right)\right\}, \end{aligned}$$

where $x_{r:n} \leq \Omega_1 < x_{r+1:n}$ for $r = 1, 2, \ldots, n$ and $X_{0:n} = -\infty$, $X_{n+1:n} = \infty$.

For estimating the parameters, we obtain the partial derivatives of the log-likelihood function with respect to the parameters. According to Van Dorp and Kotz (2002a), we can find the maximum likelihood estimates (MLE's) of parameters. We first consider the MLE's of α and β when the parameters λ, η and ξ are known. At the corner point β, the log-likelihood function for the $GOTTS-G$ distribution is not differentiable and we can not find the estimate of β in a regular way. Without loss of generality, we assume that $\lambda = 0$. So, the log-likelihood function will be

$$\begin{aligned}
\ell &= n \log \alpha + \sum_{i=1}^{n} \log(\psi'(x_i; \eta, \xi)) \\
&\quad + \log \left\{ \prod_{i=1}^{r} \left(\frac{\psi(x_{i:n}; \eta, \xi)}{\beta} \right)^{\alpha-1} \prod_{i=r+1}^{n} \left(\frac{1 - \psi(x_{i:n}; \eta, \xi)}{1 - \beta} \right)^{\alpha-1} \right\} \\
&= n \log \alpha + \sum_{i=1}^{n} \log(\psi'(x_i; \eta, \xi)) \\
&\quad + (\alpha - 1) \log \left\{ \frac{\prod_{i=1}^{r} \psi(x_{i:n}; \eta, \xi) \prod_{i=r+1}^{n} (1 - \psi(x_{i:n}; \eta, \xi))}{\beta^r (1-\beta)^{n-r}} \right\}.
\end{aligned}$$

According to Van Dorp and Kotz (2002a) and Korkmaz and Genç (2017), the MLE's of α and β are as follows

$$\hat{\alpha} = -\frac{n}{\log M(\hat{r}, \eta, \xi)},$$

$$\hat{\beta} = \psi(x_{\hat{r}:n}; \eta, \xi),$$

where $\hat{r} = arg \max M(r, \eta, \xi)$, $r \in \{1, 2, \ldots, n\}$ with

$$M(r, \eta, \xi) = \prod_{i=1}^{r-1} \frac{\psi(x_{i:n}, \eta, \xi)}{\psi(x_{r:n}, \eta, \xi)} \prod_{r+1}^{n} \frac{1 - \psi(x_{i:n}, \eta, \xi)}{1 - \psi(x_{r:n}, \eta, \xi)}.$$

By taking the derivative of the log-likelihood function with respect to parameter vector ξ and parameters η and λ, the MLE's of parameters ξ, η and λ are obtained by equating it to zero. These derivative are given as

$$\begin{aligned}
\frac{\partial \ell}{\partial \lambda} &= \sum_{i=1}^{\hat{r}} \frac{\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{\alpha-1} - 2\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{2\alpha-1}}{(1+\lambda)\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{2\alpha-1}} \\
&\quad + \sum_{i=\hat{r}+1}^{n} \frac{\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{\alpha-1} - 2\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{2\alpha-1}}{(1+\lambda)\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{2\alpha-1}},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \xi_k} &= \sum_{i=1}^{n} \frac{\psi''(x_{i:n};\eta,\xi)}{\psi'(x_{i:n};\eta,\xi)} \\
&+ \sum_{i=1}^{\hat{r}} \frac{\psi'(x_{i:n};\eta,\xi)}{\beta} . \frac{(1+\lambda)(\alpha-1)\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{\alpha-2} - 2\lambda(2\alpha-1)\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{2\alpha-2}}{(1+\lambda)\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{\psi(x_{i:n};\eta,\xi)}{\beta}\right)^{2\alpha-1}} \\
&+ \sum_{i=\hat{r}+1}^{n} \frac{-\psi'(x_{i:n};\eta,\xi)}{1-\beta} . \frac{(1+\lambda)(\alpha-1)\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{\alpha-2} - 2\lambda(2\alpha-1)\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{2\alpha-2}}{(1+\lambda)\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-\psi(x_{i:n};\eta,\xi)}{1-\beta}\right)^{2\alpha-1}},
\end{aligned}
$$

where $\psi''(t;\eta,\xi) = \frac{\partial \psi'(t;\eta,\xi)}{\partial \xi_k}$ and $\psi'(t;\eta,\xi) = \frac{\partial \psi(t;\eta,\xi)}{\partial \xi_k}$.

However, these equations are nonlinear and there are no explicit solutions. Thus, they have to be solved numerically. So, the *optim* function is used for estimating the parameters in *R* program.

D Generalized odd transmuted two-sided exponential distribution

The $GOTTS-G$ distribution is specialized by taking G as the well-known distribution. We suppose that the base distribution G has an exponential distribution with *pdf*, *cdf* and inverse *cdf* functions $g(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$, $G(x;\theta) = 1 - e^{-\frac{x}{\theta}}, x > 0, \theta > 0$ and $G^{-1}(x;\theta) = -\theta \log(1-x)$, respectively. By substituting $g(x;\theta)$ and $G(x;\theta)$ in equations $\psi(x;\eta,\xi)$ and $\psi'(x;\eta,\xi)$, the *pdf* of $GOTTS-G$ distribution can be given as

$$
f(x) = \begin{cases} \alpha\psi'(x;\eta,\theta)\left((1+\lambda)\left(\frac{\psi(x;\eta,\theta)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{\psi(x;\eta,\theta)}{\beta}\right)^{2\alpha-1}\right), & 0 < x \le \Omega_3, \\ \alpha\psi'(x;\eta,\theta)\left((1+\lambda)\left(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\right)^{2\alpha-1}\right), & \Omega_3 \le x < \Omega_4, \end{cases} \tag{D.1}
$$

and its *cdf* is given by

$$
F(x) = \begin{cases} \beta\left((1+\lambda)\left(\frac{\psi(x;\eta,\theta)}{\beta}\right)^{\alpha} - \lambda\left(\frac{\psi(x;\eta,\theta)}{\beta}\right)^{2\alpha}\right), & -\infty < x \le \Omega_3, \\ 1-(1-\beta)\left((1+\lambda)\left(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\right)^{\alpha} - \lambda\left(\frac{1-\psi(x;\eta,\theta)}{1-\beta}\right)^{2\alpha}\right), & \Omega_3 \le x < \Omega_4, \end{cases} \tag{D.2}
$$

where $\Omega_3 = -\theta \ln\left(1 - \left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\eta}}\right)$ and $\Omega_4 = -\theta \ln\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\eta}}\right)$.

We call this distribution the odd transmuted two-sided generalized exponential distribution and denote by $GOTTS-E(\alpha,\beta,\lambda,\eta,\theta)$. Figure 1 showes the graphs of the densities of the $GOTTS-E$ distribution with $0 \le \lambda \le 1$.

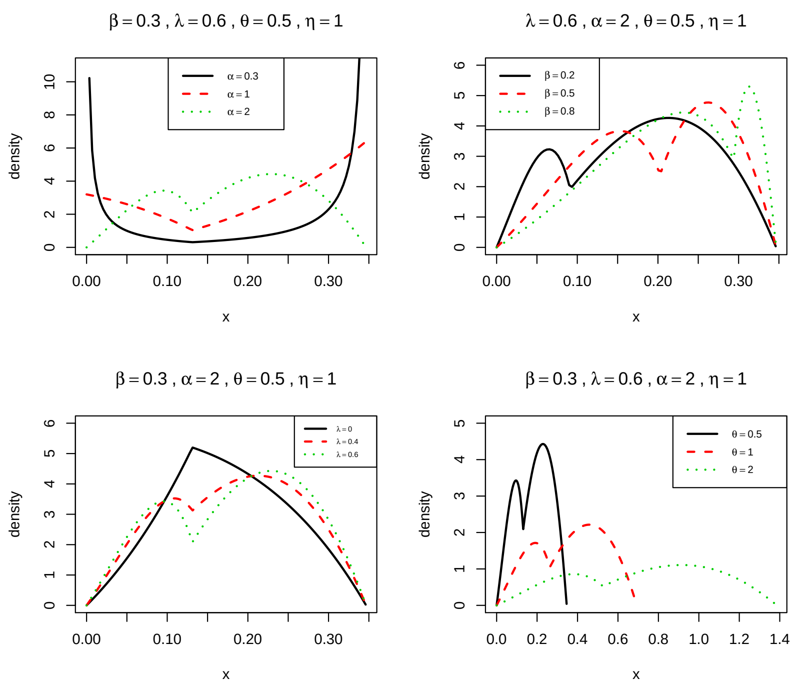

Figure 1: The graphs of the densities of the $GOTTS-E$ distribution with $0 \leq \lambda \leq 1$.

D.1 Hazard function of the $GOTTS-E$ distribution

Generally, the hazard function of a distribution is

$$r(x) = \frac{f(x;\alpha,\beta,\lambda,\eta,\xi)}{1-F(x;\alpha,\beta,\lambda,\eta,\xi)}. \tag{D.3}$$

Now, the hazard rate function of the $GOTTS-G$ can be obtained by substituting relations (B.1) and (B.2) in (D.3). In special case, when the parent distribution is exponential, one can calculate the hazard rate function of the $GOTTS-E$ function.

E Application of the $GOTTS-E$ distribution

To investigate the advantage of the proposed distribution, we consider a real data set provided by Bjerkedal (1960). The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temper- ature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the

Table 1: The MLE's of parameters for data set.

Model	Estimation	Log-likelihood	AIC	BIC
$GOTTSE$	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta}, \hat{\eta})$=(2.362, 0.855, 0.893,12.807, 0.596)	-122.286	254.571	266.726
$TTSGE$	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$=(2.330,0.953, -0.790, 1.471)	-127.144	262.287	272.011
gamma	$(\hat{\alpha}, \hat{\lambda})$=(3.492, 1.365)	-136.937	277.874	282.735
Weibull	$(\hat{\beta}, \hat{\lambda})$=(2.374,2.863)	-130.053	264.107	268.968
TSGE	$(\hat{\alpha}, \hat{\beta}, \hat{\theta})$=(3.211, 0.922, 1.691)	-130.979	267.958	275.250
GE	$(\hat{\alpha}, \hat{\lambda})$=(3.562, 0.758)	-139.841	283.681	288.543

nonstructural outer ply or failure of the heating system. The data are given below:

Failure times of 84 Aircraft Windshield

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

E.1 MLE inference and comparing with other models

We fit the proposed distribution to the real data set by MLE method and compare the results with the gamma, Weibull, two-sided generalized exponential ($TSGE$), transmuted two-sided generalized exponential ($TTSGE$) and generalized exponential (GE) distributions.

Here we provide numerical results for the real data set. For each model, Table 1 includes the MLE's of parameters, corresponding log-likelihood, Akaike information criterion (AIC) and Bayesian Akaike information criterion (BIC) for the first data set. We fit the $GOTTSE$ distribution to the real data set and compare it with the mentioned distributions. The selection criterion is that the lowest AIC and BIC statistic corresponding to the best fitted model. The $GOTTSE$ distribution provides the best fit for the data set as it has lower AIC and BIC statistic than the other competitor models. The histogram of data set, fitted *pdf* of $GOTTSE$ distribution and fitted *pdf*s of other competitor distributions for the real data set are plotted in Figure 2. The plots of empirical and fitted *cdf*s functions, $P-P$ plots and $Q-Q$ plots for the $OGTTSE$ and other fitted distributions are displayed in Figures 2. These plots also support the results in Table 1.

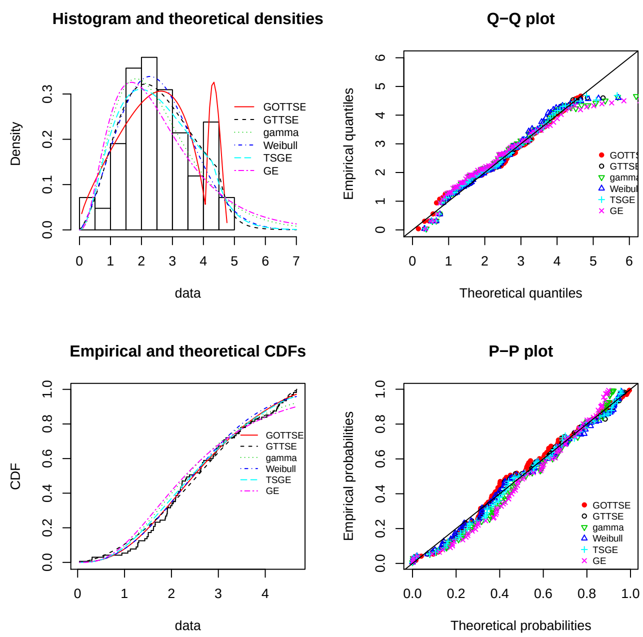

Figure 2: $P-P$ plots of fitted pdfs for the first data set.

References

[1] Alzaatreh, A., Lee, C. and Famoye, F. (2013), A new method for generating families of continuous distributions, *Metron*, **71**(1), 63-79.

[2] Bjerkedal, T. (1960), Acquisition of Resistance in Guinea Pies infected with Different Doses of Virulent Tubercle Bacilli, *American Journal of Hygiene*, **72**(1), 130-48.

[3] Kharazmi, O. and Zargar, M. (2018), A new generalized two-sided class of the distributions via new transmuted two-sided bounded distribution, *Journal of Statistical Theory and Applications*, Accepted.

[4] Korkmaz, M.C., and Genç, A. İ. (2015b), Two-Sided Generalized Exponential Distribution, *Communications in Statistics-Theory and Methods*, **44**(23), 5049-5070.

[5] Van Dorp, J.R. and Kotz, S. (2002a), The standard two-sided power distribution and its properties: with applications in financial engineering, *The American Statistician*, **56**(2), 90-99.

[6] Van Dorp, J.R. and Kotz, S. (2002b), A novel extension of the triangular distribution and its

parameter estimation, *Journal of the Royal Statistical Society: Series D (The Statistician)*, **51**(1), 63-79.

[7] Van Dorp, J.R. and Kotz, S. (2003), Generalizations of two-sided power distributions and their convolution, *Communications in Statistics-Theory and Methods*, **32**(9), 1703-1723.

Hazard Rate and Reversed Hazard Rate of k-out-of-n:F Systems in a Single Outlier Model

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Abstract: The hazard rate function of a k-out-of-n:F system is studied in the situation in which all observations except one have the same distribution. Such model is known as a single outlier model. The sensitivity of the hazard rate function with respect to the outlier is investigated in some different cases. Analogously, the reversed hazard rate function is computed. Moreover, the aging behavior of series and parallel systems are of interest in this paper.

Keywords Reliability function, Series system, Parallel system, Proportional (reversed) hazard rate model, Order statistics.

Mathematics Subject Classification (2010) : 62N05**.**

A Introduction

The k-out-of-n systems are of interest in the literature of reliability. A k-out-of-n:F system consists of n components which fails if and only if at least k of the components fail. Such systems have various applications in engineering. For more details, we refer to (13). In the special cases of $k = 1$ and $k = n$, the series and parallel systems are deduced, respectively.

The hazard rate function is an important measure to study the lifetime random variable in reliability theory, survival analysis and stochastic modeling. Let T be a random variable representing the lifetime of living organism or of a system having an absolutely continuous cumulative distribution function $F(\cdot)$, reliability function $\bar{F}(\cdot)$ and probability density function $f(\cdot)$. The hazard rate function of T is given by

$$h_F(t) = \frac{f(t)}{\bar{F}(t)}. \tag{A.1}$$

The $h_F(t)dt$ is the conditional probability that the system is failed between time t and $t + dt$, given that it was not failed up to time t. If $h_F(t)$ is increasing, constant or decreasing in t, then F is belong to the class of distributions with increasing failure rate (IFR), constant failure

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rate (CFR) of decreasing failure rate (DFR) property, respectively. There are also systems with bathtub shaped or inverse bathtub shape hazard rate functions. See, Lawless (13) for more details.

Similarly, Barlow (3) introduced the concept of reversed hazard rate function of T as

$$r_F(t) = \frac{f(t)}{F(t)}.$$

To interpret the reversed hazard rate function, note that $r_F(t)dt$ is the conditional probability that the system is failed in an infinitesimal interval of width dt preceding t, given that it failed before t. For more applications of reversed hazard rate, see (7) and (5). In general, the reversed hazard rate has been found to be useful in estimating the survival function for left censored data, see (8), (14) and (4) have presented several interesting results regarding the reversed hazard rates. If $r_F(t)$ is decreasing in t, then F is belong to the class of distributions with decreasing reversed failure rate (DRFR).

Many researchers have studied the k-out-of-n:F system assuming the lifetimes of the components are independent and identically distributed random variables. But, there are some situations in life-testing and reliability experiments in which the observations are independent but not identically distributed. Khanjari (9) investigated mean past and mean residual life functions of a parallel system with nonidentical components. Tavangar and Bayramoglu (15) studied the residual lifetimes of coherent systems with exchangeable components.

In this paper, a single outlier model is investigated in which the distribution of one of the observations is different from the others. Such model had been previously studied by (10) in both problems of estimation and prediction of future order statistics under Type-II censoring. Predicting the lifetime of a k-out-of-n:F system in this model Was also considered by (11). Indeed, it is assumed that $(n-1)$ component of a k-out-of-n:F system have the cumulative distribution function (cdf) $F(\cdot)$, whereas the last component has the cdf $G(\cdot)$. We focus on the hazard rate and reversed hazard rate of a k-out-of-n:F system as well as the special cases of series and parallel system.

The rest of paper is as follows. In Section 2, the hazard rate of a k-out-of-n:F system in is studied. In this section, the special case of series system is also studied in view of some aging properties. The reverse hazard rate of a k-out-of-n:F system has been considered in Section 3. This section includes some reliability results about the parallel systems. Finally, some conclusions are stated in Section 4.

B Hazard rate of system

Let us denote the lifetimes of the components of a k-out-of-n:F system by $T_1, \ldots, T_n$ which are independent random variables, such that $T_1, \ldots, T_{n-1}$ come from a population with the cdf $F(\cdot)$ and pdf $f(\cdot)$; moreover, T_n is an outlier from a different population with the cdf $G(\cdot)$ and pdf $g(\cdot)$. Denote the corresponding order statistics by $T_{1:n} < \cdots < T_{n:n}$. Then, it is obvious that the lifetime of a k-out-of-n:F system is $T_{k:n}$. From (2), the pdf of $T_{k:n}$ $(1 \leq k \leq n)$ in the presence of an outlier is

$$\begin{aligned}
f_{k:n}(t) \quad = \quad & \frac{(n-1)!}{(k-2)!(n-k)!}[F(t)]^{k-2}G(t)f(t)[\bar{F}(t)]^{n-k} \\
& + \frac{(n-1)!}{(k-1)!(n-k)!}[F(t)]^{k-1}g(t)[\bar{F}(t)]^{n-k} \\
& + \frac{(n-1)!}{(k-1)!(n-k-1)!}[F(t)]^{k-1}f(t)[\bar{F}(t)]^{n-k-1}\bar{G}(t), \qquad \text{(B.1)}
\end{aligned}$$

where the first and last terms vanish when $k = 1$ and $k = n$, respectively.

It is also not difficult to show that the reliability function of $T_{k:n}$ $(1 \leq k \leq n)$ is as follow

$$\begin{aligned}
\bar{F}_{k:n}(t) \quad = \quad & [\bar{F}(t)]^{n-1}\bar{G}(t)I(k=1) \\
& + \Bigg\{ \sum_{i=0}^{k-1} \binom{n-1}{i} F^i(t)[\bar{F}(t)]^{n-1-i} \\
& \quad - \binom{n-1}{k-1}[F(t)]^{k-1}[\bar{F}(t)]^{n-k}\bar{G}(t) \Bigg\} I(1 < k < n) \\
& + \big(1 - [F(t)]^{n-1}G(t)\big)I(k=n). \qquad \text{(B.2)}
\end{aligned}$$

where $I(A)$ stands for the indicator on event A. Therefore, using (A.1), (B.1) and (B.2), the hazard rate function of a k-out-of-n:F system is given by

$$h_{k:n}(t) = \frac{f_{k:n}(t)}{\bar{F}_{k:n}(t)}. \qquad \text{(B.3)}$$

For more investigation about the sensitivity of $h_{k:n}(t)$ with respect to the outlier, let us assume $F(\cdot)$ and $G(\cdot)$ to be the cdfs of exponential and weibull distributions with the cdfs

$$F(t) = 1 - e^{\theta t} \quad \text{and} \quad G(t) = 1 - e^{-(\lambda t)^{\beta}}, \quad t > 0,$$

respectively. Note that $F(t)$ is CFR; on the other hand, for all values of λ, $G(t)$ is IFR, CFR or DFR for $\beta > 1, \beta = 1$ or $\beta < 1$, respectively. The behavior of $h_{k:n}(t)$ in the presence of outlier (solid lines) and without any outlier (dash lines) is plotted in Figure 1 for $n = 5, \theta = 2$ and $\lambda = 3$ and some choices of k and β. From this figure, the following results are deduced:

- Although the hazard rate function of series (1-out-of-5:F) system is constant when there is no outlier, it is decreasing (or increasing) when an outlier with wibull distribution exists such that $\beta < 1$ (or $\beta > 1$).
- The behavior of hazard rate function of other k-out-of-5:F systems for $k = 2, 4, 5$ in presence of an outlier is similar to that when there is no outlier, but, the values of this function are so different, specially around the mode of hazard function.

Here, the aging behavior of a series system is studied, theoretically. As previously mentioned, the lifetime of a series system is presented by $T_{1:n}$, whose cdf is $F_{1:n}$.

If $F(t)$ and $G(t)$ are both IFR, CFR or DFR, then $F_{1:n}$ is also IFR, CFR or DFR, respectively.

. For $k = 1$ and substituting (B.1) and (B.2) into (C.1), the hazard rate function of a series system is given by

$$\begin{aligned} h_{1:n}(x) &= \frac{g(t)[\bar{F}(t)]^{n-1} + (n-1)f(t)[\bar{F}(t)]^{n-2}\bar{G}(t)}{[\bar{F}(t)]^{n-1}\bar{G}(t)} \\ &= h_G(x) + (n-1)h_F(x). \end{aligned} \tag{B.4}$$

Hence, the proof is complete. □

Now, suppose that the hazard rate functions of the outlier and other observations follow a proportional hazard rate model (PHRM), in which for a positive constant α, we have

$$h_G(t) = \alpha h_F(t),$$

where $h_F(\cdot)$ is the hazard rate function of $T_1, \ldots, T_{n-1}$ and $h_G(\cdot)$ is the hazard rate function of T_n. In this case, we get

$$\bar{G}(t) = [\bar{F}(t)]^{\alpha}.$$

The PHRM includes several well-known lifetime distributions such as: Exponential, Pareto, Lomax, Burr type XII, Weibull (one parameter) and so on. For more details, see for example (13). In such a model, using (B.4), the hazard rate function of a series system is as follows:

$$h_{1:n}(t) = (n-1+\alpha)h_F(t).$$

Therefore, the behavior of $h_{1:n}(t)$ is identical to $h_F(t)$. For example, if F is IFR, CFR or DFR, then $F_{1:n}$ is also IFR, CFR or DFR, respectively.

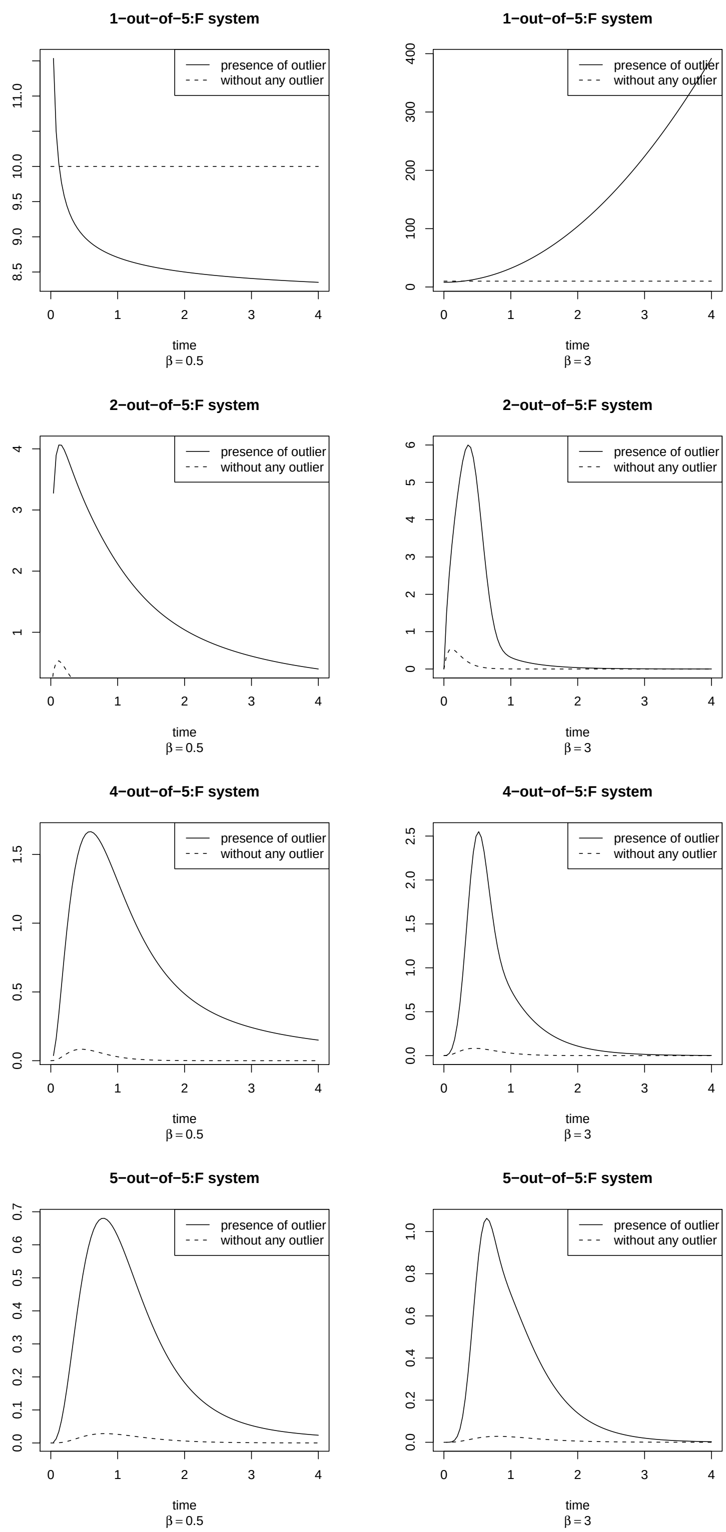

Figure 1: Plots of $h_{k:5}(t)$ for $\theta = 2$, $\beta = 0.5, 3$, $\lambda = 2$ and $k = 1, 2, 4, 5$.

C Reversed Hazard rate of system

Analogous to the previous section, reversed hazard rate function may be also considered to study the reliability properties of a k-out-of-n:F system. Using (A.2), (B.1) and (B.2), we get

$$r_{k:n}(t) = \frac{f_{k:n}(t)}{F_{k:n}(t)}. \tag{C.1}$$

Here, it is of interest to study the behavior of reversed hazard rate function of a parallel system. A parallel system, consisting of n components, is a system which functions if and only if at least one of its n components functions. As mentioned earlier in k-out-of-n:F system, where $k = n$, the system is a parallel system. Hence, the lifetime of a parallel system with n components is $T_{n:n}$ and the associated reversed hazard rate function is

$$\begin{aligned} r_{n:n}(x) &= \frac{(n-1)[F(t)]^{n-2}G(t)f(t) + [F(t)]^{n-1}g(t)}{[F(t)]^{n-1}G(t)} \\ &= (n-1)r_F(x) + r_G(x). \end{aligned}$$

Therefore, if F and G are both DRFR, then the cdf of $T_{n:n}$ is also DRFR.

If the reversed hazard rate functions of outlier and other observations satisfy

$$r_G(t) = \beta r_F(t), \tag{C.2}$$

where β is a positive parameter, then we will say that the observations obey a proportional reversed hazard rate model (PRHRM) with proportionality constant rate β. In such model $G(t) = [F(t)]^{\beta}$. This model was proposed by (6) in contrast to the celebrated PHRM. They studied the monotonicity of the reversed failure rates in the case of exponentiated Weibull, exponentiated exponential, exponentiated Pareto and exponentiated Gamma family of distributions. Some properties of reversed hazard rate of order statistics and record values are studied in (12). It is easy to show that in a PRHRM in (C.2),

$$r_{n:n}(t) = (n - 1 + \beta)r_F(t).$$

Therefore, the behavior of $r_{n:n}(t)$ is identical to $r_F(t)$. For example, if F is DRFR, then $F_{n:n}$ is also DRFR.

D Concluding Remark

A single outlier model was considered in this paper. Indeed, a system consisting n components was of interest such that $(n-1)$ components had the cdf F, whereas the last component came from the cdf G. The hazard rate function of a k-out-of-n:F system with one outlier was computed and its sensitivity was investigated with respect to the outlier, when F and G are the cdfs of exponential and weibull distributions, respectively; it was shown that the hazard rate function of a series system is completely different when an outlier exists, but it is not so sensitive for other k-out-of-n:F systems with $2 \leq k \leq n$. The sensitivity of hazard rate function of a k-out-of-n:F system may be studied for other distributions. It was also deduced that when F and G are both IFR, CFR or DFR, the cdf of a series system is IFR, CFR or DFR, respectively. Further, in a proportional hazard rate model, it was stated that the aging behavior of a series system is as the same of F. The reversed hazard rate function of a k-out-of-n:F system was also presented. It was proved that when F and G are both DRFR, then a parallel system is also DRHR. The results of this paper may be extended to other reliability concepts such as mean residual life, mean past life, reliability orderings, etc. In this paper, a single outlier model was considered. Studying about the coherent systems with more than one outlier is under progress.

References

[1] Bairamov, I. and Arnold, B.C. (2008), On the residual life lengths of the remaining components in an $(n-k+1)$-out-of-n system, *Statistics Probability Letters*, **78**, 945-952.

[2] Balakrishnan, N. (2007), Permanents, Order Statistics, Outliers, and Robustness, *Revista matemtica complutense*, **20**, 7-107.

[3] Barlow, R.E., Marshall, A.W. and Proschan, F. (1963), Properties of probability distributions with monotone hazard rate, *The Annals of Mathematical Statistics*, **34**, 375-389.

[4] Chandra, N.N. and Roy, D. (2001), Some results on reversed hazard rate, *Probability in the Engineering and informational Sciences*, **15**, 95-102.

[5] Eeckhoudt, L. and Gollier, C. (1995), Demand for risky assets and the monotone probability ratio order, *Journal of Risk and Uncertainty*, **11**, 113-122.

[6] Gupta, R.C., Gupta, P.L. and Gupta, R.D. (1998), Modeling failure time data by Lehman alternatives, *Communications in Statistics-theory and Methods*, **27**(4), 887-904.

[7] Gupta, R.D. and Nanda, A.K. (2001), Some results on (reversed) hazard rate ordering, *Communications in Statistics-theory and Methods*, **30**(11), 2447-2458.

[8] Kalbfleisch, J.D. and Lawless, J.F. (1989), Inference based on retrospective ascertainment: an analysis of the data on transfusion-related, *Journal of the American Statistical Association*, **84**, 360-372.

[9] Khanjari, S.M. (2008), Mean past and mean residual life functions of a parallel system with nonidentical components, *Communications in Statistics, Theory and Methods*, **37**, 1134–1145.

[10] Khatib, B and Razmkhah, M. (2018), Estimation and Prediction in the Presence of an Outlier under Type-II Censoring, *Journal of Statistical Computation and Simulation*, **88**(17),1-18.

[11] Khatib, B (2017), Predicting the lifetime of a k-out-of-n system in the presence of an outlier, *Proceeding of the 3rd Seminar on Reliability Theory and its Applications*, 166-171.

[12] Kundua, C., Nandaa, A. and Taizhong, H. (2009), A note on reversed hazard rate of order statistics and record values, *Journal of Statistical Planningand Inference*, **139**, 1257-1265.

[13] Lawless, J.F. (2003), *Statistical models and methods for lifetime data*, Wiley Series in Probability and Statistics, Second Edition.

[14] Sengupta, D. and Nanda, A.K. (1999), Logconcave and concave distributions in reliability, *Naval Research Logistics (NRL)*, **46**(4), 419-433.

[15] Tavangar, M. and Bayramoglu, I. (2012), On the residual lifetimes of coherent systems with exchangeable components, *Pakistan Journal of Statistics*, **28**, 303-313.

Some Results on Stress-Strength Model in General Form of Discrete Lifetime Distribution

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Abstract:

This paper deals with inference for stress-strength interference(SSI), $R = P(X < Y)$, where the distributions of X and Y follow discrete proportional hazard rate (PHR) models. In special case of Geometric PHR, maximum likelihood estimator (MLE) of R and it variances are obtained. Also approximation problem of the estimator's variance of R are discussed via Bhattacharyya lower bounds for Poisson SSI model.

Keywords Reliability function, Stress-Strength model, Telescopic form.

Mathematics Subject Classification (2010) : 62N05, 90B25.

A Introduction

$R = P(Y < X)$ is a measure of component reliability, which provides a general measure of the difference between two populations and has applications in many areas such as clinical trials, genetics, and reliability. For example, if Y is the response for a control group, and X refers to a treatment group, R is a measure of the effect of the treatment. Or, if Y is the water pressure on the dam wall, and X be the strength of the dam, then the parameter R is of very important in maintenance.

A lot of authors on various topics have done extensive research on stress-strength models and a good review of many papers on theory and applications about $R = P(Y < X)$ can be found on Kotz et al. (9) and Patowary et al. (11) in continuous cases.

On the other hand, discrete lifetime distributions and their application in real-life situations greatly increase the importance of studying the concepts of reliability for this case. A few works have been done when the stress Y and strength X are taken to be discrete random variables. The first works on discrete cases are done by Maiti (10) in the geometric case and Chaturvedi

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and Tomer (5) and Sathe and Dixit (14) in the negative binomial case. Behboodian (3) find properties of $P(X > Y)$ for binomial distribution.

B Telescopic form of discrete distributions

If X has extended exponential distribution then its pdf and reliability function are respectively given by

$$
\begin{aligned}
f(x) &= \alpha k'_\theta(x) e^{-\alpha k_\theta(x)}; \quad x > 0, \alpha > 0; && \text{(B.1)} \\
R(x) &= e^{-\alpha k_\theta(x)}, && \text{(B.2)}
\end{aligned}
$$

where $k_\theta(x)$ is a strictly increasing function of x with $k_\theta(0) = 0$ and $k_\theta(x) \to \infty$ as $x \to \infty$ (Al-Hussaini(1)). Rezaei et al. (12) have introduced a general form of any discrete distribution and named it *"Telescopic form of distribution"* by discretizing the extended exponential distribution as

$$
\begin{aligned}
p_X(x) &= q^{k_\theta(x)} - q^{k_\theta(x+1)}, \quad x = 0, 1, 2, \ldots && \text{(B.3)} \\
S_X(t) &= P(X \geq t) = q^{k_\theta(t)}.
\end{aligned}
$$

and denoted by $X \sim T(q, k_\theta)$.

Some of important distributions that belong to this family are: discrete exponential (geometric), discrete Rayleigh, discrete Weibull, discrete linear exponential, discrete Gompertz and discrete Burr III and XII.

C Discrete Stress Strength reliability Model

If X and Y are two independent discrete random variables with PMF of forms,

$$p_X(x) = q_1^{k_\theta(x)} - q_1^{k_\theta(x+1)}, \quad x = 0, 1, 2, \ldots,$$

and

$$p_Y(y) = q_2^{k_\theta(y)} - q_2^{k_\theta(y+1)}, \quad y = 0, 1, 2, \ldots,$$

respectively, we have,

$$\begin{aligned} R = P(Y \leq X) &= \sum_{x=0}^{\infty} P(Y \leq x) p_X(x) \\ &= \sum_{x=0}^{\infty} (1 - q_2^{k_\theta(x+1)})(q_1^{k_\theta(x)} - q_1^{k_\theta(x+1)}) \\ &= 1 + \sum_{x=0}^{\infty} (q_1 q_2)^{k_\theta(x+1)} - \sum_{x=0}^{\infty} q_1^{k_\theta(x)} q_2^{k_\theta(x+1)}. \end{aligned} \tag{C.1}$$

Similarly on can obtain,

$$R^* = P(Y < X) = 1 - \sum_{x=0}^{\infty} (q_1 q_2)^{k_\theta(x)} + \sum_{x=0}^{\infty} q_1^{k_\theta(x+1)} q_2^{k_\theta(x)}.$$

As special cases for Geometric distribution (i.e. $k_\theta(x) = x$) we have,

$$\begin{aligned} R = P(Y \leq X) &= \frac{1 - q_2}{1 - q_1 q_2}, \\ R^* = P(Y < X) &= \frac{q_1(1 - q_2)}{1 - q_1 q_2}, \end{aligned} \tag{C.2}$$

which is coincide with the result of Maiti (10).

As it is seen, in some discrete case, obtaining a closed functional form for parameter R is not always straightforward, since the series (C.1) are not converge for some $k_\theta(x)$. But in most cases the parameter R can be approximate by the first terms of series. Indeed the sums of (C.1) are rapidly converge to a specified quantity.

C.1 MLE of R:

For obtianing the MLE of R we should first compute the MLE of the parameters q_1 and q_2. Suppose $X_i \sim T(q, k_\theta), i = 1, \ldots, n$ with known k_θ and unknown parameter q. Then from maximum likelihood method $\hat{q}_{ml}$ is obtainable as the solution of the following equation:

$$\sum_{i=1}^{n} \frac{k_\theta(x)}{1 - q^{\Delta k_\theta(x)}} = \sum_{i=1}^{n} \frac{k_\theta(x+1)}{q^{-\Delta k_\theta(x) - 1}}, \tag{C.3}$$

which is not generally available in closed form; therefore, numerical methods like Newton-Raphson may be employed. But in discrete distributions another method called "*method of proportions*" have been proposed by Khan et al. (7). Based on this method for distributions of form (B.3) we have

$$p_X(0) = p(X = 0) = 1 - q^{k_\theta(1)}. \tag{C.4}$$

Therefore if Y be the number of zero's in the sample then the proportion $\frac{Y}{n}$ estimates the probability $p_X(0)$, so $\hat{q}_{mp}$ as the estimator of q, can be expressed in a simple form of

$$\hat{q}_{mp} = \left(1 - \frac{Y}{n}\right)^{\frac{1}{k_\theta(1)}}. \tag{C.5}$$

Our simulation results indicate that the likelihoods of $\hat{q}_{ml}$ and $\hat{q}_{mp}$ are equal to the high precision (< 0.00001), so $\hat{q}_{mp}$ can be used instead of $\hat{q}_{ml}$. Therefore, with a little ignore, the closed form of MLE of R can be expressed as

$$\hat{R} = 1 + \sum_{x=0}^{c} (\hat{q_1}_{mp}\hat{q_2}_{mp})^{k_\theta(x+1)} - \sum_{x=0}^{c} \hat{q_1}_{mp}^{k_\theta(x)} \hat{q_2}_{mp}^{k_\theta(x+1)}, \tag{C.6}$$

where c is a constant in which the series (C.1) is converge. Imani and Khorashadizadeh (6) have used this method for discrete Weibull SSI model.

D Stress Strength reliability inference in PHR models

Let X and Y are two non-negative discrete random variables with reliability functions $R(t)$ and $S(t)$ respectively, then we say X and Y satisfying proportional hazard model with resilience parameter $\theta > 0$ if,

$$S(t) = [R(t)]^\theta, \forall t = 0, 1, \ldots, \tag{D.1}$$

where $R(t)$ is regarded as baseline distribution function (15). The model (D.1) can be different specific models based on the different choices of the baseline function, such as Geometric, discrete Weibull, discrete Rayleigh and so on. In the continuous case the model (D.1) is known as the PHM where the hazard rate corresponding to the random variable Y is proportional to the hazard rate of X (i.e. $h_Y(t) = \theta h_X(t)$). However, the model (D.1) does not yield proportional hazards in the discrete setup when we use $h_X(t) = \frac{p_X(t)}{P(X \geq t)} = 1 - \frac{R(t+1)}{R(t)}$ as the hazard rate of X. Actually in model (D.1) we have, $h_Y(t) = 1 - [1 - h_X(t)]^\theta$. But if we use the alternative hazard rate function in discrete lifetime models defined by Roy and Gupta (13) as $h_X^*(t) = -\ln \frac{S(t+1)}{S(t)}$, the model (D.1) will satisfy the proportional hazard rate in discrete case.

Let X (the strength) and Y (the stress) are independent and have proportional CDF to common baseline CDF, $S_0(t)$, with proportional parameters α and β respectively. So, if the baseline distribution, $S_0(t)$, be a discrete distribution with pdf of form (B.3), we have,

$$\begin{aligned} S_X(t) &= [S_0(t)]^\alpha = q^{\alpha k_\theta(t)}, \\ S_Y(t) &= [S_0(t)]^\beta = q^{\beta k_\theta(t)}. \end{aligned}$$

So,

$$
\begin{aligned}
R = P(Y \leq X) &= \sum_{t=0}^{\infty} P(X \geq t) P(Y = t) \\
&= \sum_{t=0}^{\infty} S_0(t)^{\alpha+\beta} - \sum_{t=0}^{\infty} S_0(t)^{\alpha} S_0(t+1)^{\beta}
\end{aligned}
$$

Example (Geometric PHR model)

Suppose the SSI model with X and Y are proportional hazard rate to common baseline geometric distribution. (i.e. $S_0(t) = q^t$), then for $t = 0, 1, \ldots$,

$$
\begin{gathered}
S_X(t) = q^{\alpha t},\ p_X(t) = q^{\alpha t}(1 - q^{\alpha}), \\
S_Y(t) = q^{\beta t},\ p_Y(t) = q^{\beta t}(1 - q^{\beta}),
\end{gathered}
$$

so, the reliability parameter is given by,

$$
\begin{aligned}
R = P(Y \leq X) &= \sum_{t=0}^{\infty} S_X(t) p_Y(t) \\
&= \frac{1 - q^{\beta}}{1 - q^{\alpha+\beta}}.
\end{aligned}
$$

For estimating R by maximum likelihood (ML) approach, first, we obtain the MLEs of α and β. Suppose that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are two samples coming from $S_X(t)$ and $S_Y(t)$ respectively. Then, the log likelihood function of combining two samples is

$$
l = l(q, \alpha, \beta) = n \ln(q^{\alpha} - 1) + \alpha \ln(q) \sum_{i=1}^{n} x_i + m \ln(q^{\beta} - 1) + \beta \ln(q) \sum_{i=1}^{m} y_i,
$$

Likelihood equations are then obtained as follows,

$$
\begin{aligned}
\frac{\partial l}{\partial \alpha} &= \frac{q^{\alpha} \ln(q)}{q^{\alpha} - 1} + \overline{X} \ln(q) = 0, \\
\frac{\partial l}{\partial \beta} &= \frac{q^{\beta} \ln(q)}{q^{\beta} - 1} + \overline{Y} \ln(q) = 0, \\
\frac{\partial l}{\partial q} &= \frac{\alpha}{m} \left(\frac{1}{1 - q^{-\alpha}} + \overline{X} \right) + \frac{\beta}{n} \left(\frac{1}{1 - q^{-\beta}} + \overline{Y} \right) = 0
\end{aligned} \tag{D.2}
$$

Solving the likelihood equations with respect to α and β we get that theMLE's for α and β are

$$
\begin{aligned}
\hat{\alpha} &= \frac{\ln\left(\frac{\overline{X}}{\overline{X}+1}\right)}{\ln q}, \\
\hat{\beta} &= \frac{\ln\left(\frac{\overline{Y}}{\overline{Y}+1}\right)}{\ln q}.
\end{aligned}
$$

The equation (D.2) does not have closed solution for q, so the MLE for q can be obtain numerically. In PHM usually the parameters of the baseline distribution is known, therefor the Plug-in MLE of R is as follow,

$$\hat{R} = \frac{1 - q^{\hat{\alpha}}}{1 - q^{\hat{\alpha}+\hat{\beta}}} = \frac{\overline{Y} + 1}{\overline{X} + \overline{Y} + 1}, \tag{D.3}$$

which is not depend on q. Based on Maiti (10) the asymptotic variance of $\hat{R}$ is given by,

$$Var(\hat{R}) \approx \left(\frac{1}{n} + \frac{q^{\alpha+\beta}}{m}\right) \frac{q^{\beta}(1-q^{\alpha})^2(1-q^{\beta})^2}{(1-q^{\alpha+\beta})^4} \tag{D.4}$$

E Approximation of the Variance of R's unbiased Estimators

Sometimes due to complicated form of estimators, we can not compute their variances. In some cases the best way to approximate the variance is using lower bounds such as Cramer-Rao and its extension Bhattacharyya lower bounds.

Under some regularity conditions, the Bhattacharyya bound for any unbiased estimator of the $g(\theta)$ is defined as follows,(4)

$$Var_{\theta}(T(X)) \geq \mathbf{J}_{\theta}\mathbf{W}^{-1}\mathbf{J}_{\theta}^{t} := B_k(\theta), \tag{E.1}$$

where t refers to the transpose, $\mathbf{J}_{\theta} = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(k)}(\theta))$, $g^{(j)}(\theta) = \partial^j g(\theta)/\partial\theta^j$ for $j = 1, 2, \ldots, k$ and $\mathbf{W}^{-1}$ is the inverse of the Bhattacharyya matrix, where

$$\mathbf{W} = (W_{rs}) = \left(Cov_{\theta}\left\{\frac{f^{(r)}(X|\theta)}{f(X|\theta)}, \frac{f^{(s)}(X|\theta)}{f(X|\theta)}\right\}\right),$$

such that $E_{\theta}(\frac{f^{(r)}(X|\theta)}{f(X|\theta)}) = 0$ for $r, s = 1, 2, \ldots, k$.

If we substitute $k = 1$ in (*E.1*), then it indeed reduces to the Cramer-Rao inequality. By using the properties of the multiple correlation coefficient, it is easy to show that as the order of the Bhattacharyya matrix (k) increases, the Bhattacharyya bound becomes sharper.

Khorashadizadeh et al. (8) studied the approximation of the variance of any unbiased estimator of R in Burr XII distribution.

Example (Poisson distribution)

Let X and Y be independent r.v. modeling stress and strength, respectively, with $X \sim Poisson(\theta_1)$ and $Y \sim Poisson(\theta_2)$. then,

$$R = P(Y \leq X) = \sum_{x=0}^{\infty} \frac{e^{-\theta_1}\theta_1^x}{x!} \left(\sum_{y=0}^{x} \frac{e^{-\theta_2}\theta_2^y}{y!} \right). \tag{E.2}$$

As it was noted in Barbiero (2) the above sums are rapidly converge and the reliability R can be approximated by first terms with a very high precision. As an example the Figure 1 shows the terms of sum (E.2) for different values of θ_1 and θ_2.

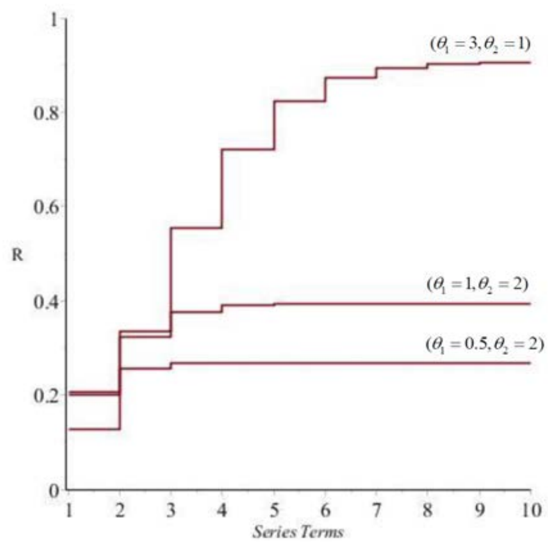

Figure 1: The converge of series (E.2) for the parameter R in Poisson distribution.

As it can seen, the finite sum of this series converges to the real value of the reliability parameter. Table 1 shows the first five Bhattacharyya lower bounds for the variance of any unbiased estimator of R for Poisson distribution.

References

[1] Al-Hussaini, E.K. (1999), Predicting observables from a general class of distributions, *Journal of Statistical Planning and Inference*, **79**(1), 79-91.

[2] Barbiero, A. (2013), Inference on reliability of Stress-Strength models for Poisson data, *Journal of Quality and Reliability Engineering*, **2013**, Article ID 530530, 8 pages.

Table 1: First five Bhattacharyya lower bounds for the variance of any unbiased estimator of R for Poisson distribution

θ_1	θ_2	R	B_1	B_2	B_3	B_4	B_5
1	2	0.394	0.08966	0.10677	0.10727	0.10756	0.10779
1	1	0.654	0.09518	0.09952	0.09966	0.09997	0.10006
2	1	0.817	0.04482	0.04518	0.04625	0.04638	0.04639
1	1.547	0.500	0.10076	0.11238	0.11240	0.11283	0.11311

[3] Behboodian, J. (2003), A look at $P(X > Y)$ in Binomial case, *Presented in fourth Probability and Stochastic Processes Seminar*, Gorgan, Iran.

[4] Bhattacharyya, A. (1946), On some analogues of the amount of information and their use in statistical estimation, *Sankhya A*, **8**, 1-14.
(1), 137-142.

[5] Chaturvedi, A. and Tomer, S.K. (2002), Classical and Bayesian reliability estimation of the negative binomial distribution, *Journal of Applied Statistical Science*, **11**(1), 33-43.

[6] Imani, M. and Khorashadizadeh, M. (2019), On estimation of Stress-Strength parameter for discrete Weibull distribution, *5th Seminar on Reliability Theory and its Applications*, University of Yazd, Iran.

[7] Khan, M.S.A., Khalique, A. and Abouammoh, A.M. (1989), On estimating parameters in a discrete Weibull distribution, *IEEE Transaction On Reliability*, **38**(3), 348-350.

[8] Khorashadizadeh, M., Nayeban, S., and Borzadaran, G.M. (2017), On the varicne of $P(Y < X)$ estimator in Burr XII distribution, *Journal of Reliability and Statistical Studies*, **10**(2).

[9] Kotz, S., Lumelskii, Y. and Pensky, M. (2003), *The Stress-Strength Model and Its Generalizations*, World Scientific Press, Singapore.

[10] Maiti, S.S. (1995), Estimation of $P(X \geq Y)$ in the geometric case, *Journal of the Indian Statistical Association*, **33**, 87-91.

[11] Patowary, A.N., Hazarika, J. and Sriwastav, G.L. (2013), Interference theory of reliability: a review, *International Journal of System Assurance Engineering and Management*, **4**(2), 146-158.

[12] Rezaei Roknabadi, A.H., Mohtashami Borzadaran, G.R. and Khorashadizadeh, M. (2009), Some aspects of discrete hazard rate function in Telescopic families, *Economic Quality Control*, **24**, 1, 35-42.

[13] Roy, D. and Gupta, R.P. (1999), Characterizations and model selections through reliability measures in the discrete case, *Statistics and Probability Letters*, **43**, 197-206.

[14] Sathe, Y.S. and Dixit, U.J. (2001), Estimation of $P[X \leq Y]$ in the negative binomial distribution, *Journal of Statistical Planning and Inference*, **93**, 83-92.

[15] Yu, Q. (2007), A note on the proportional hazards model with discontinuous data, *Statistics and Probability Letters*, **77**, 735-739.

Bayesian Inference on Stress-Strength Parameter in Burr type XII Distribution under Hybrid Progressive Censoring Samples

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Abstract: In this paper, the Bayesian inference of stress-strength parameter for Burr type XII distribution under the Type-II hybrid progressive censored samples is considered. The problem is solved in three cases. In first case, assuming that stress and strength have the unknown common first shape parameter and different second shape parameters, the Bayes estimate of stress-strength parameter is derived by two approximation method: Lindley's approximation and MCMC method. In second case, assuming that stress and strength have the known common first shape parameter and unknown different second shape parameters, the exact Bayes estimate of stress-strength parameter is derived. In third case, assuming that all parameters are different and unknown, the Bayesian inference of stress-strength parameter is derived by MCMC method. We use one Monte Carlo simulation study to compare the performance of different methods.

Keywords Type-II hybrid progressive censored sample, Stress-strength model, Burr type XII distribution, Bayesian inference.

Mathematics Subject Classification (2010) : 62N05, 62F15, 62F10.

A Introduction

Statistical inference about the stress-strength parameter, $R = P(X < Y)$, is one of the most important problem in reliability theory and statistics and has been done from the frequentist and Bayesian viewpoints. In spite of the fact that, many papers have studied the stress-strength models in complete samples, much consideration has not been paid to censored data (see (3)).

Type-I and Type-II censoring schemes are two most fundamental schemes and by mixing of theses two schemes, hybrid scheme is derived. Unfortunately, none of above schemes cannot remove active units during the experiment. So, the progressive censoring scheme is mentioned.

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Combining hybrid and progressive schemes, hybrid progressive censoring (HPC) scheme is provided which introduced by Kundu and Joarder (4) and can be described as follows. Let N units are put on the test with censoring scheme $(R_1, \ldots, R_n)$ and pausing time $T^* = \min\{X_{n:n:N}, T\}$, where $X_{1:n:N} \leq \ldots \leq X_{n:n:N}$ be a progressive censoring scheme and $T > 0$ is a fixed time. It is obvious that if $X_{n:n:N} < T$ then we finish the test at time $X_{n:n:N}$ and $\{X_{1:n:N}, \ldots, X_{n:n:N}\}$ is the observed sample. Otherwise, if $X_{J:n:N} < T < X_{J+1:n:N}$ then we finish the test at time T and $\{X_{1:n:N}, \ldots, X_{J:n:N}\}$ is the observed sample. We denote a HPC sample with $\{X_1, \ldots, X_J\}$ under the scheme $\{N, n, T, J, R_1, \ldots R_n\}$. The likelihood function of the HPC samples is as follows:

$$L(\theta) \propto \prod_{i=1}^{J} f(x_i)[1 - F(x_i)]^{R_i}[1 - F(T)]^{R_J^*}.$$

where $R_{J_1}^* = N - J - \sum_{i=1}^{J} R_i$.

Burr type XII (Bur) distribution with the first and second shape parameters λ and α, has the probability density function as $f(x) = \lambda\alpha x^{\lambda-1}(1 + x^{\lambda})^{-\alpha-1},\ x, \alpha, \lambda > 0$. Some recent works on this distribution can be found in (8) and (6). In this paper, we obtain the Bayesian inference of the $R = P(X < Y)$ based on HPC sample, when X and Y are two independent random variables from the Bur distribution.

B Bayesian inference of R with unknown common λ

If $X \sim Bur(\lambda, \alpha)$ and $Y \sim Bur(\lambda, \beta)$, then the stress-strength parameter can be obtained as

$$R = P(X < Y) = \frac{\alpha}{\alpha + \beta}.$$

In this section, the Bayesian inference of R is considered under squared error loss functions, when α, β and λ are independent gamma random variables. Based on the observed censoring samples, the joint posterior density function is as follows:

$$\pi(\alpha, \beta, \lambda|\text{data}) \propto L(\text{data}|\alpha, \beta, \lambda)\pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda) \tag{B.1}$$

where $\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha},\ \alpha, a_1, b_1 > 0,\ \pi_2(\beta) \propto \beta^{a_2-1}e^{-b_2\beta},\ \beta, a_2, b_2 > 0$ and $\pi_3(\lambda) \propto \lambda^{a_3-1}e^{-b_3\lambda},\ \lambda, a_3, b_3 > 0$. As we see, from equation (B.1), the Bayes estimate cannot be obtained in a closed form. So, we should approximate it by applying two methods:

- Lindley's approximation,
- MCMC method.

B.1 Lindley's approximation

Lindley (5) introduced one of the most numerical techniques to derive the Bayes estimate. If $U(\theta)$ be a function of $\theta = (\theta_1, \theta_2, \theta_3)$, Lindley's approximation of it, $\mathbb{I}\big(\text{data}\big)$, is

$$\begin{aligned}\mathbb{I}\big(\text{data}\big) = u + (u_1 d_1 + u_2 d_2 + u_3 d_3 + d_4 + d_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) \\ + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})],\end{aligned}$$

calculated at $\widehat{\theta} = (\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3)$, where $\ell(\theta)$ is the logarithm of the likelihood function, and $\rho(\theta)$ is the logarithm of the prior density of θ. Also, $u_i = \partial u(\theta)/\partial\theta_i$, $u_{ij} = \partial^2 u(\theta)/\partial\theta_i\partial\theta_j$, $\ell_{ijk} = \partial^3\ell(\theta)/\partial\theta_i\partial\theta_j\partial\theta_k$, $\rho_j = \partial\rho(\theta)/\partial\theta_j$, and $\sigma_{ij} = (i,j)$th element in the inverse of matrix $[-\ell_{ij}]$ all evaluated at the MLE of the parameters. Moreover,

$$\begin{aligned}
d_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \quad i = 1, 2, 3, \\
d_4 &= u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}, \\
d_5 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}), \\
A &= \ell_{111}\sigma_{11} + 2\ell_{121}\sigma_{12} + 2\ell_{131}\sigma_{13} + 2\ell_{231}\sigma_{23} + \ell_{221}\sigma_{22} + \ell_{331}\sigma_{33}, \\
B &= \ell_{112}\sigma_{11} + 2\ell_{122}\sigma_{12} + 2\ell_{132}\sigma_{13} + 2\ell_{232}\sigma_{23} + \ell_{222}\sigma_{22} + \ell_{332}\sigma_{33}, \\
C &= \ell_{113}\sigma_{11} + 2\ell_{123}\sigma_{12} + 2\ell_{133}\sigma_{13} + 2\ell_{233}\sigma_{23} + \ell_{223}\sigma_{22} + \ell_{333}\sigma_{33}.
\end{aligned}$$

In our case, for $(\theta_1, \theta_2, \theta_3) \equiv (\alpha, \beta, \lambda)$, we have

$$\begin{aligned}
\rho_1 &= \frac{a_1 - 1}{\alpha} - b_1, \quad \rho_2 = \frac{a_2 - 1}{\beta} - b_2, \quad \rho_3 = \frac{a_3 - 1}{\lambda} - b_3, \\
\ell_{11} &= -\frac{J_1}{\alpha^2}, \qquad \ell_{22} = -\frac{J_2}{\beta^2}, \qquad \ell_{12} = \ell_{21} = 0, \\
\ell_{13} = \ell_{31} &= -\sum_{i=1}^{J_1}(R_i + 1)x_i^\lambda \frac{\log(x_i)}{1 + x_i^\lambda} - R^*_{J_1} T_1^\lambda \frac{\log(T_1)}{1 + T_1^\lambda}, \\
\ell_{23} = \ell_{32} &= -\sum_{j=1}^{J_2}(S_j + 1)y_j^\lambda \frac{\log(y_j)}{1 + y_j^\lambda} - S^*_{J_2} T_2^\lambda \frac{\log(T_2)}{1 + T_2^\lambda}, \\
\ell_{33} &= -\frac{J_1 + J_2}{\lambda^2} - \sum_{i=1}^{J_1}\big(\alpha(R_i + 1) + 1\big)x_i^\lambda \Big(\frac{\log(x_i)}{1 + x_i^\lambda}\Big)^2 - \alpha R^*_{J_1} T_1^\lambda \Big(\frac{\log(T_1)}{1 + T_1^\lambda}\Big)^2 \\
&\quad - \sum_{j=1}^{J_2}\big(\beta(S_j + 1) + 1\big)y_j^\lambda \Big(\frac{\log(y_j)}{1 + y_j^\lambda}\Big)^2 - \beta S^*_{J_2} T_2^\lambda \Big(\frac{\log(T_2)}{1 + T_2^\lambda}\Big)^2.
\end{aligned}$$

σ_{ij}, $i,j=1,2,3$ are obtained by using ℓ_{ij}, $i,j=1,2,3$ and

$$\ell_{111}=\frac{2J_1}{\alpha^3}, \qquad \ell_{222}=\frac{2J_2}{\beta^3}$$

$$\ell_{133}=\ell_{331}=\ell_{313}=-\sum_{i=1}^{J_1}(R_i+1)x_i^{\lambda}\Big(\frac{\log(x_i)}{1+x_i^{\lambda}}\Big)^2-R^*_{J_1}T_1^{\lambda}\Big(\frac{\log(T_1)}{1+T_1^{\lambda}}\Big)^2,$$

$$\ell_{233}=\ell_{332}=\ell_{323}=-\sum_{j=1}^{J_2}(S_j+1)y_j^{\lambda}\Big(\frac{\log(y_j)}{1+y_j^{\lambda}}\Big)^2-S^*_{J_2}T_2^{\lambda}\Big(\frac{\log(T_2)}{1+T_2^{\lambda}}\Big)^2,$$

$$\begin{aligned}\ell_{333}=&\frac{2J_1}{\lambda^3}-\sum_{i=1}^{J_1}\big(\alpha(R_i+1)+1\big)x_i^{\lambda}(1-x_i^{\lambda})\Big(\frac{\log(x_i)}{1+x_i^{\lambda}}\Big)^3-\alpha R^*_{J_1}T_1^{\lambda}(1-T_1^{\lambda})\Big(\frac{\log(T_1)}{1+T_1^{\lambda}}\Big)^3\\ &+\frac{2J_2}{\lambda^3}-\sum_{j=1}^{J_2}\big(\beta(S_j+1)+1\big)y_j^{\lambda}(1-y_j^{\lambda})\Big(\frac{\log(y_j)}{1+y_j^{\lambda}}\Big)^3-\beta S^*_{J_2}T_2^{\lambda}(1-T_2^{\lambda})\Big(\frac{\log(T_2)}{1+T_2^{\lambda}}\Big)^3,\end{aligned}$$

and other $\ell_{ijk}=0$. Hence,

$$d_4=u_{12}\sigma_{12}, \qquad d_5=\frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22}),$$

$$A=\ell_{111}\sigma_{11}+\ell_{331}\sigma_{33}, \quad B=\ell_{222}\sigma_{22}+\ell_{332}\sigma_{33}, \quad C=2\ell_{133}\sigma_{13}+2\ell_{233}\sigma_{23}+\ell_{333}\sigma_{33}.$$

So, the approximate Bayes estimate of R, under the squared error loss function is obtained by setting $u(\theta)=R=\frac{\alpha}{\alpha+\beta}$. Also, $u_3=0$, $u_{i3}=0$, $i=1,2,3$ and

$$u_{11}=\frac{-2\beta}{(\alpha+\beta)^3},\ u_{12}=u_{21}=\frac{\alpha-\beta}{(\alpha+\beta)^3},\ u_{22}=\frac{2\alpha}{(\alpha+\beta)^3}.$$

Consequently, under the squared error loss function, the Bayes estimate of R is

$$\begin{aligned}\widehat{R}^{Lin}=&R+[u_1d_1+u_2d_2+d_4+d_5]+\frac{1}{2}[A(u_1\sigma_{11}+u_2\sigma_{12})\\ &+B(u_1\sigma_{21}+u_2\sigma_{22})+C(u_1\sigma_{31}+u_2\sigma_{32})].\end{aligned} \tag{B.2}$$

Notice that all parameters are calculated at $(\widehat{\alpha},\widehat{\beta},\widehat{\lambda})$.

B.2 MCMC method

From the equation (B.1), the posterior pdfs of of α, β and λ can be derived as:

$$\begin{aligned}\alpha|\lambda,\text{data}&\sim\Gamma\big(J_1+a_1,b_1+V(\lambda)\big),\\ \beta|\lambda,\text{data}&\sim\Gamma\big(J_2+a_2,b_2+U(\lambda)\big),\\ \pi(\lambda|\alpha,\beta,\text{data})&\propto\Bigg(\prod_{i=1}^{J_1}x_i^{\lambda-1}(1+x_i^{\lambda})^{-\alpha(R_i+1)-1}\Bigg)\Bigg(\prod_{j=1}^{J_2}y_j^{\lambda-1}(1+y_j^{\lambda})^{-\beta(S_j+1)-1}\Bigg)\\ &\quad\times\lambda^{J_1+J_2+a_3-1}e^{-\lambda b_3}(1+T_1^{\lambda})^{-\alpha R^*_{J_1}}(1+T_2^{\lambda})^{-\beta S^*_{J_2}}.\end{aligned}$$

where

$$V(\lambda) = \sum_{i=1}^{J_1} (R_i + 1)\log(1 + x_i^{\lambda}) + R_{J_1}^{*}\log(1 + T_1^{\lambda}),$$

$$U(\lambda) = \sum_{j=1}^{J_2} (S_j + 1)\log(1 + y_j^{\lambda}) + S_{J_2}^{*}\log(1 + T_2^{\lambda}). \tag{B.3}$$

It is observed that generating samples from the posterior pdf of λ should be done by the Metropolis-Hastings method. So, we propose the following algorithm of Gibbs sampling:

1. Start with initial values $(\alpha_{(0)}, \beta_{(0)}, \lambda_{(0)})$.
2. Set $t = 1$.
3. Generate $\lambda_{(t)}$ from $\pi(\lambda|\alpha_{(t-1)}, \beta_{(t-1)}, \text{data})$, using Metropolis-Hastings method.
4. Generate $\alpha_{(t)}$ from $\Gamma\big(n + a_1, b_1 - V(\lambda_{(t-1)})\big)$.
5. Generate $\beta_{(t)}$ from $\Gamma\big(m + a_2, b_2 - U(\lambda_{(t-1)})\big)$.
6. Evaluate $R_t = \frac{\alpha_t}{\alpha_t + \beta_t}$.
7. Set $t = t + 1$.
8. Repeat T times, steps 3-7.

Therefore, the Bayes estimate of R, under the squared error loss functions is:

$$\widehat{R}^{MB} = \frac{1}{T}\sum_{t=1}^{T} R_t. \tag{B.4}$$

Also, the $100(1-\gamma)\%$ HPD credible interval of R can be constructed, using the method of Chen and Shao (1).

C Bayesian inference of R with known common λ

In this section, the Bayesian inference of R is considered under the squared error loss function, when α and β are independent gamma random variables. Based on the observed censoring samples, the joint posterior density function is as follows:

$$\pi(\alpha, \beta|\lambda, \text{data}) = \frac{\big(\alpha(V(\lambda) + b_1)\big)^{J_1 + a_1}\big(\beta(U(\lambda) + b_2)\big)^{J_2 + a_2}}{\alpha\beta\Gamma(J_1 + a_1)\Gamma(J_2 + a_2)} e^{-\alpha(V(\lambda) + b_1) - \beta(U(\lambda) + b_2)}, \tag{C.1}$$

where $V(\cdot)$ and $U(\cdot)$ are given in (B.3). So, the Bayes estimate of R under the squared error loss function, should be achieved by solving the following integral:

$$\widehat{R}^B = \int_0^\infty \int_0^\infty \frac{\alpha}{\alpha+\beta} \times \pi(\alpha, \beta|\lambda, \text{data}) d\alpha d\beta.$$

By applying the idea of Kizilaslan and Nadar (2), the exact Bayes estimate is obtained as:

$$\widehat{R}^B = \begin{cases} \dfrac{(1-z)^{J_1+a_1}(J_1+a_1)}{w} \; {}_2F_1(w, J_1+a_1+1; w+1, z) & \text{if } \; |z|<1, \\ \\ \dfrac{(J_1+a_1)}{w(1-z)^{J_2+a_2}} \; {}_2F_1(w, J_2+a_2; w+1, \dfrac{z}{1-z}) & \text{if } \; z<-1, \end{cases} \tag{C.2}$$

where $w = J_1 + J_2 + a_1 + a_2$, $z = 1 - \dfrac{V(\lambda)+b_1}{U(\lambda)+b_2}$ and

$${}_2F_1(\alpha, \beta; \gamma, z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} dt, \;\; |z|<1$$

is hypergeometric series, which is quickly calculated and readily available in standard software such as MATLAB. Furthermore, the $100(1-\gamma)\%$ Bayesian interval of R can be constructed as (L, U), where L and U should be satisfied, respectively, in

$$\int_0^L f_R(R) dR = \frac{\gamma}{2}, \quad \int_0^U f_R(R) dR = 1 - \frac{\gamma}{2}, \tag{C.3}$$

where $f_R(R)$, using change-of-variable method, can be earned by (C.1) as

$$f_R(R) = \frac{(1-z)^{J_1+a_1} R^{J_1+a_1-1}(1-R)^{J_2+a_2-1}(1-Rz)^{-w}}{B(J_1+a_1, J_2+a_2)}, \quad 0<R<1.$$

D Bayesian inference of R in general case

If $X \sim Bur(\lambda_1, \alpha)$ and $Y \sim Bur(\lambda_2, \beta)$, then the stress-strength parameter can be obtained as

$$R = P(X<Y) = 1 - \int_0^\infty \beta\lambda_2 y^{\lambda_2-1}(1+y^{\lambda_2})^{-\beta-1}(1+y^{\lambda_1})^{-\alpha} dy.$$

In this section, the Bayesian inference of R is considered under squared error loss functions, when α, β, λ_1 and λ_2 are independent gamma random variables. Like in section 2, as the Bayes estimate of R can not be evaluated in a closed form, it is approximated by MCMC method. From the joint posterior density function, we can be derived the posterior pdfs of α, β, λ_1 and

λ_2 as follows:

$$\alpha|\lambda_1, \text{data} \sim \Gamma\big(J_1 + a_1, b_1 + V(\lambda_1)\big),$$
$$\beta|\lambda_2, \text{data} \sim \Gamma\big(J_2 + a_2, b_2 + U(\lambda_2)\big),$$
$$\pi(\lambda_1|\alpha, \text{data}) \propto \left(\prod_{i=1}^{J_1} x_i^{\lambda_1 - 1}(1 + x_i^{\lambda_1})^{-\alpha(R_i+1)-1}\right)\lambda_1^{J_1+a_3-1}e^{-\lambda_1 b_3}(1 + T_1^{\lambda_1})^{-\alpha R^*_{J_1}},$$
$$\pi(\lambda_2|\beta, \text{data}) \propto \left(\prod_{j=1}^{J_2} y_j^{\lambda_2 - 1}(1 + y_j^{\lambda_2})^{-\beta(S_j+1)-1}\right)\lambda_2^{J_2+a_4-1}e^{-\lambda_2 b_4}(1 + T_2^{\lambda_2})^{-\beta S^*_{J_2}}.$$

It is observed that generating samples from the posterior pdfs of λ_1 and λ_2 should be done by the Metropolis-Hastings method. So, we propose the following algorithm of Gibbs sampling:

1. Start with initial values $(\alpha_{(0)}, \beta_{(0)}, \lambda_{1(0)}, \lambda_{2(0)})$.
2. Set $t = 1$.
3. Generate $\lambda_{1(t)}$ from $\pi(\lambda_1|\alpha_{(t-1)}, \text{data})$, using Metropolis-Hastings method.
4. Generate $\lambda_{2(t)}$ from $\pi(\lambda_2|\beta_{(t-1)}, \text{data})$, using Metropolis-Hastings method.
5. Generate $\alpha_{(t)}$ from $\Gamma\big(J_1 + a_1, b_1 + V(\lambda_{1(t-1)})\big)$.
6. Generate $\beta_{(t)}$ from $\Gamma\big(J_2 + a_2, b_2 + U(\lambda_{2(t-1)})\big)$.
7. Evaluate $R_t = 1 - \int_0^\infty \beta_{(t)}\lambda_{2(t)}y^{\lambda_{2(t)}-1}(1 + y^{\lambda_{2(t)}})^{-\beta_{(t)}-1}(1 + y^{\lambda_{1(t)}})^{-\alpha_{(t)}}dy$.
8. Set $t = t + 1$.
9. Repeat T times, steps 3-8.

Therefore, the Bayes estimate of R, under the squared error loss functions is:

$$\widehat{R}^{MB} = \frac{1}{T}\sum_{t=1}^{T} R_t. \tag{D.1}$$

Also, the $100(1-\gamma)\%$ HPD credible interval of R can be constructed, using the method of Chen and Shao (1).

E Simulation Study

We consider the performance of different Bayes estimates, under HPC schemes by using the Monte Carlo simulations. The different estimates, in terms of mean squared errors (MSEs) are compared together and the different confidence intervals, in terms of average confidence lengths and coverage percentages are compared together. Based on 3000 replications, all results are gathered. Also, the used censoring schemes are as:

$$
\begin{aligned}
&\text{Scheme 1: } R_1 = \ldots = R_n = \frac{N-n}{n}, \\
&\text{Scheme 2: } R_{2k} = \frac{N-n}{n} - 1,\ R_{2k-1} = \frac{N-n}{n} + 1,\ k = 1, \ldots, \frac{n}{2}, \\
&\text{Scheme 3: } R_{2k} = \frac{2(N-n)}{n},\ \ R_{2k-1} = 0, \qquad\qquad k = 1, \ldots, \frac{n}{2}.
\end{aligned}
$$

In the first case, with unknown common λ, the parameter values $(\alpha, \beta, \lambda) = (2, 2, 2)$ are used to obtain the simulation results. Also, the Bayesian inference is considered by assuming two priors as Prior 1: $a_j = 0,\ b_j = 0,\ j = 1, 2, 3$, Prior 2: $a_j = 0.5,\ b_j = 0.5,\ j = 1, 2, 3$. Under the above hypotheses, the MSEs of Bayesian estimates of R, via Linldey's approximation and MCMC method are derived by (B.2) and (B.4), respectively. Also, we derived the 95% HPD intervals for R. The simulation results are given in Table 1.

In the second case, with known common λ, the parameter values $(\alpha, \beta) = (2.5, 2.5)$ are used to obtain the simulation results. Also, the Bayesian inference is considered by assuming two priors as Prior 3: $a_j = 0,\ b_j = 0,\ j = 1, 2$, Prior 4: $a_j = 0.5,\ b_j = 0.5,\ j = 1, 2$. Under the above hypotheses, the Bayes estimate and Bayesian intervals of R are derived by (C.2) and (C.3), respectively. The results are provided in Table 1.

In the third case, with unknown different λ_1 and λ_2, the parameter values $(\alpha, \beta, \lambda_1, \lambda_2) = (3, 3, 3, 3)$ are used to obtain the simulation results. Also, the Bayesian inference are considered by assuming two priors as Prior 5: $a_j = 0,\ b_j = 0,\ j = 1, 2, 3, 4$, Prior 6: $a_j = 0.5,\ b_j = 0.5,\ j = 1, 2, 3, 4$. Under the above hypotheses, the MSEs of Bayesian estimates of R are derived by (D.1). Also, we derived the 95% HPD intervals for R. The simulation results are given in Table 1.

From Table 1, we observed that the best performance, in terms of MSE, belong to informative priors (priors 2, 4 and 6). Furthermore, in first case, performance of Bayes estimates which obtained by MCMC method are generally better than those obtained by Lindleys approximation. Also, we observed that the best performance among the different intervals belong to HPD intervals based on informative priors (priors 2, 4 and 6).

To tell the truth, from Table 1, with increasing n for fixed N and T, and also with increasing T for fixed N and n, MSEs and average confidence lengths decrease and the associated coverage percentages increase in all cases.

Table 1: Simulation results

(N, n, T)	C.S	Unknown common λ							
		MCMC						Lindley	
		Prior 1			Prior 2			Prior 1	Prior 2
		MSE	length	C.P	MSE	length	C.P	MSE	MSE
(40,10,1)	(1,1)	0.0103	0.4085	0.911	0.0080	0.4018	0.916	0.0148	0.0086
	(2,2)	0.0110	0.4127	0.910	0.0072	0.4025	0.913	0.0149	0.0092
	(3,3)	0.0122	0.4120	0.909	0.0073	0.4031	0.918	0.0166	0.0104
	(1,2)	0.0115	0.4078	0.910	0.0080	0.3999	0.914	0.0146	0.0095
(60,10,2)	(1,1)	0.0094	0.4066	0.918	0.0052	0.3970	0.924	0.0133	0.0081
	(2,2)	0.0091	0.4054	0.917	0.0060	0.3999	0.920	0.0121	0.0077
	(3,3)	0.0100	0.4044	0.918	0.0061	0.4004	0.926	0.0121	0.0083
	(1,2)	0.0102	0.4016	0.919	0.0052	0.3974	0.925	0.0133	0.0087
(40,10,2)	(1,1)	0.0092	0.4067	0.918	0.0061	0.3970	0.926	0.0111	0.0078
	(2,2)	0.0059	0.4069	0.920	0.0043	0.3999	0.925	0.0078	0.0050
	(3,3)	0.0080	0.4019	0.917	0.0062	0.3971	0.924	0.0125	0.0067
	(1,2)	0.0087	0.4062	0.915	0.0066	0.3991	0.926	0.0127	0.0073
(60,20,2)	(1,1)	0.0041	0.2993	0.940	0.0036	0.2945	0.944	0.0050	0.0041
	(2,2)	0.0047	0.2961	0.941	0.0043	0.2934	0.943	0.0054	0.0046
	(3,3)	0.0043	0.2982	0.942	0.0040	0.2951	0.945	0.0056	0.0045
	(1,2)	0.0050	0.2961	0.940	0.0045	0.2956	0.944	0.0060	0.0049

References

[1] Chen, M. H. and Shao, Q. M. (1999), Monte Carlo estimation of Bayesian Credible and HPD intervals. *Journal of Computational and Graphical Statistics*, **8**, 69-92.

[2] Kizilaslan, F. and Nadar, M. (2016), Estimation of reliability in a multicomponent stress-strength model based on a bivariate Kumaraswamy distribution. *Statistical Papers*, Accepted. DOI: 10.1007/s00362-016-0765-8.

[3] Kohansal, A. (2017), On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample, *Statistical Papers*, Accepted. DOI: 10.1007/s00362-017-0916-6.

[4] Kundu, D. and Joarder, A. (2006), Analysis of type-II progressively hybrid censored data, *Computational Statistics and Data Analysis*, **50**, 2509-2528.

[5] Lindley, D.V. (1980), Approximate Bayesian methods, *Trabajos de Estadistica*, **3**, 281-288.

[6] Maurya, R.K., Tripathi, Y.M., Rastogi, M.K. and Asgharzadeh, A. (2017), Parameter estimation for a Burr Type XII distribution under progressive censoring, *American Journal of Mathematical and Management Sciences*, **36**, 259-276.

(N, n, T)	C.S	Known common λ					
		Exact					
		Prior 3			Prior 4		
		MSE	length	C.P	MSE	length	C.P
(40,10,1)	(1,1)	0.0109	0.4062	0.903	0.0086	0.4008	0.915
	(2,2)	0.0117	0.4053	0.903	0.0092	0.4000	0.919
	(3,3)	0.0105	0.4065	0.908	0.0082	0.4016	0.912
	(1,2)	0.0116	0.4059	0.907	0.0091	0.4004	0.911
(40,20,1)	(1,1)	0.0065	0.3031	0.930	0.0057	0.3003	0.935
	(2,2)	0.0063	0.3054	0.931	0.0055	0.3021	0.934
	(3,3)	0.0056	0.3031	0.934	0.0049	0.2999	0.935
	(1,2)	0.0068	0.3046	0.932	0.0060	0.3014	0.934
(60,20,1)	(1,1)	0.0058	0.2985	0.930	0.0051	0.2953	0.933
	(2,2)	0.0058	0.2999	0.933	0.0052	0.2969	0.933
	(3,3)	0.0057	0.2986	0.934	0.0051	0.2955	0.935
	(1,2)	0.0061	0.3000	0.932	0.0054	0.2967	0.935
(60,20,2)	(1,1)	0.0053	0.2980	0.941	0.0047	0.2948	0.944
	(2,2)	0.0057	0.2965	0.943	0.0050	0.2942	0.946
	(3,3)	0.0052	0.2985	0.940	0.0046	0.2943	0.943
	(1,2)	0.0049	0.2965	0.943	0.0043	0.2941	0.943

(N, n, T)	C.S	General Case					
		MCMC					
		Prior 5			Prior 6		
		MSE	length	C.P	MSE	length	C.P
(60,10,1)	(1,1)	0.0088	0.4110	0.910	0.0077	0.4023	0.910
	(2,2)	0.0107	0.4066	0.911	0.0098	0.4003	0.919
	(3,3)	0.0103	0.4098	0.907	0.0090	0.4039	0.910
	(1,2)	0.0143	0.4076	0.909	0.0124	0.3995	0.911
(40,10,2)	(1,1)	0.0064	0.4023	0.918	0.0056	0.3983	0.929
	(2,2)	0.0094	0.3991	0.918	0.0081	0.3944	0.925
	(3,3)	0.0056	0.4078	0.917	0.0048	0.4009	0.927
	(1,2)	0.0093	0.4059	0.919	0.0080	0.3999	0.925
(60,10,2)	(1,1)	0.0086	0.4083	0.917	0.0076	0.4001	0.927
	(2,2)	0.0100	0.4052	0.918	0.0090	0.3998	0.924
	(3,3)	0.0072	0.4038	0.917	0.0065	0.3995	0.926
	(1,2)	0.0102	0.4000	0.919	0.0088	0.3940	0.925
(40,20,2)	(1,1)	0.0041	0.3002	0.940	0.0039	0.2968	0.944
	(2,2)	0.0046	0.3017	0.942	0.0043	0.3001	0.945
	(3,3)	0.0052	0.2967	0.939	0.0049	0.2945	0.942
	(1,2)	0.0040	0.3040	0.942	0.0037	0.3012	0.945

[7] Ng, H.K.T., Kundu, D. and Chan, P.S. (2009), Statistical analysis of exponential lifetimes under an adaptive Type-II progressively censoring scheme, *Naval Research Logistics*, **56**, 687-698.

[8] Rao, G.S., Aslam, M. and Kundu, D. (2014), Burr-XII distribution parametric estimation and estimation of reliability of multicomponent stress-strength, *Communications in Statistics - Theory and Method*, **44**, 4953-4961.

A Class of Mean Residual Life Regression Models with Censored Survival Data Under Case-Cohort Design

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Abstract: A general class of mean residual life models is studied for analysing survival data under the case-cohort design. Martingale estimating equations are proposed for estimation of the regression parameters and the baseline mean residual life function. It is shown that the resulting regression estimators are asymptotically normal, with variance-covariance matrix that has a closed form.

Keywords Censored data, Estimating equation, Mean residual life, Case-cohort design.

Mathematics Subject Classification (2010) : 62N01, 62N02**.**

A Introduction

The mean residual life function is of interest in many fields such as actuarial studies, reliability research, survival analysis, demography, and other disciplines. For a non-negative survival time T with finite expectation, the mean residual life function (MRLF) at time $t \geq 0$ is defined as $m(t) = E(T - t|T > t)$. This function can sometimes serve as more desirable tool than the survival function and the hazard function. For instance, it may be more informative to tell a breast cancer patient how long she can survive, on average given her survival up to time t as compared with her instantaneous survival chance.

To study the effects of covariates on the MRLF, the proportional mean residual life model by (11) may be used:

$$m(t|Z) = m_0(t)\exp(Z^\top \beta), \tag{A.1}$$

where $m(t|Z)$ is the MRLF corresponding to the p-vector covariate Z, $m_0(t)$ is some unknown baseline MRLF when $Z = 0$, and β is an unknown vector of regression parameters. For inference on the parameters in model (A.1), (9) developed estimation procedures for β in model (A.1), when $m_0(t)$ is unknown, but mainly for uncensored survival data. To accommodate censoring,

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(5) extended the estimation procedure of (9) to censored survival data using the inverse probability of censoring weighting technique to estimate the parameters in model (A.1). (3) employed counting process theory to develop new inference procedures for the regression analysis of model (A.1) when censoring is present and $m_0(t)$ is not known. (10) considered the proportional restricted mean residual life model and discussed a goodness-of-fit test of the model. A class of additive mean residual life model was also proposed by (4) and (2). In these articles, the authors discussed various estimation methodologies with or without right censoring. Later, (14) considered a more general class of mean residual life regression models given by

$$m(t|Z) = m_0(t)g(Z^\top \beta), \tag{A.2}$$

where $g(.)$ is a pre-specified non-negative link function and assumed to be twice continuously differentiable. Choices of g include $g(t) = 1+t$, $g(t) = \exp(t)$ and $g(t) = \log(1+\exp(t))$. Selection of an appropriate link function g may be based on prior data or the desiring interpretation of the regression parameters.

However, the above methods for inference on the parameters of the mean residual life models can not be used when some covariates are missing. Epidemiologic cohort studies and disease prevention trials typically require a large cohort to provide definitive information about covariate effects for relatively rare disease. Often it is expensive to collect the covariates of interest in large cohort studies. To provide a cost effective way of conducting such cohort studies, (12) suggested the case-cohort design. Under this design, a random sample is selected from the full cohort, named the subcohort. Survival times and censoring indicators are collected for the cohort and covariate information is collected only for the subjects in the subcohort and all the cases who experience the event of interest. The case-cohort design was later studied in a modified version by (13) and (7) using the Cox proportional hazards model. Other regression models were also studied, for example the additive hazards model ((6)), the proportional odds model ((1)) and the semiparametric transformation regression model ((8)).

Although, some authors have studied statistical inference for the hazards models under case-cohort design, it seems to have been no work for the mean residual life models under case-cohort design. Thus in this paper, we have studied a class of mean residual life regression models (A.2) under case-cohort design. The proposed models are generalization of the proportional mean residual life model with more choices of the link function $g(.)$. We develop inference procedures for estimating the parameters of models (A.2), using the martingale theory.

The rest of the paper is organized as follows. Section 2 is devoted to semiparametric inference procedures for estimating the non-parametric component $m_0(t)$ and parametric component β in model (A.2) by applying martingale estimating equations under case-cohort design. Section 3 gives the asymptotic properties of the proposed regression estimators with theoretical proofs.

B Estimating equations

Let C be the potential censoring time, and assume that the survival time T is independent of C given the $p \times 1$ covariate vector Z. To avoid lengthy technical discussion of the tail behavior of the limiting distributions, we further assume that $\Pr(C \geq \tau) > 0$, where $0 < \tau = \inf\{t : \Pr(T \geq t) = 0\} < \infty$. The observed data for a cohort of n independent subjects is denoted by $(\tilde{T}_i, \delta_i, Z_i)$, $i = 1, 2, \ldots, n$, where $\tilde{T}_i = \min(T_i, C_i)$, $\delta_i = I(T_i \leq C_i)$. Here, $I(.)$ is the indicator function. Define the counting process $N_i(t) = I(\tilde{T}_i \leq t, \delta_i = 1)$ and at-risk process $Y_i(t) = I(\tilde{T}_i \geq t)$. Consider the filtration defined by $\mathcal{F}_t = \sigma\{N_i(u), Y_i(u), Z_i : 0 \leq u \leq t; i = 1, \ldots, n\}$, then $M_i(t) = N_i(t) - \int_0^t Y_i(u) d\Lambda(u|Z_i)$ are zero-mean martingale with respect to $\mathcal{F}_t$, where $\Lambda(t|Z_i)$ denotes the cumulative hazard function of T_i given Z_i. (14) proposed the following two estimating equations to estimate $m_0(t)$ and β respectively, in model (A.2)

$$\sum_{i=1}^{n} \left[m_0(t) dN_i(t) - Y_i(t)\{g(Z_i^\top \beta)^{-1} dt + dm_0(t)\} \right] = 0 \quad (0 \leq t \leq \tau), \tag{B.1}$$

$$\sum_{i=1}^{n} \int_0^\tau \frac{g^{(1)}(Z_i^\top \beta)}{g(Z_i^\top \beta)} Z_i \left[m_0(t) dN_i(t) - Y_i(t)\{g(Z_i^\top \beta)^{-1} dt + dm_0(t)\} \right] = 0, \tag{B.2}$$

where $g^{(1)}(t) = dg(t)/dt$. Under the case-cohort design in which the covariate information is not observed for entire cohort, the estimating equations (B.1) and (B.2) are not suitable for estimating the parameters of model (A.2).

To develop the estimating equations for estimation of the regression parameter of model (A.2) under the case-cohort design, we need some additional notation. We define the size of the cohort n and the size of subcohort $\tilde{n}$ which is selected from the full cohort by the simple random sampling. Let d_i be the subcohort indicator with $d_i = 1$ if the subject is included in the subcohort and $d_i = 0$ otherwise. We assume that $\Pr(d_i = 1) = p$ which means each subject has the same probability p to be selected into the subcohort, and since we are sampling without replacement $p = \tilde{n}/n$. The survival time $\tilde{T}_i$ and the failure status δ_i are observed for all subjects in the full cohort. However, we only observe Z_i for subjects in the subcohort, where $d_i = 1$, and all the cases outside the subcohort, where $\delta_i = 1$ and $d_i = 0$. Therefore, the observed data can be summarized as $[\tilde{T}_i, \delta_i, d_i, \{\delta_i + (1 - \delta_i) d_i\} Z_i]$, $(i = 1, \cdots, n)$. Here d_i is independent of $(\tilde{T}_i, \delta_i, Z_i)$, for $i = 1, \cdots, n$, while the d_i's are dependent because of the sampling without replacement.

Similar to (8), for each subject in the full cohort, we define a weight $\pi_i = \delta_i + (1 - \delta_i)d_i/p$ by the inverse selection probabilities. Then we propose the following estimating equations by incorporating the weight π_i to estimate $m_0(t)$ and β, respectively

$$\sum_{i=1}^{n} \pi_i \big[m_0(t)dN_i(t) - Y_i(t)\{g(Z_i^\top \beta)^{-1}dt + dm_0(t)\}\big] = 0 \quad (0 \le t \le \tau), \tag{B.3}$$

$$\sum_{i=1}^{n} \int_0^\tau \frac{g^{(1)}(Z_i^\top \beta)}{g(Z_i^\top \beta)} Z_i \pi_i \big[m_0(t)dN_i(t) - Y_i(t)\{g(Z_i^\top \beta)^{-1}dt + dm_0(t)\}\big] = 0. \tag{B.4}$$

In fact, the estimating equation (B.3) is a first-order linear ordinary differential equation in $m_0(t)$

$$\left\{\frac{\sum_{i=1}^n \pi_i dN_i(t)}{\sum_{i=1}^n \pi_i Y_i(t)}\right\} m_0(t) - dm_0(t) = Q(t;\beta)dt,$$

where $Q(t;\beta) = \sum_{i=1}^n \pi_i Y_i(t) g(Z_i^\top \beta)^{-1} / \sum_{i=1}^n \pi_i Y_i(t)$. It thus has the closed form solution

$$\hat{m}_0(t;\beta) = \hat{S}(t)^{-1} \int_t^\tau \hat{S}(u)Q(u;\beta)du,$$

where $\hat{S}(t) = exp\{-\int_0^t \sum_{i=1}^n \pi_i dN_i(u)/\sum_{i=1}^n \pi_i Y_i(u)\}$.

To obtain an estimator for β, we replace $m_0(t)$ with $\hat{m}_0(t;\beta)$ in equation (B.4). Then it is straight-forward to show that the resulting equation (B.4) is equivalent to

$$U(\beta, \hat{m}_0(t;\beta)) = \sum_{i=1}^{n} \int_0^\tau \pi_i \big\{h(Z_i^\top \beta)Z_i - \bar{Z}(t)\big\}\big[\hat{m}_0(t;\beta)dN_i(t) - Y_i(t)g(Z_i^\top \beta)^{-1}dt\big], \tag{B.5}$$

where $h(t) = g^{(1)}(t)/g(t)$, and $\bar{Z}(t) = \sum_{i=1}^n \pi_i Y_i(t) h(Z_i^\top \beta) Z_i / \sum_{i=1}^n \pi_i Y_i(t)$. Let $\hat{\beta}$ denote the solution to $U(\beta, \hat{m}_0(t;\beta)) = 0$. The corresponding estimator of $m_0(t)$ is given by $\hat{m}_0(t) = \hat{m}_0(t;\hat{\beta})$.

C Asymptotic normality of regression parameters

In order to study the asymptotic properties of $\hat{\beta}$, some notations and regularity conditions are required. For any $t \in (0, \tau]$, define

$$\tilde{Z}(t) = \frac{\hat{S}(t)}{\sum_{i=1}^n \pi_i Y_i(t)} \int_0^t \hat{S}(u)^{-1} \sum_{i=1}^{n} \pi_i \big\{h(Z_i^\top \beta)Z_i - \bar{Z}(u)\big\} dN_i(u).$$

Let $\mu(t)$ and $\tilde{\mu}(t)$ be the limits of $\bar{Z}(t)$ and $\tilde{Z}(t)$, respectively. The regularity conditions are also as the following:

(C1) $\Pr(C \geq \tau) > 0$, and $N(\tau)$ is bounded almost surely.

(C2) The covariate Z is bounded.

(C3) $m_0(t)$ is continuously differentiable on $[0, \tau]$.

(C4) $A = E\big[\int_0^\tau \pi_i\{h(Z_i^\top\beta)Z_i - \mu(t)\}^{\otimes 2} Y_i(t) g(Z_i^\top\beta)^{-1} dt\big]$ is non-singular, where $a^{\otimes 2}$ denotes $aa^\top$ for a vector a.

In order to prove the asymptotic normality of $\hat{\beta}$, first we need to prove the asymptotic normality of $U(\beta; \hat{m}_0(t;\beta))$. Note that

$$\sum_{i=1}^{n} \pi_i \big[m_0(t)dN_i(t) - Y_i(t)\{g(Z_i^\top\beta)^{-1}dt + dm_0(t)\}\big] = \sum_{i=1}^{n} \pi_i m_0(t) dM_i(t),$$

and

$$\sum_{i=1}^{n} \pi_i \big[\hat{m}_0(t;\beta)dN_i(t) - Y_i(t)\{g(Z_i^\top\beta)^{-1}dt + d\hat{m}_0(t;\beta)\}\big] = 0.$$

Then it follows that

$$\frac{\sum_{i=1}^{n} \pi_i dN_i(t)}{\sum_{i=1}^{n} \pi_i Y_i(t)} \big\{\hat{m}_0(t;\beta) - m_0(t)\big\} - d\big\{\hat{m}_0(t;\beta) - m_0(t)\big\} = -\frac{\sum_{i=1}^{n} \pi_i m_0(t) dM_i(t)}{\sum_{i=1}^{n} \pi_i Y_i(t)},$$

which is a first-order linear ordinary differential equation in $\hat{m}_0(t;\beta) - m_0(t)$. It thus has the closed-form solution given by

$$\hat{m}_0(t;\beta) - m_0(t) = -\hat{S}(t)^{-1} \sum_{i=1}^{n} \int_t^\tau \hat{S}(u) \frac{m_0(u)\pi_i}{\sum_{i=1}^{n} \pi_i Y_i(u)} dM_i(u). \tag{C.1}$$

A decomposition of $n^{-1/2}U(\beta; \hat{m}_0(t;\beta))$ can be considered of the form

$$\begin{aligned} n^{-1/2}U(\beta; \hat{m}_0(t;\beta)) =& n^{-1/2} \sum_{i=1}^{n} \int_0^\tau \pi_i \big\{h(Z_i^\top\beta)Z_i - \bar{Z}(t)\big\} m_0(t) dM_i(t) \\ &+ n^{-1/2} \sum_{i=1}^{n} \int_0^\tau \pi_i \big\{h(Z_i^\top\beta)Z_i - \bar{Z}(t)\big\}\big\{\hat{m}_0(t;\beta) - m_0(t)\big\} dN_i(t) \end{aligned}$$

Thus, by the representation of $\hat{m}_0(t;\beta) - m_0(t)$ in equation (C.1), it can be written that

$$n^{-1/2}U(\beta; \hat{m}_0(t;\beta)) = n^{-1/2} \sum_{i=1}^{n} \int_0^\tau \pi_i \big\{h(Z_i^\top\beta)Z_i - \bar{Z}(t) - \tilde{Z}(t)\big\} m_0(t) dM_i(t),$$

where

$$\tilde{Z}(t) = \frac{\hat{S}(t)}{\sum_{i=1}^{n} \pi_i Y_i(t)} \int_0^t \hat{S}(u)^{-1} \sum_{i=1}^{n} \pi_i \big\{h(Z_i^\top\beta)Z_i - \bar{Z}(u)\big\} dN_i(u).$$

By the uniform strong law of large numbers, it can be written that

$$
\begin{aligned}
n^{-1/2}U(\beta;\hat{m}_0(t;\beta)) =& n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\pi_i\big\{h(Z_i^{\top}\beta)Z_i-\mu(t)-\tilde{\mu}(t)\big\}m_0(t)dM_i(t)+o_p(1)\\
=& n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\big\{h(Z_i^{\top}\beta)Z_i-\mu(t)-\tilde{\mu}(t)\big\}m_0(t)dM_i(t) \qquad \text{(C.2)}\\
&+n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}(\pi_i-1)\big\{h(Z_i^{\top}\beta)Z_i-\mu(t)-\tilde{\mu}(t)\big\}m_0(t)dM_i(t)+o_p(1) \qquad \text{(C.3)}
\end{aligned}
$$

with $\mu(t)$ and $\tilde{\mu}(t)$ are the limit in probability of $\bar{Z}(t)$ and $\tilde{Z}(t)$, respectively. The two terms on the right-hand side of the above equation are uncorrelated. The first term (C.2) can be decomposed as a sum of independent and identically distributed terms as $n^{-1/2}\sum_{i=1}^{n}\xi_i+o_p(1)$, where $\xi_i=\int_0^{\tau}\pi_i\big\{h(Z_i^{\top}\beta)Z_i-\mu(t)-\tilde{\mu}(t)\big\}m_0(t)dM_i(t)$. Hence using the multivariate central limit theorem, term (C.2) converges in distribution to a normal distribution with mean 0 and variance-covariance $\Sigma_1=E\{\xi_i^{\otimes 2}\}$. The second term (C.3) can be written as

$$
\begin{aligned}
&-n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}(1-\delta_i)(1-d_i/p)\big\{h(Z_i^{\top}\beta)Z_i-\mu(t)-\tilde{\mu}(t)\big\}m_0(t)dM_i(t)+o_p(1)\\
=&-n^{-1/2}\sum_{i=1}^{n}\eta_i(1-d_i/p)+o_p(1),
\end{aligned}
$$

where $\eta_i=(1-\delta_i)\int_0^{\tau}\{h(Z_i^{\top}\beta)Z_i-\mu(t)-\tilde{\mu}(t)\}m_0(t)dM_i(t)$. Then it can be seen that $E\{\eta_i(1-d_i/p)\}=E\{\eta_iE(1-d_i/p|\mathcal{F}_i)\}=0$, and $var\{\eta_i(1-d_i/p)\}=E\{\eta_i^{\otimes 2}var(1-d_i/p|\mathcal{F}_i)\}=(1-p)/pE\{\eta_i^{\otimes 2}\}=\Sigma_2$. Hence $n^{-1/2}U(\beta;\hat{m}_0(t;\beta))$ is asymptotically normal with mean zero and variance-covariance matrix $\Sigma=\Sigma_1+\Sigma_2$.

Since the censoring time C is independent of T and Z, and $\int_t^{\tau}S(u|Z)g(Z^{\top}\beta)^{-1}du=m_0(t)S(t|Z)$, then under model (A.2), it follows from the uniform strong law of large numbers that

$$
\begin{aligned}
\frac{\partial\hat{m}_0(t;\beta)}{\partial\beta}=&-\hat{S}(t)^{-1}\int_t^{\tau}\frac{\hat{S}(u)}{\sum_{i=1}^{n}\pi_iY_i(u)/n}\big\{n^{-1}\sum_{i=1}^{n}\pi_iY_i(u)h(Z_i^{\top}\beta)g(Z_i^{\top}\beta)^{-1}Z_idu\big\}\\
=&-S(t)^{-1}E\big\{h(Z_i^{\top}\beta)Z_i\int_t^{\tau}S(u|Z_i)g(Z_i^{\top}\beta)^{-1}du\big\}+o_p(1)\\
=&-m_0(t)\mu(t)+o_p(1). \qquad \text{(C.4)}
\end{aligned}
$$

Let $\hat{A} = n^{-1}\partial U(\beta; \hat{m}_0(t;\beta))/\partial\beta$, and $h^{(1)}(t) = dh(t)/dt$. Then it follows from (C.4) that

$$
\begin{aligned}
\hat{A} =& n^{-1}\sum_{i=1}^{n}\int_0^\tau \pi_i\big\{h^{(1)}(Z_i^\top\beta)Z_i^{\otimes 2} - \frac{\sum_{i=1}^n \pi_i Y_i(t)h^{(1)}(Z_i^\top\beta)Z_i^{\otimes 2}}{\sum_{i=1}^n \pi_i Y_i(t)}\big\}\big[\hat{m}_0(t;\beta)dN_i(t) - Y_i(t)g(Z_i^\top\beta)^{-1}dt\big]\\
&+n^{-1}\sum_{i=1}^{n}\int_0^\tau \pi_i\big\{h(Z_i^\top\beta)Z_i - \bar{Z}(t)\big\}\big[\frac{\partial\hat{m}_0(t;\beta)}{\partial\beta}dN_i(t) - Y_i(t)h(Z_i^\top\beta)Z_i g(Z_i^\top\beta)^{-1}dt\big]\\
=& n^{-1}\sum_{i=1}^{n}\int_0^\tau \pi_i\big\{h^{(1)}(Z_i^\top\beta)Z_i^{\otimes 2} - \frac{\sum_{i=1}^n \pi_i Y_i(t)h^{(1)}(Z_i^\top\beta)Z_i^{\otimes 2}}{\sum_{i=1}^n \pi_i Y_i(t)}\big\}\big[m_0(t)dM_i(t) + Y_i(t)dm_0(t)\big]\\
&-n^{-1}\sum_{i=1}^{n}\int_0^\tau \pi_i\big\{h(Z_i^\top\beta)Z_i - \bar{Z}(t)\big\}\big[\mu(t)\big\{m_0(t)dM_i(t) + Y_i(t)g(Z_i^\top\beta)^{-1}dt + Y_i(t)dm_0(t)\big\}\big]\\
&-n^{-1}\sum_{i=1}^{n}\int_0^\tau \pi_i\big\{h(Z_i^\top\beta)Z_i - \bar{Z}(t)\big\}Y_i(t)h(Z_i^\top\beta)Z_i g(Z_i^\top\beta)^{-1}dt + o_p(1) = A + o_p(1),
\end{aligned}
$$

where $A = E\big[\int_0^\tau \pi_i\{h(Z_i^\top\beta)Z_i - \mu(t)\}^{\otimes 2}Y_i(t)g(Z_i^\top\beta)^{-1}dt\big]$. Therefore, the asymptotic distribution of $\hat{\beta}$ follows from a Taylor series expansion of $U(\hat{\beta})$ at β which gives

$$
\begin{aligned}
n^{1/2}(\hat{\beta}-\beta) =& -A^{-1}n^{-1/2}U(\beta, \hat{m}_0(t;\beta)) + o_p(1)\\
=& -A^{-1}n^{-1/2}\sum_{i=1}^{n}\int_0^\tau \pi_i\big\{h(Z_i^\top\beta)Z_i - \mu(t) - \tilde{\mu}(t)\big\}m_0(t)dM_i(t) + o_p(1).
\end{aligned}
$$

Thus $n^{1/2}(\hat{\beta}-\beta)$ is asymptotically normal with zero mean and variance-covariance matrix $A^{-1}\Sigma A^{-1}$, which can be consistently estimated by $\hat{A}^{-1}\hat{\Sigma}\hat{A}^{-1}$, where

$$
\begin{aligned}
\hat{A} =& \frac{1}{n}\sum_{i=1}^{n}\pi_i\int_0^\tau \{h(Z_i^\top\hat{\beta})Z_i - \bar{Z}(t)\}^{\otimes 2}Y_i(t)g(Z_i^\top\hat{\beta})^{-1}dt,\\
\hat{\Sigma} =& \hat{\Sigma}_1 + \hat{\Sigma}_2,\\
\hat{\Sigma}_1 =& \frac{1}{n}\sum_{i=1}^{n}\Big[\int_0^\tau \pi_i\{h(Z_i^\top\hat{\beta})Z_i - \bar{Z}(t) - \tilde{Z}(t)\}\hat{m}_0(t)d\hat{M}_i(t)\Big]^{\otimes 2},\\
\hat{\Sigma}_2 =& \frac{1-p}{p}\frac{1}{n}\sum_{i=1}^{n}\Big[(1-\delta_i)\int_0^\tau \{h(Z_i^\top\hat{\beta})Z_i - \bar{Z}(t) - \tilde{Z}(t)\}\hat{m}_0(t)d\hat{M}_i(t)\Big]^{\otimes 2}.
\end{aligned}
$$

References

[1] Chen, H.Y. (2001b), Weighted semiparametric likelihood method for fitting a proportional odds regression model to data from the case-cohort design, *Journal of the American Statistical Association*, **96**(456), 1446-1457.

[2] Chen, Y.Q. (2007), Additive expectancy regression, *Journal of the American Statistical Association*, **102**(477), 153-166.

[3] Chen, Y.Q. and Cheng, S.C. (2005). Semiparametric regression analysis of mean residual life with censored survival data. *Biometrika*, **92**(1), 19-29.

[4] Chen, Y.Q. and Cheng, S.C. (2006), Linear life expectancy regression with censored data, *Biometrika*, **93**(2), 303-313.

[5] Chen, Y.Q., Jewell, N.P., Lei, X., and Cheng, S.C. (2005), Semiparametric estimaion of proportional mean residual life model in presence of censoring, *Biometrics*, **61**(1), 170-178.

[6] Kulich, M. and Lin, D.Y. (2000), Additive hazards regression for case-cohort studies, *Biometrika*, **87**(1), 73-87.

[7] Lin, D.Y. and Ying, Z. (1993), Cox regression with incomplete covariate measurements, *Journal of the American Statistical Association*, **88**(424), 1341-1349.

[8] Lu, W. and Tsiatis, A.A. (2006), Semiparametric transformation models for the case-cohort study, *Biometrika*, **93**(1), 207-214.

[9] Maguluri, G. and Zhang, C.H. (1994). Estimation in the mean residual life regression model, *Journal of the Royal Statistical Society, Series B (Methodological)*, **56**(3)m 477-489.

[10] Mansourvar, Z., Martinussen, T., and Scheike, T.H. (2015), Semiparametric regression for restricted mean residual life under right censoring, *Journal of Applied Statistics*, bf 42(12), 2597-2613.

[11] Oakes, D. and Dasu, T. (1990), A note on residual life, *Biometrika*, **77**(2), 409-410.

[12] Prentice, R.L. (1986), A case-cohort design for epidemiologic cohort studies and disease prevention trials, *Biometrika*, **73**(1), 1-11.

[13] Self, S.G. and Prentice, R.L. (1988), Asymptotic distribution theory and efficiency results for case-cohort studies, *The Annals of Statistics*, **16**(1), 64-81.

[14] Sun, L., and Zhao, Q. (2010), A class of mean residual life regression models with censored survival data, *Journal of Statistical Planning and Inference*, **140**(11), 3425-3441.

Reliability of Weighted-k-out-of-n Systems Consisting m Types of Components with Randomly Chosen Components in Each Types

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Abstract: In this paper, we consider weighted-k-out-of-n system in which $m \geq 2$ type of components each with its own positive integer-valued weight ω_i, $(i = 1, \cdots, m)$. The random lifetimes of components are from two cases: (1) the identically distributed and dependent random lifetimes of components, (2) the exchangeable and dependent random lifetimes of components in the same type with independent classes (i.e. the product copula is used). It was assumed that the random numbers $N_i, N_i = 0, 1, \cdots, n_i$ of components are chosen from class C_i for type $i(i = 1, \cdots, m)$. The structure of dependency of the system component lifetimes is modelled by copula function. The reliability of the system is obtained as a mixture of the reliability of weighted-k-out-of-n systems consisting m types of components with fixed number of them in terms of the probability mass function of the random vector $(N_1, \cdots, N_{m-1})$.

Keywords Copulas, Reliability, Weighted-$(k_1, k_2, \cdots, k_m)$-out-of-n system.

Mathematics Subject Classification (2010) : 62N05, 90B25.

A Introduction

In the most of real life systems, the total contribution of the components plays an important role and must be exceeding a predefined performance level. In many situations, the components contribute variously to the system's capacity. The weighted systems with unequal weights for each components are introduced by Wu and Chen(18) to deal with this situation which has been studied in the literature. A system including n components with their different positive integer weights is known as weighted-k-out-of-n:G system when it works if and only if the sum total weights of functioning components is exceeding a given threshold k.

Chen and Yang (1) developed the existing algorithms to calculate the system reliability of one-stage weighted-k-out-of-n model to two-stage weighted-k-out-of-n models. Samaniego and

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Shaked (16) presented a review on weighted-k-out-of-n systems. Navarro et al. (12) extended the signature-based representations of the reliability functions of coherent systems to systems with heterogeneous components. Eryilmaz (4) studied the reliability properties of a k-out-of-n system with random weights for components. Rahmani et al. (14) defined the weighted importance (WI) measure for k-out-of-n system with random weights that depends only on the distribution of component weights and also, Meshkat and Mahmoudi (10) generalized this measure for two component i and j and the relation of these measures is investigated with Birnbaum reliability importance measure. Eryilmaz and Sarikaya (6) studied the special case of weighted-k-out-of-n:G system containing two types of components, each group having different weights and reliabilities such that one group has the common weight ω and reliability p_1, while the other has the common weight ω^* and reliability p_2. They also obtained the non-recursive equations for the system reliability, survival function and Mean Time To Failure (MTTF).

The ordinary k-out-of-n system operates if at least k components work. In these kind of systems all components perform same tasks with an equal portion to the performance of the entire system. In a more general setting, the system consisting multiple types of components having different functions may be existed that different numbers of components of each type may be required for the proper operation of the whole system. Recently, Eryilmaz (7) introduced the $(k_1, k_2, \cdots, k_m)$-out-of-n system including n_i components of type i for $i = 1, \cdots, m$ and $n = \sum_{i=1}^{m} n_i$. The corresponding system is assumed to work if at least k_1 components of type 1, k_2 components of type 2, $\cdots, k_m$ components of type m function. Its reliability and the setup of weighted-$(k_1, k_2, \cdots, k_m)$-out-of-n system is also defined and studied. In this system, it is assumed that the random lifetimes of components of the same type are exchangeable and dependent, and that the random lifetimes of components of different type are dependent. That is, there are two levels of dependence: The first level dependence defines the dependence between the components of same type, and the second level of dependence is a dependence among different types of components.

Let $(T_1^{(i)}, \cdots, T_{n_i}^{(i)})$ be the vector of lifetimes of type i and ω_i denote the weight of all components of type i. The weighted-$(k_1, k_2, \cdots, k_m)$-out-of-n system is functioning if and only if the total weight of functioning components of type 1 is at least k_1, the total weight of functioning components of type 2 is at least $k_2, \cdots$ and the total weight of functioning components of type m is at least k_m. The total weight of functioning components of type i at time $t (\geq 0)$ is

$$W_i(t) = \sum_{j=1}^{n_i} \omega_i I(T_j^{(i)} > t),$$

and the total weight of the system at time $t(\geq 0)$ can be defined by

$$W_n(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \omega_i I(T_j^{(i)} > t),$$

where $I(T_j^{(i)} > t)$ is an indicator function (which is 1 if $T_j^{(i)} > t$ and 0 if $T_j^{(i)} \leq t$).

Recently, the weighted-k-out-of-n systems with dependent components have attracted great deal of attention in reliability studies. One of applied methods for modelling this dependence is to use copula functions. Nelsen (13) is provided an introduction and wide investigation on the theory of copulas. Jia and Cui (9) proposed a copula-based method for analysing the reliability of supply chains. Tang et al. (17) presented a copula-based method to investigate the impact of copulas for modelling bivariate distributions on system reliability under incomplete probability information. Eryilmaz (5) applied a multivariate copula-based method for dynamic reliability modelling of weighted-k-out-of-n systems is applied.

Using the copula is a useful viewpoint to describe the dependence between the random variables. According to Sklars theorem ((13)), any joint distribution function H of a random vector $(X_1, \cdots, X_n)$ with the marginal distribution functions $F_1, \cdots, F_n$ can be written as following

$$H(x_1, \cdots, x_n) = C_\alpha(F_1(x_1), \cdots, F_n(x_n)), \quad \forall x_i \in \mathbb{R}, \tag{A.1}$$

where $C_\alpha : [0,1]^n \to [0,1]$ is the copula function in which α is the copula parameter describing the dependency between $X_1, \cdots, X_n$. In addition, if the marginals $F_1, \cdots, F_n$ are continuous, then the copula C_α given by following is unique

$$C_\alpha(u_1, \cdots, u_n) = H(F_1^{-1}(u_1), \cdots, F_n^{-1}(u_n)),$$

where $F_i^{-1}(u) = \inf\{x : F_i(x) \geq u\}$. Conversely, if the marginal distributions of $X_1, \cdots, X_n$ and the copula function are known, then the joint distribution of random vector $(X_1, \cdots, X_n)$ can be determined by (A.1). Indeed, the copula parameter α determines the properties of $C_\alpha(u_1, \cdots, u_n)$. In reliability, the dependence structure among the component lifetimes are commonly positive, therefore the FGM family of copulas with $\alpha > 0$ is an appropriate choice for positive dependence. The n-variate form of FGM family of copulas is defined as

$$C_\alpha(u_1, \cdots, u_n) = \left(\prod_{i=1}^{n} u_i\right)\left\{1 + \alpha \prod_{i=1}^{n}(1 - u_i)\right\}, \quad \alpha \in (-1, 1).$$

In some situations, a random strategy might be a better option when the choice must be between some kinds of units such that one is more reliable than the others. Crescenzo (2), Crescenzo and Pellerey (3), Navarro et al. (11) and Hazra et al. (8) presented some useful applications of the random strategy. Salehi et al. (15) investigate the reliability and stochastic properties of weighted-k-out-of-n systems which consist of a random number of components when the components are from two different types. The structure of dependency of the components is assumed that is modelled by a copula function.

In this study, we consider the mentioned general setup of weighted-k-out-of-n system in which $m \geq 2$ types of components each with its own positive integer-valued weight ω_i, $(i = 1, \cdots, m)$ when the random lifetimes of components are from two cases: (1) the identically distributed and dependent random lifetimes of components, (2) the exchangeable and dependent random lifetimes of components in the same type with independent classes (i.e. the product copula is used). The aim of this paper is to investigate the reliability of this system which consist of the random number of components in each different types. We assume that among the n number of the components of the system, the random numbers $N_i, N_i = 0, 1, \cdots, n_i$ of components are chosen from class C_i for type $i(i = 1, \cdots, m)$. The structure of dependency of the system component lifetimes is modelled by copula function. The reliability of the system was expressed as a mixture of the reliability of weighted-k-out-of-n systems consisting m types of components with fixed number of each types in terms of the probability mass function of the random vector $(N_1, \cdots, N_{m-1})$.

The remainder of the paper is arranged as follows: In Section 2, for mentioned weighted-k-out-of-n system, the description of system and the reliability of the system lifetime is provided. Also, one example is illustrated the result. The concluding remarks are provided in Section 3.

B The system model

In this section, we consider the setup of weighted-k-out-of-n system with m types of components. The components of same type are assumed to have the same weight. This system is supposed to work with performance level k if and only if the total weight of functioning components of all types is at least k.

The components belong to m distinct classes $C_1 = \{T_1^{(1)}, \cdots, T_{n_1}^{(1)}\}, C_2 = \{T_{n_1+1}^{(2)}, \cdots, T_{n_1+n_2}^{(2)}\}, \cdots$ and $C_m = \{T_{n-n_m+1}^{(m)}, \cdots, T_n^{(m)}\}$ with sizes $n_1, n_2, \cdots$ and n_m, respectively. Suppose $F_1, \cdots$ and F_m denote the distribution functions of the components lifetime in $C_1, \cdots$ and C_m with corresponding weights $\omega_1, \cdots$ and ω_m, respectively. Indeed, components of the system are chosen from m distinct classes of components such that n_i of them is from type i with weight ω_i and lifetime distribution F_i. If $T_1^{(1)}, \cdots, T_{n_1}^{(1)}, \cdots, T_{n-n_m+1}^{(m)}, \cdots, T_n^{(m)}$ denote the lifetimes of the system components in the m classes, so the total weight of functioning components of type i at time $t(\geq 0)$ is

$$W_i(t) = \sum_{j=1}^{n_i} \omega_i I(T_j^{(i)} > t),$$

and the total weight of the system at time $t(\geq 0)$ can be defined as

$$W_n(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \omega_i I(T_j^{(i)} > t),$$

where $I(T_j^{(i)} > t)$ is an indicator function. Considering T as the lifetime of the system, then it is defined as

$$T = \inf\{t : W_n(t) < k\}.$$

and the system reliability is

$$R(t) = P(T > t) = P(W_n(t) \geq k), \qquad \forall t \geq 0.$$

Suppose that the dependence between $T_1^{(1)}, \cdots, T_{n_1}^{(1)}, \cdots, T_{n-n_m+1}^{(m)}, \cdots, T_n^{(m)}$ is modelled by the n-dimensional copula function C. Now, we consider two cases: (1) the identically distributed and dependent random lifetimes of components, (2) the exchangeable and dependent random lifetimes of components in the same type with independent classes (i.e. the product copula is used). Therefore

$$\begin{aligned}
H_1(t_1, \cdots, t_n) &= C(F_1(t_1), \cdots, F_1(t_{n_1}), \cdots, F_m(t_{n-n_m+1}) \cdots, F_m(t_n)),\\
H_2(t_1, \cdots, t_n) &= C(F_1(t_1), \cdots, F_1(t_{n_1})) \times \cdots \times C(F_m(t_{n-n_m+1}) \cdots, F_m(t_n)).
\end{aligned}$$

Here, we consider that the size of m classes $C_1, \cdots$ and C_m are random. Let $N_1, \cdots$ and N_{m-1} be a random variables with support contained in $\{0, 1, \cdots, n\}$. If $n_1, \cdots$ and n_{m-1} (the numbers of components from $C_1, \cdots$ and C_{m-1}) is selected randomly according to the random variables

$N_1, \cdots$ and N_{m-1}, then the system reliability function can be given as

$$
\begin{aligned}
&R^1_{N_1,\cdots,N_{m-1}}(t) = P(T_{N_1,\cdots,N_{m-1}} > t) \\
&= \sum_{n_1=0}^{n} \cdots \sum_{n_{m-1}=0}^{n} P\left(\omega_1 \sum_{j=1}^{n_1} I(T_j^{(1)} > t) + \cdots + \omega_m \sum_{j=1}^{n_m} I(T_j^{(m)} > t) > k\right) \\
&\quad \times P(N_1 = N_1, \cdots, N_{m-1} = n_{m-1}) \\
&= \sum_{n_1=0}^{n} \cdots \sum_{n_{m-1}=0}^{n} \underset{\substack{\omega_1 y_1 + \cdots + \omega_m y_m \geq k \\ 0 \leq y_i \leq n_i,\ i=1,\cdots,m}}{\sum \cdots \sum} \binom{n_1}{y_1} \cdots \binom{n_m}{y_m} \sum_{l_1=0}^{y_1} \cdots \sum_{l_m=0}^{y_m} (-1)^{l_1+\cdots+l_m} \\
&\quad \times \binom{y_1}{l_1} \cdots \binom{y_m}{l_m} C(\underbrace{F_1(t), \cdots, F_1(t)}_{n_1-y_1+l_1}, \cdots, \underbrace{F_m(t), \cdots, F_m(t)}_{n_m-y_m+l_m}) \\
&\quad \times P(N_1 = N_1, \cdots, N_{m-1} = n_{m-1}).
\end{aligned}
$$

$$
\begin{aligned}
&R^2_{N_1,\cdots,N_{m-1}}(t) \\
&= \sum_{n_1=0}^{n} \cdots \sum_{n_{m-1}=0}^{n} \underset{\substack{\omega_1 y_1 + \cdots + \omega_m y_m \geq k \\ 0 \leq y_i \leq n_i,\ i=1,\cdots,m}}{\sum \cdots \sum} \binom{n_1}{y_1} \cdots \binom{n_m}{y_m} \sum_{l_1=0}^{y_1} \cdots \sum_{l_m=0}^{y_m} (-1)^{l_1+\cdots+l_m} \\
&\quad \times \binom{y_1}{l_1} \cdots \binom{y_m}{l_m} \prod_{i=1}^{m} C(\underbrace{F_i(t), \cdots, F_i(t)}_{n_i-y_i+l_i}) P(N_1 = N_1, \cdots, N_{m-1} = n_{m-1}).
\end{aligned}
$$

Consider a weighted-7-out-of-6 system consisting 3 types of components with the weights $\omega_1 = 2$, $\omega_2 = 1$ and $\omega_3 = 3$. Also, suppose that the components have the lifetime distributions $F_1(t) = 1 - \frac{1}{t}$, $F_2(t) = 1 - \left(\frac{2}{t}\right)^3$ and $F_3(t) = 1 - \left(\frac{3}{t}\right)^2$. Let FGM copula models the dependence structure among components. Assume that the number of components from C_1 and C_2 is selected randomly according to the random variables N_1 and N_2 which $(N_1, N_2, N_3) \sim Multi(6, 0.2, 0.3, 0.5)$ with

$$
P(N_1 = n_1, \cdots, N_m = n_m) = \binom{n}{n_1, \cdots, n_m} \prod_{i=1}^{m} p_i^{n_i}, \qquad \sum_{i=1}^{m} p_i = 1,\ \sum_{i=1}^{m} n_i = n.
$$

Hence

$$
\begin{aligned}
R^{(1)}_{N_1,N_2}(t) = &\sum_{n_1=0}^{6} \sum_{n_2=0}^{6} P\left(2 \sum_{j=1}^{n_1} I(T_j^{(1)} > t) + \sum_{j=1}^{n_2} I(T_j^{(2)} > t) + 3 \sum_{j=1}^{n_3} I(T_j^{(3)} > t) > 7\right) \\
&\times P(N_1 = n_1, N_2 = n_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n_1=0}^{6}\sum_{n_2=0}^{6} \sum_{\substack{2y_1+y_2+3y_3\geq 7\\ 0\leq y_i\leq n_i,\ i=1,\cdots,3}}\cdots\sum \binom{n_1}{y_1}\binom{n_2}{y_2}\binom{n_3}{y_3}\sum_{l_1=0}^{y_1}\cdots\sum_{l_4=0}^{y_4}(-1)^{l_1+\cdots+l_4} \\
&\times\left(1-\frac{1}{t}\right)^{n_1-y_1+l_1}\left(1-\frac{8}{t^3}\right)^{n_2-y_2+l_2}\left(1-\frac{9}{t^2}\right)^{n_3-y_3+l_3} \\
&\times\left(1+\alpha\left(\frac{1}{t}\right)^{n_1-y_1+l_1}\left(\frac{8}{t^3}\right)^{n_2-y_2+l_2}\left(\frac{9}{t^2}\right)^{n_3-y_3+l_3}\right) \\
&\times P(N_1=n_1,N_2=n_2).
\end{aligned}
$$

$$
\begin{aligned}
R^{(2)}_{N_1,N_2}(t) &= \sum_{n_1=0}^{6}\sum_{n_2=0}^{6} \sum_{\substack{2y_1+y_2+3y_3\geq 7\\ 0\leq y_i\leq n_i,\ i=1,\cdots,3}}\cdots\sum \binom{n_1}{y_1}\binom{n_2}{y_2}\binom{n_3}{y_3}\sum_{l_1=0}^{y_1}\cdots\sum_{l_4=0}^{y_4}(-1)^{l_1+\cdots+l_4} \\
&\times\left(1-\frac{1}{t}\right)^{n_1-y_1+l_1}\left(1+\alpha\left(\frac{1}{t}\right)^{n_1-y_1+l_1}\right) \\
&\times\left(1-\frac{8}{t^3}\right)^{n_2-y_2+l_2}\left(1+\alpha\left(\frac{8}{t^3}\right)^{n_2-y_2+l_2}\right) \\
&\times\left(1-\frac{9}{t^2}\right)^{n_3-y_3+l_3}\left(1+\alpha\left(\frac{9}{t^2}\right)^{n_3-y_3+l_3}\right)P(N_1=n_1,N_2=n_2).
\end{aligned}
$$

C Conclusion

In this paper, we studied reliability of weighted-k-out-of-n system in which $m \geq 2$ types of components each with its own positive integer-valued weight ω_i, $(i = 1, \cdots, m)$. The random lifetimes of components are from two cases: (1) the identically distributed and dependent random lifetimes of components, (2) the exchangeable and dependent random lifetimes of components in the same type with independent classes (i.e. the product copula is used). It was assumed that the random numbers $N_i, N_i = 0, 1, \cdots, n_i$ of components are chosen from class C_i for type $i(i = 1, \cdots, m)$. The structure of dependency of the system component lifetimes is modelled by copula function. The reliability of the system was expressed as a mixture of the reliability of weighted-k-out-of-n systems consisting m types of components with fixed number of them in terms of the probability mass function of the random vector $(N_1, \cdots, N_{m-1})$.

References

[1] Chen, Y. and Yang, Q. (2005), Reliability of two-stage weighted k-out-of-n systems with components in common, *IEEE Transactions on Reliability*, **54**, 431-440.

[2] Di Crescenzo, A. (2006), A Parrondo paradox in reliability theory, *Math Sci*, **32**, 17-22.

[3] Di Crescenzo, A. and Pellerey, F. (2011), Stochastic comparisons of series and parallel systems with randomized independent components, *Operations Research Letters*, **39**(5), 380-384.

[4] Eryilmaz, S. (2013), On reliability analysis of a k-out-of-n system with components having random weights, *Reliability Engineering and System Safety*, **109**, 41-44.

[5] Eryilmaz, S. (2014), Multivariate copula based dynamic reliability modeling with application to weighted-k-out-of-n systems of dependent components, *Structural Safety*, **51**, 23-28.

[6] Eryilmaz, S. and Sarikaya, K. (2014), Modeling and analysis of weighted-k-out-of-n:G system consisting of two different types of components, *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, **228**(3), 265-271.

[7] Eryilmaz, S. (2019), $(k_1, k_2, \cdots, k_m)$-out-of-n system and its reliability, *Journal of Computational and Applied Mathematics*, **346**, 591-598.

[8] Hazra, N.K., Finkelstein, M. and Cha, J.H. (2017), On optimal grouping and stochastic comparisons for heterogeneous items, *Journal of Multivariate Analysis*, **160**, 146-156.

[9] Jia, X. and Cui, L. (2012), Reliability research of k-out-of-n:G supply chain system based on copula, *Communications in Statistics-Theory and Methods*, **41**(21), 4023-4033.

[10] Meshkat, R. S. and Mahmoudi, E. (2017), Joint reliability and weighted importance measures of a k-out-of-n system with random weights for components, *Journal of Computational and Applied Mathematics*, **326**, 273-283.

[11] Navarro, J., Pellerey, F. and Di Crescenzo, A. (2015), Orderings of coherent systems with randomized dependent components, *European Journal of Operational Research*, **240**(1), 127-139.

[12] Navarro, J., Samaniego, F.J. and Balakrishnan, N. (2011), Signature-based representations for the reliability of systems with heterogeneous components. *Journal of Applied Probability*, **48**, 856-867.

[13] Nelsen, R.B. (2006), *An introduction to copulas.* Springer, New York.

[14] Rahmani, R. A., Izadi, M. and Khaledi, B. E. (2016), Importance of components in k-out-of-n system with components having random weights, *Journal of Computational and Applied Mathematics*, **296**, 1-9.

[15] Salehi, M., Shishebor, Z. and Asadi, M. (2018), On the reliability modeling of weighted-k-out-of-n systems with randomly chosen components, *Metrika*, 1-17.

[16] Samaniego, F. J. and Shaked, M. (2008), Systems with weighted components, *Statistics and Probability Letters*, **78**, 815-823.

[17] Tang, X.S., Li, D.Q., Zhou, C.B., Phoon, K.K. and Zhang, L.M. (2013), Impact of copulas for modeling bivariate distributions on system reliability, *Structural Safety*, **44**, 80-90.

[18] Wu, J.S. and Chen, R.J. (1994), An algorithm for computing the reliability of a weighted-k-out-of-n system, *IEEE Transactions on Reliability*, **43**, 327-328.

Bayesian Analysis of Masked Data with Non-ignorable Missing Mechanism

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Abstract: The problem of estimation lifetime parameters in the presence of masked data is considered for a series system. Maximum Likelihood Estimations are derived and compared under missing not at random (MNAR) and missing at random (MAR) mechanisms. The results show superior performance of our approach when non-ignorable missing mechanism is occurred. Morevere, the Bayesian approach is expanded for estimation of model parameters. Bayesian analysis led to less biasness for parameters estimation than classical analysis. The proposed method is illustrated through a real example.

Keywords Masked Data, Non-ignorable Missing Data, Markov chain Monte Carlo Method.
Mathematics Subject Classification (2010) : 62M05**,** 49N30**,** 65C40**.**

A Introduction

The exact cause of failure is important in reliability estimation of a series system. However, in many cases, because of some reasons, such as cost and time limitations, the exact cause of failure is not recognized, but only is identified that belongs to a smaller set of causes. These data are called to be Masked data (Miyakawa;1984, Basu, **et al**;1999).

Some works have been done on classical statistical inference of Masked data. For example: Miyakawa (1984), Usher and Hodgson (1988) and Lin **et al**. (1993). Also some authors considered Bayesian analysis for Masked data, such as: Reiser **et al**. (1995), Berger and Sun (1993), Mukhopadhyay and Basu (1997), Basu **et al**. (1999), Mukhopadhyay (2006) and Xu **et al**. (2014).

Missing data appears when data value is not observed for a variable. Missing data have different mechanisms with respect to missingness reasons. If missingness depends only on observed values, missing mechanism is called missing at random (MAR), while if missingness depends on

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both observed and missing values, missing mechanism is called missing not at random (MNAR) (Little & Rubin, 2002).

In this work we aim to estimate the parameters of interest in Bayesian framework, introducing a missing indicator according to the masking status of each observed failure time. A MNAR mechanism is assumed and the Bayesian estimates are derived. We conducted some simulation studies, which show superior results for our proposed model when masking has a non-ignorable mechanism. The rest of the paper is as follow. In Section 2, the model assumptions are introduced, and the general formulation of the likelihood functions are given. In Section 3, the auxiliary variables are introduced, and the Bayesian analysis is discussed. In section 4, The proposed methodology is presented using numerical example. Finally we conclude the paper in Section 5.

B Model Assumptions and Likelihood Function

A. Assumptions

Suppose that we have r series systems under the test such that all of them have equal components, say J components. Assume that at the end of the test we observe failure data, $t_1, t_2, ..., t_r$, but the exact cause of failure might be unknown, and only we know that belongs to the Minimum Random Subset (MRS) of $\{1, 2, ..., J\}$. Let M_i be the observed MRS corresponding to the failure time $t_i, i = 1, 2, ..., r$ for ith system. The set M_i essentially includes components that are possible to be cause for system failure. If M_i be a singleton set, then the data are competing risks data. While if $M_i = \{1, ..., J\}$ then the system is called to be completely masked. We define the binary variable R_i which takes 1, when M_i is a singleton set and 0 for masked data (when M_i has more than one element). Thus, the observed data are

$$(t_1, M_1, R_1), (t_2, M_2, R_2), ..., (t_r, M_r, R_r). \tag{B.1}$$

The model used in this paper is based on the following assumptions:

- Let $T_1, T_2, ..., T_J$ be the lifetimes of J independent components, also assume that system fails only due to one of the J components, therefor system failure time is $T = min(T_1, ..., T_J)$.
- T_l, the failure time of the lth component, follows a distribution in continuous distribution family with density and reliability functions denoted by $f_l(t), R_l(t)$.

- $Pr(M = M_i | T = t_i, K_i = l)$ is called the masking probability, where K_i denotes the exact cause of failure of ith system. In this article, we assume $Pr(M = M_i | T = t_i, K_i = l) = Pr(M = M_i | K_i = l) = p_l(M_i)$, that is, the masking probability is independent of failure time, but is dependent to the causes of failure.

- $p_l(M_i)$s have some constraints. Suppose M be the all of nonempty subsets of $\{1, ..., J\}$ that have $2^J - 1$ members. Define $M_l = \{M_0 \in M : l \in M_0, l \in \{1, ..., J\}\}$ thus

$$p_l(M_i) = P(M = M_i | K_i = l) = 0 \qquad \forall M_i \in M_l^c = M - M_l$$

and

$$\sum_{M_i \in M} p_l(M_i) = \sum_{M_i \in M_l} p_l(M_i) = 1, l = 1, ..., J. \tag{B.2}$$

Denote $p_l = \{P_l(M_i) : M_i \in M_l\}, l = 1, 2, ..., J$ then $p = (p_1, ..., p_J)$.

- If T be the system failure time, the reliability function is given by

$$R(t) = R(t; \theta) = P(T > t) = \prod_{l=1}^{J} [1 - F_l(t)] \tag{B.3}$$

Where $\theta = (\theta_1, ..., \theta_J)$, and θ_l is parameters set related to component l.

- Suppose K be a random variable which indicates the cause of failure. Then the joint probability distribution function of (T, K) is given by

$$f_{T,K}(t, l) = f_l(t) \prod_{j \neq l} [1 - F_j(t)]. \tag{B.4}$$

- R_i is a Bernoulli variable with success probability

  p($R_i = 1 | k_i = j I_{\{j \in M_i\}}, M_i, t_i) = h(\beta_0 + \beta_1 k_i + \beta_2 t_i)$,

where h(.) is some appropriate link function (e.g. logit, probit, clog-log,...). When $\beta_1 = 0$, the missing is ignorable and missing mechanism is MAR.

B. Likelihood Function

The likelihood function for data (B.1) can be written as follow:

$$\begin{aligned} L(\theta, p, \beta | t, M, R) &= \prod_{i=1}^{r} [\sum_{j \in M_i} P(R_i | t_i, M_i, K_i = j) p(M_i | t_i, K_i = j) f_{T,K}(t_i, j)] \\ &= \prod_{i=1}^{r} [\sum_{j \in M_i} P(R_i | t_i, M_i, K_i = j) p_j(M_i) f_{T,K}(t_i, j)] \end{aligned} \tag{B.5}$$

Where $\beta = (\beta_0, \beta_1)$, and θ is the vector of parameters related to liftime distribution.
If the missing mechanism is at random ($\beta_1 = 0$) then the above likelihood is reduced to:

$$L_R(\theta) \propto \prod_{i=1}^{r} \left[f(R_i|t_i) \sum_{j \in M_i} p_j(M_i) f_{T,K}(t_i, j) \right] \tag{B.6}$$

Where simple masked data analysis is used.

C Bayesian Analysis

First, we simplify the likelihood function using the auxiliary random variables. Define $I_{ij} = I(T_j = t_i)$ for $1 \leq i \leq r$ and $1 \leq j \leq J$, where $I(.)$ is the indicator variable such that shows the exact cause-of-failure. $I_{ij} = 1$ means that the ith system failed due to component j where $j \in M_i$. If $M_i = \{j\}$ is a singleton set, that is the failure cause is known, then $I_{ij} = 1$ and $I_{ij'} = 0, j' \neq j$. Therefore likelihood function (B.5) can be rewritten as follow, respectively:

$$\begin{aligned} L(\theta, p, \beta|t, M, R) &= \prod_{i=1}^{r} [\prod_{j \in M_i} (P(R_i|t_i, M_i, K_i = j) p_j(M_i) f_{T,K}(t_i, j))^{I_{ij}}] \\ &= \prod_{i=1}^{r} [\prod_{j=1}^{J} (P(R_i|t_i, M_i, K_i = j) p_j(M_i) f_{T,K}(t_i, j)^{I_{ij}})] \end{aligned} \tag{C.1}$$

Then, we consider the Dirichlet distribution, $D(\gamma_j)$, as prior distribution for p_j, where γ_j is the 2^{J-1} dimensional vector. The choice of prior distributions for other parameters will be s-dependent on the CDF that is considered for T_l.
If $\pi(\theta)$, $\pi(\beta)$ and $\pi_l(p_l)$ be the priors for parameters θ, β and p_l respectively, then the joint density function of (t, M, I, R) is

$$p(t, M, I, R) = p(\theta, p, \beta|t, M, I, R) \pi(\theta) \pi(\beta) \prod_{l=1}^{J} \pi_l(p_l) \tag{C.2}$$

The likelihood function for Exponential distribution with parameter α_l for lth component based on C.1 is as follow:

$$\begin{aligned} L(\theta, p, \beta|t, M, R) &= \prod_{i=1}^{r} [\prod_{j=1}^{J} (P(R_i|t_i, M_i, K_i = j)^{I_{ij}} p_j(M_i)^{I_{ij}} \alpha_j^{I_{ij}})^{I_{ij}}] \\ &\quad \times exp\{-\sum_{i=1}^{r} \sum_{l=1}^{J} \alpha_l t_i\}. \end{aligned} \tag{C.3}$$

D Numerical Example

In this section, we would try to illustrate the proposed method by a simulation study.

A. Exponential Distribution

In this subsection we assume 100 series systems with two components where the lifetime of components follow Exponential distribution with parameters α_1 and α_2 for first and second component, respectively. We have generated non-ignorable missing mechanism according to the logistic regression $logit(p(R_i = 1|k_i = j)) = \beta_0 + \beta_1 k_i$. The masking probabilities of the data are p_1 and p_2, where $p_1 = p_1(\{1,2\})$ and $p_2 = p_2(\{1,2\})$. Let $\alpha_1 = 0.3$, $\alpha_2 = 0.7$, $\beta_0 = -0.1$, $\beta_1 = 0.5$, $p_1 = 0.1$ and $p_2 = 0.2$.

The simulated data are listed in Table 1.
The results of maximum likelihood estimation of parameters α_1 and α_2 are presented in Table 2. In Table 2, there are the true value of the parameters and the amount of biasness for α_1 and α_2 (denoted by $B\alpha_1$ and $B\alpha_2$, respectively) in 1000 iterations of each simulation study. According to the results, MNAR model leads to less biased estimators compared with the usual MAR model.

Table 1: The simulated data

(t,k,R)
(1.501,1,1),(2.386,1,1),(0.849,2,1),(2.223,1,0),(3.055,1,1),(0.444,0,0),(1.878,1,0),(4.101,2,1),
(1.965,2,0),(0.252,1,0),(1.938,2,1),(0.221,1,0),(0.282,1,1),(2.006,2,1),(0.511,2,1),(3.057,1,1),
(0.563,2,1),(1.308,2,1),(0.160,1,0),(1.341,2,1),(2.546,2,1),(0.040,0,0),(1.863,1,0),(1.207,2,1),
(1.497,1,1),(1.717,0,0),(0.705,2,1),(3.170,1,1),(0.067,1,1),(1.808,1,1),(0.319,2,0),(3.444,2,1),
(0.676,2,1),(0.566,2,1),(0.960,1,1),(0.299,0,0),(2.111,0,0),(0.210,1,1),(0.433,2,1),(0.868,2,1),
(0.275,2,1),(2.029,2,0),(3.218,2,1),(0.584,1,1),(1.221,2,1),(0.224,0,0),(0.485,1,1),(0.333,0,0),
(0.919,2,1),(0.209,2,1),(0.816,1,1),(1.488,2,1),(1.234,2,1),(1.792,0,0),(1.681,2,1),(0.291,2,1),
(0.815,1,0),(0.444,2,1),(2.776,2,1),(0.718,1,0),(0.847,2,1),(1.362,2,1),(2.438,2,0),(1.735,2,1),
(1.481,2,1),(0.471,2,1),(0.545,0,0),(0.688,1,1),(1.489,2,1),(2.274,1,0),(1.095,1,1),(0.265,2,1),
(0.166,2,1),(0.557,1,1),(0.181,2,1),(0.544,2,0),(2.207,2,1),(0.246,2,1),(0.645,1,1),(0.095,1,0),
(0.090,2,1),(0.195,2,0)(0.486,2,1),(0.203,2,1),(0.215,2,1),(0.248,2,1),(1.310,2,1),(0.826,2,1),
(0.198,2,1),(1.634,0,0),(0.689,2,1),(0.357,1,0),(3.419,0,0),(1.148,2,0),(0.607,2,1),(1.249,2,1),
(1.259,1,1),(0.921,2,1),(0.071,2,1),(2.169,2,1)

Table 2: The results of simulation analysis

	α_1	α_2	β_1	β_0	Bα_1	Bα_2
MAR	0.3	0.7	-0.1	0.5	0.038	0.033
MNAR					0.023	0.018

Now we introduce some proper priors for parameters in the MNAR model and obtain Bayesian estimates for the parameters using MCMC method with masking probabilities $p_1 = 0.1$, $p_2 = 0.2$ and true values $a_1 = 0.3$, $a_2 = 0.7$. We consider the following prior set

$$\begin{aligned} &a_1 \sim gamma(0.9, 3), \qquad a_2 \sim gamma(0.49, 0.7), \qquad \beta_0 \sim norm(-0.1, 1000), \\ &\beta_1 \sim norm(0.5, 1000) \qquad p_1 \sim Beta(0.8, 7.2) \qquad p_2 \sim Beta(0.01, 0.05). \end{aligned} \tag{D.1}$$

Using 10,000 iterations of Gibbs sampling with burn-in 2,000 iterations and length of the thinning interval 5, the Bayes estimates of the parameters based on (D.1) in 1,600 posterior samples are listed in table 3. Also standard deviation (SD), lower bound (LCI) and upper bound (UCI) of credible interval is calculated.

Table 3: Bayes estimates of parameters

Parameter	True Value	mean	SD	LCI	UCI
p_1	0.1	0.388	0.079	0.241	0.545
p_2	0.2	0.340	0.058	0.224	0.460
a_1	0.3	0.295	0.056	0.195	0.415
a_2	0.7	0.606	0.075	0.474	0.756
b_0	-0.1	-0.138	0.301	-0.199	-0.077
b_1	0.5	0.418	0.028	0.364	0.471

E Conclusion

In this paper, we have introduced a new approach for handle masked data. We have used a generalized linear model to illustrate relationship between masking probability and exact cause of failure in a missingness framework. MCMC sampling method has been used to obtain Bayes estimates of model parameters. Finally, simulation study demonstrates the usefulness of our methods.

References

[1] Levuliene, R. (2002), Semiparametric estimation and goodness-of-fit tests for tire wear and failure time data, *Nonlinear Analysis: Modelling and Control*, **7**, 61-95.

[2] Little, R.J.A. and Rubin, D.B. (2002), *Statistical Analysis With Missing Data*, 2nd Edition, Joh Wiley & Sons, New York.

[3] Misaii, H., Eftekhari Mahabadi, S. (2016), *Analysis of Masked Data with nonignorable Missing Mechanism*, 2nd Seminar on Reliability Theory and its Applications.

[4] Mukhopadhyay, C. (2006), Maximum likelihood analysis of masked series system lifetime data, *Journal of statistical planning and inference*, **136**, 803-838.

[5] Xu, A. Basu, S. and Tang, Y. (2014), A Full Bayesian Approach for Masked Data in Step-Stress Accelerated Life Testing, *IEEE Transactions on Reliability*, **63**(3), 798-806.

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On the Properties of a Reliability Dependent Model

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Abstract: In this paper, a model for two dependent series-parallel systems with random number of sub-systems is introduced and the dependence structure of the proposed model is studied. Moreover, the dependency measures including the Kendall's tau and Spearman's rho are investigated. Furthermore, some of the bivariate reliability indexes such as the bivariate mean residual life and the bivariate reversed hazard rate are presented.

Keywords Kendall's tau, Spearman's rho, Bivariate reversed hazard rate, mean reversed residual lives.

A Introduction

Constructing flexible families of lifetime distributions is one of the most interesting topics in reliability. Several methods have been presented to construct such families. One of them was introduced by Marshall and Olkin (1997) in which the authors considered the component-wise maximum (minimum) of N independent and identical bivariate random vectors when N is a discrete random variable. Some other researchers such as Kundu and Gupta (2014), Zhang et al. (2016) and Roozegar and Nadarajah (2017) used the same method to construct flexible lifetime distributions.

Modeling dependence has attracted the researchers attention in recent decades. The form of stochastic dependence is a useful way of formulating properties of a dependent model. There has been several dependence concepts defined in the literature. Let (X, Y) be a random vector with the bivariate survival distribution function $\bar{F}(.,.)$ and the marginal survival distributions $\bar{F}_1(.)$ and $\bar{F}_2(.)$, respectively. Then

(i) It is said that X and Y are positively quadrant dependent (denoted by $PQD(X, Y)$) if and only if $\bar{F}(x, y) \geq \bar{F}_1(x)\bar{F}_2(y)$ for all $x, y \geq 0$.

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(ii) The random variable Y is said to be right tail increasing (denoted by $RTI(Y \mid X)$) when $\bar{F}(x,y)/\bar{F}_1(x)$ is non-decreasing in x for all $y \geq 0$.

(iii) The random vector (X,Y) is said to be right corner set increasing (denoted by $RCSI(X,Y)$) if $P(X > x, Y > y \mid X > x', Y > y')$ is non-decreasing in x' and in y' for all x and y, equivalently if $\dfrac{\partial^2 \log \bar{H}}{\partial x \partial y} \geq 0$.

See for example, Nelsen (2006), for dependence concepts.

Kendall's tau τ and Spearman's rho ρ are two well-known measures of dependence that express the strength of association between random variables. For the pair (X,Y) they are given by, respectively,

$$\tau = 4 \int_0^\infty \int_0^\infty \bar{F}(x,y) d\bar{F}(x,y) - 1,$$

and

$$\rho = 12 \int_0^\infty \int_0^\infty \left(\bar{F}(x,y) - \bar{F}_1(x)\bar{F}_2(y) \right) d\bar{F}_1(x) d\bar{F}_2(y). \tag{A.1}$$

For more information about the structure and measure of dependence, we refer the reader to Joe (1997) and Nelsen (2006). In reliability theory, several indexes has been introduced for characterizing the properties of survival models. The mean residual life function characterizes the stochastic behavior of survival over time. For non-negative continuous random variables X and Y the bivariate mean residual life is given by $m(x,y) = (m_1(x,y), m_2(x,y))$, where

$$m_1(x,y) = E\left(Y - y \mid X > x, Y > y\right) = \frac{\int_x^\infty \bar{F}(u,y) du}{\bar{F}(x,y)}$$

and

$$m_2(x,y) = E\left(X - x \mid X > x, Y > y\right) = \frac{\int_y^\infty \bar{F}(x,u) du}{\bar{F}(x,y)},$$

for more details we refer the reader to Arnold and Zahedi (1988), Nair and Asha (2008) and Kolev and et al. (2017). Roy (2002) introduced the bivariate vector $r_{1,2}(x,y) = (r_1(x,y), r_2(x,y))$ as the bivariate reversed hazard rate where

$$r_1(x,y) = r(x \mid Y < y) = \frac{\partial \log F(x,y)}{\partial x},$$

and

$$r_2(x,y) = r(y \mid X < x) = \frac{\partial \log F(x,y)}{\partial y}.$$

One could refer to Domma (2010) for more details.

In this paper, we are going to model the lifetime distribution of two dependent series-parallel systems with random number of sub-systems and we are going to investigate the dependency and reliability indexes of the corresponding model. The general form of the proposed model is discussed in Section 2. In Section 3, we are going to study the dependency properties of the model. The reliability indexes of the model are obtained in Section 4. Concluding remarks are given in Section 5.

B The Proposed Model

It is known that systems with parallel structure succeed if at least one of it's components work correctly and a system with series structure succeed if all of it's components work correctly. Series-parallel (parallel-series) systems are composed of fixed number of parallel (series) sub-systems which are connected in series (parallel). Here, we intend to introduce a lifetime model constructed from two dependent series-parallel systems with random number of sub-systems. With this in mind, consider two series-parallel systems with random number of sub-systems N, which N is a positive integer-valued random variable with two values 1 and 2 with probability $1-\theta$ and θ, respectively, where $0 \leq \theta \leq 1$. Let $Z_{i,j}^{(1)}$ and $Z_{i,j}^{(2)}$ for $j = 1, \dots, m$ and $i = 1, \dots, N$ be the lifetimes of the j-th component of the i-th sub-system in system(I) and system(II), respectively. Also, assume that $Z_{i,j}^{(r)}$ are iid with common marginal cdf $F_r(.)$ for $j = 1, \dots, m$, $i = 1, \dots, N$, $r = 1, 2$. Furthermore, suppose that $Z_{i,j}^{(1)}$, $Z_{i,j}^{(2)}$ and N are independent random variables. Let $X_i = \max\{Z_{i,1}^{(1)}, \dots, Z_{i,m}^{(1)}\}$ for $i = 1, \dots, N$ be the lifetime of the i-th sub-system of system (I) and $Y_l = \max\{Z_{i,1}^{(2)}, \dots, Z_{i,m}^{(2)}\}$ for $l = 1, \dots, N$ be the lifetime of the l-th sub-system of system (II), respectively. Moreover, suppose that $T_1 = \min\{X_1, \dots, X_N\}$ and $T_2 = \min\{Y_1, \dots, Y_N\}$ are the lifetimes of system (I) and system (II), respectively. Then the bivariate survival function of (T_1, T_2) is as the following

$$
\begin{aligned}
\bar{H}_{T_1,T_2}(x,y) &= (1-\theta)P(T_1 > x, T_2 > y \mid N = 1) + \theta P(T_1 > x, T_2 > y \mid N = 2) \\
&= (1-\theta)P(X_1 > x, Y_1 > y) + \theta \prod_{i=1}^{2} P(X_i > x, Y_i > y).
\end{aligned}
$$

Since for $j = 1, \dots, m$ the random variables $Z_{i,j}^{(1)}$ and $Z_{i,j}^{(2)}$ are independent the random variables X_i and Y_i for $i = 1, 2$ are independent too. Thus, the corresponding survival function of (T_1, T_2) is found to be

$$
\bar{H}_{T_1,T_2}(x,y) = P(X_1 > x)P(Y_1 > y)\left(1 - \theta(1 - P(X_2 > x)P(Y_2 > y))\right). \tag{B.1}
$$

Given that $Z_{i,j}^{(1)}$ for $i = 1, 2$ and $j = 1, \dots, m$ are iid random variables with corresponding cdf $F_1(.)$ and $Z_{l,j}^{(2)}$ for $l = 1, 2$ and $j = 1, \dots, m$ are iid random variables with corresponding cdf

$F_2(.)$, for $i = 1, 2$ we have

$$\begin{aligned} P(X_i > x) &= P(\max\{Z_{i,1}^{(1)}, \dots, Z_{i,m}^{(1)}\} > x) \\ &= 1 - P(\max\{Z_{i,1}^{(1)}, \dots, Z_{i,m}^{(1)}\} \leq x) = (1 - F_1^m(x)) \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} P(Y_i > y) &= P(\max\{Z_{i,1}^{(2)}, \dots, Z_{i,m}^{(2)}\} > x) \\ &= 1 - P(\max\{Z_{i,1}^{(2)}, \dots, Z_{i,m}^{(2)}\} \leq x) = (1 - F_2^m(y)). \end{aligned} \tag{B.3}$$

By exploiting (B.2) and (B.3) in (B.1) the survival function of (T_1, T_2) can be reexpressed as

$$\bar{H}_{T_1,T_2}(x, y) = (1 - F_1^m(x))(1 - F_2^m(y))[1 - \theta(1 - (1 - F_1^m(x))(1 - F_2^m(y)))]. \tag{B.4}$$

In the special case let $m = 1$ in (B.4), then

$$\bar{H}_{T_1,T_2}(x, y) = \bar{F}_1(x)\bar{F}_2(y)[1 - \theta(1 - \bar{F}_1(x)\bar{F}_2(y))]. \tag{B.5}$$

That is a member of the model proposed in Mirhosseini et al. (2015). It should be noted that the corresponding density function of (B.4) is found to be

$$h(x, y) = m^2 f_1(x) f_2(y) F_1^{m-1}(x) F_2^{m-1}(y) \left((1 - \theta) + 4\theta(1 - F_1^m(x))(1 - F_2^m(y))\right).$$

Now, let us take $F_1(x) = 1 - e^{-\alpha_1 x}$ and $F_2(y) = 1 - e^{-\alpha_2 y}$ in (B.4), then

$$\begin{aligned} \bar{H}_{T_1,T_2}(x, y) =& (1 - (1 - e^{-\alpha_1 x})^m)(1 - (1 - e^{-\alpha_2 y})^m) \\ & \times [1 - \theta\left(1 - (1 - (1 - e^{-\alpha_1 x})^m)(1 - (1 - e^{-\alpha_2 y})^m)\right)]. \end{aligned} \tag{B.6}$$

Moreover, if $m = 1$ we have

$$\bar{H}_{T_1,T_2}(x, y) = e^{-(\alpha_1 x + \alpha_2 y)}[1 - \theta\left(1 - e^{-(\alpha_1 x + \alpha_2 y)}\right)]. \tag{B.7}$$

And the corresponding density function of (B.6) is given by

$$\begin{aligned} h(x, y) =& m^2 \alpha_1 \alpha_2 e^{-\alpha_1 x} e^{-\alpha_2 y} (1 - e^{-\alpha_1 x})^{m-1} (1 - e^{-\alpha_2 y})^{m-1} \\ & \times \left((1 - \theta) + 4\theta(1 - (1 - e^{-\alpha_1 x})^m)(1 - (1 - e^{-\alpha_2 y})^m)\right). \end{aligned} \tag{B.8}$$

The pdf (B.8) is displayed in Figure 2 when $m = 10$, $\alpha_1 = 5$, $\alpha_2 = 10$, $\theta = 0.5$.

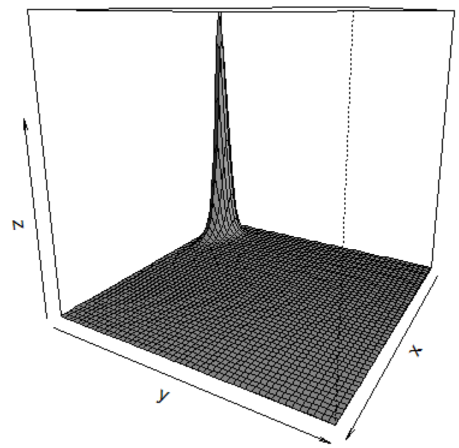

Figure 1: The pdf in (B.8) when $m = 10$, $\alpha_1 = 5$, $\alpha_2 = 10$, $\theta = 0.5$.

C Dependency Properties

In the problem of the dependence modeling, the researchers are interested to know about the dependence structure of the model which illustrates the attitude of two random variables that are related to each other. In the next proposition, some concepts of dependence are presented for the proposed model. Let (T_1, T_2) be a random vector distributed as (B.4), then

i) $PQD(T_1, T_2)$;

ii) $RTI(T_2 \mid T_1)$;

iii) $RCSI(T_1, T_2)$.

. i) Since

$$\bar{H}(x, y) - \bar{H}_1(x)\bar{H}_2(y) = \theta(1-\theta)F_1^m(x)F_2^m(y)(1 - F_1^m(x))(1 - F_2^m(y)) \geq 0,$$

we can conclude that $PQD(T_1, T_2)$.

ii) Let $\phi(x, y) = \dfrac{\bar{H}(x, y)}{\bar{H}_1(x)}$ then

$$\frac{\phi(x, y)}{\partial x} = \frac{m\theta(1-\theta)f_1(x)F_1^{m-1}(x)F_2^m(y)(1 - F_2^m(y))}{(1 - \theta F_1^m(x))^2} \geq 0.$$

This proves that $RTI(T_2 \mid T_1)$.

iii) By simple calculation from (B.4), we have

$$\frac{\partial^2 \log \bar{H}}{\partial x \partial y} = \frac{m^2\theta(1-\theta)f_1(x)F_1^{m-1}(x)f_2(y)F_2^{m-1}(y)}{(1 - \theta(1 - (1 - F_1^m(x))(1 - F_2^m(y))))^2} \geq 0.$$

So, the random vector (T_1, T_2) is RCSI. $\square$

It is interesting to note that the dependence measures quantify the strength of dependence between two random variables. In the next proposition, the Kendall's tau and Spearman's rho of model (B.6) are given. Consider (T_1, T_2) as a random vector distributed as (B.6), then

i) $\tau = \dfrac{2}{9}\theta(1-\theta)$;

ii) $\rho = \dfrac{12\theta(1-\theta)}{\alpha_1\alpha_2}\left(\sum_{j=1}^{m}\dfrac{1}{2m-j+1}\sum_{i=1}^{m}\dfrac{1}{2m-i+1}\right)$.

Figure 1 displays the Kendall's tau and Spearman's rho of (B.6) when $m = 10$, $\alpha_1 = 5$ and $\alpha_2 = 10$. As we can see these two dependence measures increase by increasing θ and after reaching their maximum value at $\theta = 0.5$, they decrease.

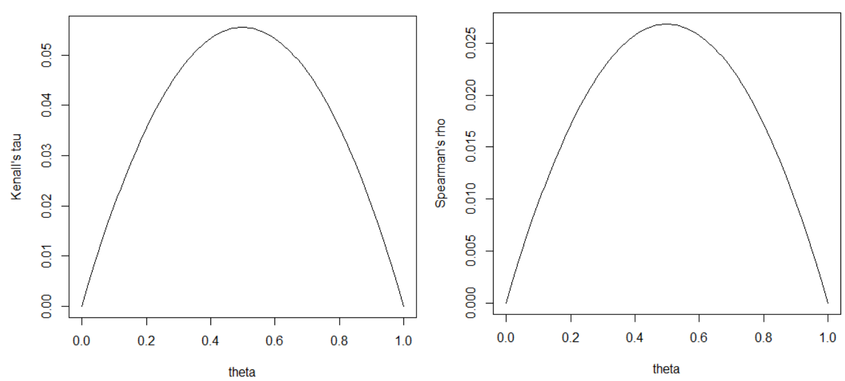

Figure 2: The Kendall's tau and Spearman's rho of the proposed model.

D Reliability Properties

In this section we will investigate some reliability properties. The probability that a system fails before the other one is an important issue in reliability theory. The next proposition provides this probability for the proposed model. Suppose that (T_1, T_2) have the bivariate distribution as given in (B.6), then

$$P(T_1 < T_2) = \frac{2\alpha_2^3 + (2\theta+5)\alpha_1\alpha_2^2 + 2(1-\theta)\alpha_1^2\alpha_2}{(\alpha_1+\alpha_2)(\alpha_2+2\alpha_1)(\alpha_1+2\alpha_2)}. \tag{D.1}$$

Figure 2 illustrates the attitude of $P(T_1 < T_2)$ with respect to α_1, α_2 and θ. As we can see $P(T_1 < T_2)$ increase with respect to θ when $m = 10, \alpha_1 = 5, \alpha_2 = 10$. Moreover, $P(T_1 < T_2)$

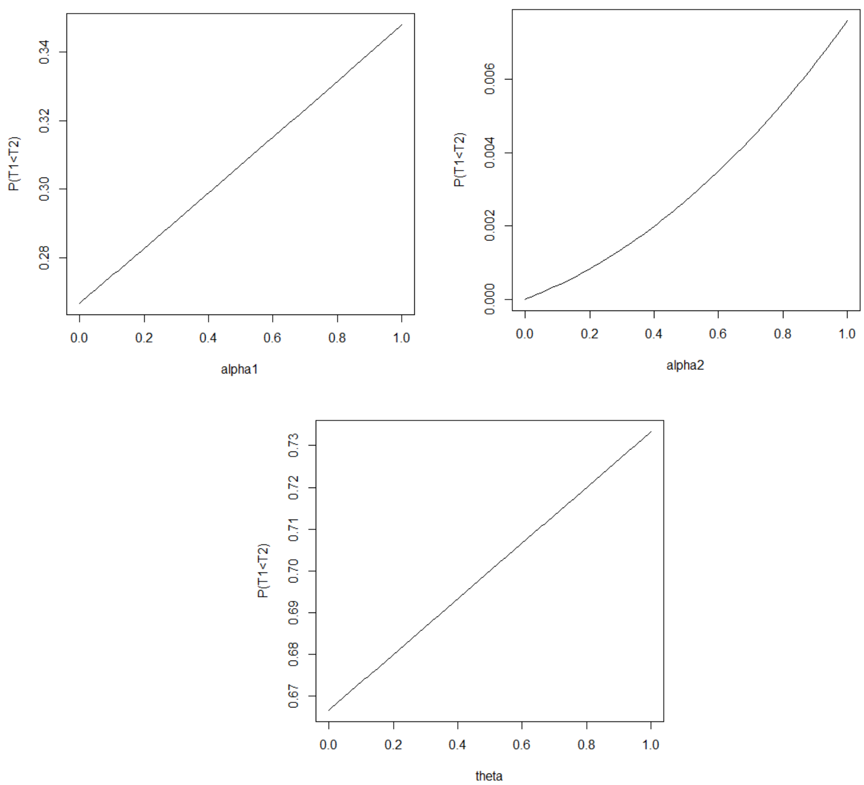

Figure 3: The probability that a system I fail before system II.

increase with respect to α_2 when $m = 10, \alpha_1 = 5, \theta = 0.5$. In addition,when $m = 10, \alpha_2 = 10, \theta = 0.5$, $P(T_1 < T_2)$ increase with respect to α_1.

The bivariate reversed hazard rate of the proposed model is the next result. Let (T_1, T_2) be distributed as (B.6) then the corresponding bivariate reversed hazard rate is given by $r_{1,2}(x, y) = (r_1(x, y), r_2(x, y))$ where

$$r_1(x,y) = m\alpha_1 e^{-\alpha_1 x}(1-e^{-\alpha_1 x})^{m-1}\{\frac{2\theta(1-e^{-\alpha_1 x})^m + (1-\theta)(1-e^{-\alpha_2 y})^m}{H(x,y)} - \frac{2m\theta(1-(1-e^{-\alpha_1 x})^m)(1-(1-e^{-\alpha_2 y})^m)}{H(x,y)}\}$$

and

$$r_2(x,y) = m\alpha_2 e^{-\alpha_2 y}(1-e^{-\alpha_2 y})^{m-1}\{\frac{2\theta(1-e^{-\alpha_2 y})^m + (1-\theta)(1-e^{-\alpha_1 x})^m}{H(x,y)} - \frac{2m\theta(1-(1-e^{-\alpha_2 y})^m)(1-(1-e^{-\alpha_1 x})^m)}{H(x,y)}\}.$$

In the special case, if $m = 1$, then

$$r_1(x, y) = \frac{\alpha_1 e^{-\alpha_1 x} \left(2\theta(1 - e^{-\alpha_1 x}) + (1 - \theta)(1 - e^{-\alpha_2 y}) - 2\theta e^{-(\alpha_1 x + \alpha_2 y)}\right)}{H(x, y)}$$

and

$$r_2(x, y) = \frac{\alpha_2 e^{-\alpha_2 y} \left(2\theta(1 - e^{-\alpha_2 y}) + (1 - \theta)(1 - e^{-\alpha_1 x}) - 2\theta e^{-(\alpha_1 x + \alpha_2 y)}\right)}{H(x, y)},$$

where

$$\begin{aligned} H(x, y) = 1 &- e^{-\alpha_1 x} \left((1 - \theta) + \theta e^{-\alpha_1 x}\right) - e^{-\alpha_2 y} \left((1 - \theta) + \theta e^{-\alpha_2 y}\right) \\ &+ e^{-(\alpha_1 x + \alpha_2 y)} \left((1 - \theta) + \theta e^{-(\alpha_1 x + \alpha_2 y)}\right). \end{aligned}$$

It should be mentioned that $r_1(x, y)\triangle x$ is an approximation for the probability for the failure of the first component in interval $(x, \triangle x)$ given that it has failed before x and the second component has failed before y.

The bivariate mean residual life is another ageing measure which is stated in the next proposition. Suppose that the random vector (T_1, T_2) is distributed as in (B.6) then the vector of mean residual life is $m_{1,2}(x, y) = (m_1(x, y), m_2(x, y))$ where

$$\begin{aligned} m_1(x, y) = &\frac{(1 - \theta(1 - e^{-\alpha_2 y})^m) \sum_{i=1}^m (\dfrac{1 - (1 - e^{-\alpha_1 x})^{m-i+1}}{m - i + 1})}{(1 - (1 - e^{-\alpha_1 x})^m) (1 - \theta(1 - (1 - (1 - e^{-\alpha_1 x})^m)(1 - (1 - e^{-\alpha_2 y})^m)))} \\ &- \frac{\theta(1 - (1 - e^{-\alpha_2 y})^m) \sum_{i=1}^m (\dfrac{(1 - e^{-\alpha_1 x})^{2m-i+1} - 1}{2m - i + 1})}{(1 - (1 - e^{-\alpha_1 x})^m) (1 - \theta(1 - (1 - (1 - e^{-\alpha_1 x})^m)(1 - (1 - e^{-\alpha_2 y})^m)))} \end{aligned}$$

and

$$\begin{aligned} m_2(x, y) = &\frac{(1 - \theta(1 - e^{-\alpha_1 x})^m) \sum_{i=1}^m (\dfrac{1 - (1 - e^{-\alpha_2 y})^{m-i+1}}{m - i + 1})}{(1 - (1 - e^{-\alpha_2 y})^m) (1 - \theta(1 - (1 - (1 - e^{-\alpha_2 y})^m)(1 - (1 - e^{-\alpha_1 x})^m)))} \\ &- \frac{\theta(1 - (1 - e^{-\alpha_1 x})^m) \sum_{i=1}^m (\dfrac{(1 - e^{-\alpha_2 y})^{2m-i+1} - 1}{2m - i + 1})}{(1 - (1 - e^{-\alpha_2 y})^m) (1 - \theta(1 - (1 - (1 - e^{-\alpha_2 y})^m)(1 - (1 - e^{-\alpha_1 x})^m)))}. \end{aligned}$$

In the special case, if $m = 1$, then

$$m_1(x, y) = \frac{2\left(1 - \theta(1 - e^{-\alpha_2 y})\right) e^{-\alpha_1 x} - \theta e^{-\alpha_2 y} \left((1 - e^{-\alpha_1 x})^2 - 1\right)}{2e^{-\alpha_1 x} \left(1 - \theta(1 - e^{-(\alpha_1 x + \alpha_2 y)})\right)}$$

and

$$m_2(x, y) = \frac{2\left(1 - \theta(1 - e^{-\alpha_1 x})\right) e^{-\alpha_2 y} - \theta e^{-\alpha_1 x} \left((1 - e^{-\alpha_2 y})^2 - 1\right)}{2e^{-\alpha_2 y} \left(1 - \theta(1 - e^{-(\alpha_1 x + \alpha_2 y)})\right)}.$$

E Conclusions

In this paper, a dependent model consisting two series-parallel systems that contain a random number of parallel sub-systems with fixed components connected in series was presented. Moreover, the dependence structure and some dependence measures such as the Kendall's tau and Spearman's rho were calculated. Furthermore, some bivariate reliability properties such as the bivariate hazard rate and the vector of mean residual life was investigated.

References

[1] Arnold, B. and Zahedi H. (1988), On multivariate mean remaining life functions, *Journal of Multivariate Analysis* **25**, 1-9.

[2] Domma, F. (2011), Bivariate reversed hazard rate, notions, and measures of dependence and their relationships, *Communications in Statistics-Theory and Methods*, **40**, 989-999.

[3] Joe, H. (1997) *Multivariate Models and Dependence Concepts*, Chapman and Hall London, .

[4] Kolev, N. Ngoc, N. and Ju, Y.T. (2017), Bivariate teissier distributions, *In: Rykov V., Singpurwalla N., Zubkov A. (eds) Analytical and Computational Methods in Probability Theory. Lecture Notes in Computer Science. Springer* **10684**, 279-290.

[5] Kundu, D. and Gupta, A. (2014), On bivariate Weibull-geometric distribution, *Journal of Multivariate Analysis*, **123**, 19-29.

[6] Marshall, I. and Olkin, A.W. (1997), A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika*, **84**, 641-652.

[7] Mirhosseini, S.M., Amini, M. and Dolati, A. (2015), On a general structure of the bivariate FGM type distributions. *Applications of Mathematics*, **60**, 91-108.

[8] Nair, N. and asha, G. (2008), Some characterizations based on bivariate reversed mean residual life, *Prob Stat Forum*, **1**, 1-14.

[9] Nelsen, R.B. (2006), *An Introduction to Copulas*, Springer.

[10] Roozegar, R. Nadarajah, S. (2017), New classes of power series bivariate copulas, *Journal of Computational and Applied Mathematics*, **326**, 235-246.

[11] Roy, D. (2002), On characterization of model approach for generating bivariate life distributions using reversed hazard rates, *Journal of the Japan Statistical Society*, **2**, 239-245.

[12] Zhang, K., Lin, J. and Xu, P. (2016), A new class of copulas involving geometric distribution: Estimation and applications, *Mathematics and Economics*, **66**, 1-10.

A Dynamic Predictive Maintenance Policy for Inverse Gaussian Process

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Abstract: Condition based maintenance (CBM) is a practical and effective way to guarantee the product availability. The optimization strategy of CBM is widely studied. Despite the gamma process, the inverse Gaussian (IG) process is new in this concept. Here, we deal with a dynamic condition-based maintenance of single-unit systems where the deterioration is governed by an IG process. To be more realistic, the parameters of the model are considered to be unknown. We employ the Bayes method to use the available information of degradation paths and update the information about parameters during the time.

Keywords Condition-based maintenance, Remaining useful life, Inverse Gaussian process, Bayesian update.

Mathematics Subject Classification (2010) : 90B25.

A Introduction

Maintenance is an important method for products to guarantee availability. Many maintenance strategies have been proposed; Wang (11) made a clear classification of different possible strategies. The great improvement in technology which enables accurate online measurements of degradation levels made engineers to tend towards condition-based maintenance (CBM) policies.

Degradation modeling plays an important role in maintenance decision-making. Many kinds of degradation models have been developed in CBM. Alaswad and Xiang (1) classifies CBM policies depending on the deterioration model. The major part of their paper discussed the maintenance strategies in which the degradation evolution is described by stochastic processes;

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the Wiener, the gamma and the inverse Gaussian (IG) processes received special mention. Examples of CBM policies based on the Wiener process are investigated by (4), (6), and (14). A vast amount of literature is devoted to the use of the gamma process in maintenance modeling; e.g. see (5; 3; 9; 8; 10).

Another stochastic process appropriate for modeling degradation and for maintenance studies is the IG process. Wang and Xu (12) originally introduced this process to reliability literature. Ye and Chen (13) then gave a meaningful physical interpretation for modeling degradation by showing that the IG process is a limit of a compound Poisson process. The IG process has greate flexibility in modeling degradation data. However, the literature on CBM policies is scarce (see e.g. (2; 7)). Chen et al.(2) investigated the optimal CBM policy with periodic inspections when the system degradation follows an IG process with a random-drift model, while Li et al. (7) obtained the optimal CBM strategy by maximizing the availability under cost contraints.

In this paper, we discuss the condition-based maintenance of a single unit system whose degradation follows an IG process. The considered system has degradation parameters that are unknown but a prior information like expert judgment is available. An adaptive Bayesian method is employed to update the information about parameters after inspections as the degradation state of the system is measured. In order to reduce unnecessary inspections and to control maintenance actions, an aperiodic maintenance policy is considered. The time interval between two successive inspections is scheduled based on the remaining useful life (RUL) of the system. We show how to employ Bayes theorem to obtain successive updates for posterior distribution of the parameters, thus increasing our knowledge of the model parameters with inspection results. We also study the effect of these updates on the RUL function, used in inspection planning, as well as on the behavior of the cost function.

The remainder of the paper is organized as follows. Section B provides a basic description of the system and states the main assumptions of the maintenance policy. The system state and long run maintenance cost is described Section C for the case of unknown model parameters. Section D describes the procedure of Bayesian updating and its use in adapting current information. Different maintenance policies are introduced in Section E In order to demonstrate the validity of the proposed maintenance policies, they are compared in Section F. Finally, Section G contains a conclusion.

B System and Maintenance Decision Rule: Descriptions and Main Assumptions

Consider a single component system which is subjected to degradation. Let X_t denote the degradation state of the system at time t. In the absence of repair or replacement actions, the evolution of the system degradation is assumed to be strictly increasing. Then X_t can be modeled by an increasing stochastic process. Moreover, other assumptions are considered:

- The initial state X_0 is 0.
- The system is failed if its degradation crosses a critical threshold level L.
- The system failure is not self-announcing and if it fails, it remains failed until the next inspection. This down time imposes some extra cost.

To avoid the occurrence of system failure, a preventive threshold $M < L$ is chosen such that corrective action is taken when the degradation level crosses this threshold.

B.1 Stochastic Degradation Process

We assume that the stochastic degradation process X_t can be modeled by an IG process such that

1. $X_0 = 0$ with probability one;
2. X_t has independent non-overlapping increments;
3. Each increment follows an IG distribution, that is, $X_t - X_s \sim IG(\mu(t-s), \lambda(t-s)^2)$ for all $t > s$; where the probability distribution function of $IG(\mu t, \lambda t^2)$ is defined as follows:

$$f_{\mu t,\lambda t^2}(x) = \sqrt{\frac{\lambda t^2}{2\pi x^3}} \exp\{-\frac{\lambda}{2x}(\frac{x}{\mu} - t)^2\}, x > 0, \mu, \lambda > 0. \tag{B.1}$$

B.2 Predictive Maintenance Decision Rule

Let $\{T_n\}_{n\in\mathbb{N}}$ be the aperiodic sequence of inspection times ($T_0 = 0$). At each inspection, one must take the required maintenance decision according to the condition of the system at the time. We assume that the maintenance actions are performed in a negligible time and T_n^- refers to the time just before the maintenance. The possible scenarios which can arise are:

- If $X_{T_n^-} \geq L$, the system has failed and is correctively replaced.

- If $M \leq X_{T_n^-} < L$, the system has not yet failed but it has deteriorated to the extent that it cannot function properly. In this case a preventive action is taken.
- If $X_{T_n^-} < M$, the system is still properly functioning and there is no need for replacement. The system is left as it is.

Both preventive and corrective replacements are perfect and reset the system to "as good as new". Hence after the inspection we have:

$$X_{T_n} = \begin{cases} 0 & \text{if } X_{T_n^-} \geq L, \\ 0 & \text{if } M \leq X_{T_n^-} < L, \\ X_{T_n^-} & \text{if } X_{T_n^-} < M. \end{cases}$$

In all cases, the inspections scheduling is carried out based on the RUL which is defined as the duration for which a system will work before it fails. The main idea of a RUL based inspection plan is that the next inspection time is chosen so that the probability of failure before the inspection remains lower than a value p $(0 < p < 1)$), where p is a decision variable to be jointly optimized with M. Hence:

$$T_{n+1} = T_n + \tau_p(X_{T_n}),$$

where $\tau_p(X_{T_n}) = \{\triangle t : \Pr(X_{T_n+\triangle t} \geq L | X_{T_n}) = p\}$. In other words, T_{n+1} is the p-quantile the remaining useful life distribution. This inspection plan provides a reliability (safety) level equal to $(1-p)$.

B.3 Cost Function

The inspections are planned discretely and each of them incurs a cost C_i. At each inspection, a preventive or corrective action is performed if necessary, with costs C_p and C_c respectively. Clearly, $C_c > C_p$. Moreover, since failure can only be detected through inspection, there is a system downtime after failure and an additional cost at a rate of C_d is incurred from the failure time until the next replacement time. Hence the cumulative maintenance cost is:

$$C(t) = C_i N_i(t) + C_p N_p(t) + C_c N_c(t) + C_d d(t),$$

where $N_i(t)$, $N_p(t)$, and $N_c(t)$ are respectively the number of inspections, the number of preventive replacements, and the number of corrective replacements in $[0, t]$. Furthermore, $d(t)$ is the total time passed in a failed state in $[0, t)$. Now two objective cost functions can be considered:

- The expected cost function on a finite horizon time T_{end}; i.e., $EC_{T_{\text{end}}}{=}E[C(T_{\text{end}})]$. For comparison between strategies, we use $EC_{T_{\text{end}}}/T_{\text{end}}$.
- The expected cost of the system per unit of time, or long-run average cost per time unit, defined by
$$EC_{\infty} = \lim_{t\to\infty} \frac{E[C(t)]}{t}.$$

It is well known that if the degradation process has the regenerative property,

$$EC_{\infty} = \frac{E[C(S_1)]}{E(S_1)}, \tag{B.2}$$

where S_1 is the first replacement time. This means that the long-run average cost per time unit is equal to the ratio of the expected cost on the first regenerative cycle, S_1, over the expected length of the regenerative cycle for almost any realization of the process.

C System State and Long Run Maintenance Cost

In this section, we derive an expression for the long run expected maintenance cost when the parameters of the degradation are unknown. The considered maintenance policy is based on the maintenance decision rule given in subsection B.2.

Let $\{Y_n = X_{T_n}\}_{n\in\mathbb{N}}$ be the discrete-time random process describing the system state at each inspection time. Grall et al. (5) derived the properties of the process $\{X_t\}_{t\geq 0}$ and the embedded chain $\{Y_n\}_{n\in\mathbb{N}}$, when $\{X_t\}_{t\geq 0}$ is a gamma process and the parameters of the models are known. Here, we use the IG process instead of gamma process.

In practice, the parameters of the model are not known and Bayesian approach can be helpfull. In this case, the available information about the parameters is used as a prior distribution and then this information is updated using Bayesian methods. Here, for ease of calculation we use conjugate priors . Hence, suppose λ has the gamma density function

$$f(\lambda) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\{-\lambda/\beta\}, \tag{C.1}$$

and let $\delta = 1/\mu$ have the conditional normal density function with mean ξ and variance σ^2/λ

$$f(\delta|\lambda) = \sqrt{\frac{\lambda}{2\pi\sigma^2}} \exp\{-\frac{\lambda(\delta-\xi)^2}{2\sigma^2}\}. \tag{C.2}$$

Then the joint prior distribution of (δ, λ) is given by $f(\delta, \lambda) = f(\delta|\lambda)f(\lambda)$. It is worthwhile to mention that to avoid the probability of obtaining negative degradation slopes, we suppose that $P(\delta \leq 0)$ is negligible.

The objective is to determine the distribution of the time until the degradation signal reaches the failure threshold, given the knowledge of the condition state of the system at time t. The CDF of remaining useful life can be obtained by:

$$F_R(r|X_t) = 1 - \sqrt{\frac{\beta}{2\pi}} \frac{\Gamma(\alpha+1/2)r}{\Gamma(\alpha)} \int_0^{L-X_t} z^{-3/2}(\sigma^2 z+1)^{-1/2} \left(1 + \frac{\beta(\xi z - r)^2}{2z(\sigma^2 z+1)}\right)^{-(\alpha+1/2)} \mathrm{d}z, \tag{C.3}$$

where X_t is the observed degradation at time t.

Then, the time between inspections given the current degradation value is the p-quantile of the distributions given in (C.3).

For unknown parameters and given priors, the properties of $\{X_t\}_{t\geq 0}$ and $\{Y_n = X_{T_n}\}_{n\in\mathbb{N}}$ are listed as follows.

- The process $\{X_t\}_{t\geq 0}$ is a regenerative and semi-regenerative process with regeneration time and semi-regeneration times, S_1 and T_1, respectively.

- The process $\{Y_n\}_{n\in\mathbb{N}}$ is a Markov chain that takes values in $[0, M)$ and has transition probability density function (conditioned on the current state)

  $$\Pr(\mathrm{d}y|x) = \overline{G}_{\tau_p(x)}(M-x)\delta_0(\mathrm{d}y) + g_{\tau_p(x)}(y-x)I_{\{x\leq y<M\}}\mathrm{d}y,$$

  where

  $$g_{\tau_p(x)}(z) = \sqrt{\frac{\beta}{2\pi}} \frac{\Gamma(\alpha+1/2)\tau_p(x)}{\Gamma(\alpha)} z^{-3/2}(\sigma^2 z+1)^{-1/2} \left(1 + \frac{\beta(\xi z - \tau_p(x))^2}{2z(\sigma^2 z+1)}\right)^{-(\alpha+1/2)},$$

  and

  $$\overline{G}_{\tau_p(x)}(z) = \int_z^\infty g_{\tau_p(x)}(u)du.$$

- Similarly, the unique stationary probability distribution of $\{Y_n\}_{n\in\mathbb{N}}$ is given by:

  $$\pi(\mathrm{d}x) = a\delta_0(\mathrm{d}x) + (1-a)b(x)\mathrm{d}x, \tag{C.4}$$

  with

  $$a = \frac{1}{1+\int_0^M B(x)\mathrm{d}x}, \qquad b(y) = \frac{a}{1-a}B(y), \tag{C.5}$$

  where

  $$B(y) = g_{\tau_p(0)}(y) + \int_0^y B(x)g_{\tau_p(x)}(y-x)\mathrm{d}x. \tag{C.6}$$

- In this case, the expected cost of the system per unit of time can also be assessed by:

$$
\begin{aligned}
EC_\infty &= \lim_{t\to\infty} \frac{E[C(t)]}{t} = \frac{E_\pi[C(T_1)]}{E_\pi[T_1]} \\
&= \frac{C_i E_\pi[N_i(T_1)]}{E_\pi[T_1]} + \frac{C_p E_\pi[N_p(T_1)]}{E_\pi[T_1]} + \frac{C_c E_\pi[N_c(T_1)]}{E_\pi[T_1]} + \frac{C_d E_\pi[d(T_1)]}{E_\pi[T_1]}.
\end{aligned} \tag{C.7}
$$

The corresponding expectations are computed as follows:

$$
\begin{aligned}
E_\pi(N_p(T_1)) &= P_\pi(M \le X_{T_1^-} < L) = \int_0^M \left[\overline{G}_{\tau_p(x)}(M-x) - \overline{G}_{\tau_p(x)}(L-x)\right] \pi(\mathrm{d}x), \\
E_\pi(N_c(T_1)) &= P_\pi(X_{T_1^-} \ge L) = \int_0^M \overline{G}_{\tau_p(x)}(L-x)\pi(\mathrm{d}x), \\
E_\pi(d(T_1)) &= \int_0^M \left[\int_0^{\tau_p(x)} \overline{G}_s(L-x)\mathrm{d}s\right] \pi(\mathrm{d}x), \\
E_\pi(T_1) &= \int_0^M \tau_p(x)\pi(\mathrm{d}x).
\end{aligned}
$$

D Degradation Model Update

The previous section gives probabilistic characteristics of the system evolution under a given aperiodic maintenance policy. It allows us to derive the long run expected maintenance cost and hence optimize the decision variables, p and M uch that the long-run expected maintenance cost reaches its minimum value. As the prior information could be not so near to dynamic of a system, such optimized values of p and M may differ from the optimized vlues which can be obtained if the parameter were known. Then in this section, we propose using Bayes method to update the information about parameters of model with available degradation data as follows.

Proposition: Suppose $Data = \{X_{t_0}, X_{t_1}, \ldots, X_{t_k}\}$ are the observed degradation data at times $t_0, t_1, \ldots, t_k$, then given the observed data, the posterior distributions are:

$$
\begin{aligned}
f(\delta|\lambda, Data) &= \sqrt{\frac{\lambda}{2\pi{\sigma^*}^2}} \exp\{-\frac{\lambda(\delta-\xi^*)^2}{2{\sigma^*}^2}\}, \\
f(\lambda|Data) &= \frac{\lambda^{\alpha^*-1}}{\Gamma(\alpha^*){\beta^*}^{\alpha^*}} \exp\{-\lambda/\beta^*\}.
\end{aligned}
$$

The updated hyperparameters are $\xi^* = B/A$, $\sigma^* = A^{-1/2}$, $\alpha^* = \alpha + k/2$, and $\beta^* = (1/\beta + 1/D)^{-1}$; where $\triangle x_i = X_{t_i} - X_{t_{i-1}}$, $\triangle t_i = t_i - t_{i-1}$ and

$$
A = \sum_{i=1}^k \triangle x_i + \frac{1}{\sigma^2}, \qquad B = \sum_{i=1}^k \triangle t_i + \frac{\xi}{\sigma^2},
$$

$$
C = \sum_{i=1}^k \frac{(\triangle t_i)^2}{\triangle x_i} + \frac{\xi^2}{\sigma^2}, \qquad D = \frac{1}{2}(C - \frac{B^2}{A}).
$$

This Bayesian procedure allows us to update current information about δ and λ with each new observation. Updating can take place at each inspection or at the end of each cycle with a specified number of inspections. It is notable that these updates affect the RUL function and consequently the next inspection time.

E Different Maintenance Policies

The maintenance decision rule described in B.2 depends on two decision variables which are the value of p for the p-quantile of the RUL and the preventive threshold M.

Here, we consider options for maintenance policies. With consideration of previous sections, there are different maintenance policies for unknown parameters which can be compared with known parameter case to see which one is more appropriate. We listed them as follows:

Maintenance policy 1: in this case, using a given proir information, the EC_∞ can be assessed with the method in C. Then the optimal value of EC_∞ and the corresponding optimal values of M and p can be obtained. This policy is so dependaent to the choice of prior distribution.

Maintenance policy 2: in this case, the Bayesian update is invovled. The hyperparameters are updated at each inspection time. The expression for the p-quantile of the RUL is modified accordingly. No analytical expression is available for EC_∞ when using this policy. To illustrate what can be obtained and assess the best results possible, $EC_{T_{\text{end}}}$ can be estimated by Monte Carlo simulation and then be minimized. This policy have good properties to reach the optimal values. But real values μ_{real} and λ_{real} are required for such simulations which is a drawback.

Maintenance policy 3: to overcome the above mentioned difficulty, this policy proposes a sub-optimal procedure in which the decision variables are optimized for each cycle according to the available knowledge at the beginning of the cycle and used for one renewal cycle, i.e. until the next update. The optimization is based on the long-run cost rate EC_∞ as given in C and is conducted using the most recent updated values of hyperparameters.

In the next section, these policies are analyzed on a finite time horizon and compared numerically using the estimate of $EC_{T_{\text{end}}}$ as a criterion.

F Simulation Study

To illustrate the procedure, a simulation study with a predefined parameters is conducted and the outcome is compared with the outcome in the case of unknown parameter. A system is considered on a limited time horizon (from $t = 0$ to $t = T_{\text{end}}$ with $T_{\text{end}} = 200$). The limited time horizon allows us to discuss the adaptation performance of the policies for unknown parameters. All samples of degradation increments are generated from an IG process with fixed parameters $\mu_{\text{real}} = 2$ and $\lambda_{\text{real}} = 8$. The failure level is set at $L = 9$. In the case of unknown parameters parameters a prior distribution is assumed as described in C. Initial values considered for the hyperparameters are as follows:

$$\alpha = 1.5,\ \beta = \frac{2}{3},\ \xi = 1, \text{ and } \sigma = \frac{1}{\sqrt{3}}. \tag{F.1}$$

F.1 Comparison of Maintenance policies

In the case of known parameters, the optimal EC_{∞} and the optimal values of p and M are as follows:

$$p^* = 0.028,\ M^* = 7.02, \text{ and }\ EC^*_{\infty} = 1.313. \tag{F.2}$$

In the case of unknown parameters, the optimized values for the Maintenance policy 1 are:

$$p^* = 0.115,\ M^* = 6.49, \text{ and }\ EC^*_{\infty} = 1.19, \tag{F.3}$$

which differ from the optimal value in (F.2).

According to Maintenance policy 2, the optimal values are:

$$p^* = 0.036,\ M^* = 7, \text{ and }\ \widehat{EC}^*_{T_{\text{end}}}/T_{\text{end}} = 1.322. \tag{F.4}$$

A comparison of the values with values from (F.2) reveals the efficiency of the Bayesian update in the maintenance decision rule. The estimated minimum values of $EC_{T_{\text{end}}}$ are very close to those obtained for known decision variables which means that the influence of the prior weakens over the time and that the Bayesian update is efficient from the maintenance point of view.

The Maintenance policy 3 allows decision variables to be optimized sequentially. But different simulation shows that this sequence of optimal values tend to the otimal values in the case of known parameters. For this simulation study, we can see the optimal values are close to (F.2).

$$p^* = 0.032,\ M^* = 6.82, \text{ and }\ \widehat{EC}^*_{T_{\text{end}}}/T_{\text{end}} = 1.318. \tag{F.5}$$

Obviously, the Maintenance policy 3 has a good ability to find the optimal values of decision variables and then minimize the cost while it has no same drawback like the Maintenance policy 2.

G Conclusion

This paper investigated optimizing the CBM strategy for single-unit systems with unknown degradation parameters assuming that the degradation is governed by an IG process and that the unknown parameters jointly follow a prior distribution with specified hyperparameters. The Bayes method is employed to update information about parameters over time. Simulation studies are conducted to illustrate the behavior of the policy and optimal maintenance variables are obtained using cost as a criterion.

References

[1] Alaswad, S. and Xiang, Y. (2017), A review on condition-based maintenance optimization models for stochastically deteriorating system, *Reliability Engineering & System Safety*, **157**, 54-63.

[2] Chen, N., Ye, Z.S., Xiang, Y. and Zhang, L. (2015), Condition-based maintenance using the inverse Gaussian degradation model, *European Journal of Operational Research*, **243**(1), 190-199.

[3] Dieulle, L., Bérenguer, C., Grall, A. and Roussignol, M. (2003), Sequential condition-based maintenance scheduling for a deteriorating system, *European Journal of Operational Research*, **150**(2), 451-461.

[4] Elwany, A.H., Gebraeel, N.Z. and Maillart, L.M. (2011), Structured replacement policies for components with complex degradation processes and dedicated sensors, *Operations Research*, **59**(3), 684-695.

[5] Grall, A., Dieulle, L., Bérenguer, C. and Roussignol, M. (2002), Continuous-time predictive-maintenance scheduling for a deteriorating system, *IEEE Transactions on Reliability*, **51**(2), 141-150.

[6] Guo, C., Wang, W., Guo, B. and Si, X. (2013), A maintenance optimization model for mission-oriented systems based on Wiener degradation, *Reliability Engineering & System Safety*, **111**, 183-194.

[7] Li, R., Liu, X., Song, Y., Cai, Z. and Li, J. (2017, July), An optimal condition based maintenance strategy using inverse-Gaussian degradation process, In Reliability Systems Engineering (ICRSE), 2017 Second International Conference on (pp. 1-5), IEEE.

[8] Mercier, S. and Castro, I.T. (2013), On the modelling of imperfect repairs for a continuously monitored gamma wear process through age reduction, *Journal of Applied Probability*, **50**(4), 1057-1076.

[9] Tan, L., Cheng, Z., Guo, B. and Gong, S. (2010), Condition-based maintenance policy for gamma deteriorating systems, *Journal of Systems Engineering and Electronics*, **21**(1), 57-61.

[10] van Noortwijk, J.M. (2009), A survey of the application of gamma processes in maintenance, *Reliability Engineering & System Safety*, **94**(1), 2-21.

[11] Wang, H. (2002), A survey of maintenance policies of deteriorating systems, *European Journal of Operational Research*, **139**(3), 469-489.

[12] Wang, X. and Xu, D. (2010), An inverse Gaussian process model for degradation data, *Technometrics*, **52**(2), 188-197.

[13] Ye, Z.S. and Chen, N. (2014), The inverse Gaussian process as a degradation model, *Technometrics*, **56**(3), 302-311.

[14] Zhang, M., Gaudoin, O. and Xie, M. (2015), Degradation-based maintenance decision using stochastic filtering for systems under imperfect maintenance, *European Journal of Operational Research*, **245**(2), 531-541.

The Stochastic Properties of General Conditional Random Variables of the Dependent Components in the $(n-m+1)$-out-of-n Systems

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Abstract: In this paper, the general conditional residual lifetime and inactivity time of the $(n-m+1)$-out-of-n systems with arbitrary dependent components are considered. Also, we investigate some stochastic properties of the general conditional residual lifetime and inactivity time of the system when the components are exchangeable.

Keywords Residual lifetime, Inactivity time, Exchangeability, Multivariate totally positive, Reliability.

Mathematics Subject Classification (2010) : 62N05, 60E15, 60E05.

A Introduction

The study of reliability properties of $(n-m+1)$-out-of-n systems, which have important role in many fields of engineering, has attracted most attention of several engineers and system designers. In the last decades, researchers have shown great interest to the study of two conditional random variables residual lifetime (RL) and inactivity time (IT) of $(n-m+1)$-out-of-n systems. An $(n-m+1)$-out-of-n system works if at least $(n-m+1)$ of the n components work, and fails if at least m components fail. Some properties of the lifetime of $(n-m+1)$-out-of-n systems have been investigated in the literature. See, for example, Asadi and Bayramoglu (2006), Khaledi and Shaked (2007), Tavangar and Bairamov (2015) and Tavangar (2016).

Consider a coherent system with lifetime T whose component lifetimes are $X_1, X_2, \ldots, X_n$. In recent years, several authors have studied the residual lifetime and inactivity time of the systems where the component lifetimes are assumed to be independent and identically distributed. Recently, some researchers have also considered general residual lifetime and general inactivity

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time. The conditional RL and IT of $(n-m+1)$-out-of-n system are defined respectively, as:

$$(X_{m:n}-t \mid X_{r:n} \le t < X_{k:n}), \qquad 1 \le r < k \le m \le n,$$
$$(t-X_{s:n} \mid X_{r:n} \le t < X_{m:n}), \qquad 1 \le s \le r < m \le n.$$

Many researchers have been studied these conditional random variables of $(n-m+1)$-out-of-n systems under different conditions on the components. Most of the results have been obtained under the independence of component lifetimes. Among others, we can refer to Asadi and Bayramoglu (2006), Khaledi and Shaked (2007), Li and Zhang (2008), Zhao et al. (2008), Salehi and Asadi (2012), Parvardeh and Balakrishnan (2013) and Tavangar (2016). Among the authors who have studied the reliability properties of the coherent systems with exchangeable components, one can find in Navarro and Rubio (2011), Tavangar and Asadi (2015), Salehi and Hashemi-Bosra (2017), Salehi and Tavangar (2019).

This paper is organized as follows. In Section 2, we first introduce some the concepts and tools that will be used in the article. In Section 3, we present some stochastic ordering results for the conditional RL and IT of m-out-of-n systems.

B Preliminaries

Let the vector $\boldsymbol{X}=(X_1,X_2,\dots,X_n)$, that denote the component lifetimes of the system, has an arbitrary joint distribution function $F(t_1,t_2,\dots,t_n)$ with corresponding joint reliability function $\bar{F}(t_1,t_2,\dots,t_n)$. We denote the vector of order statistics corresponding to X_i's by $(X_{1:n},X_{2:n},\dots,X_{n:n})$. It is known that the lifetime of the system is the mth order statistic, i.e. $X_{m:n}$, with reliability function

$$P(X_{m:n}>t)=\sum_{i=0}^{m-1}\sum_{\boldsymbol{\pi}\in C_i} P\{A_i^{(t,\boldsymbol{\pi})}\}, \tag{B.1}$$

where $\boldsymbol{\pi}=(\pi_1,\pi_2,\dots,\pi_n)$, $A_i^{(t,\boldsymbol{\pi})}$ is the event $[X_{\pi_1}\le t, X_{\pi_2}\le t,\dots,X_{\pi_i}\le t, X_{\pi_{i+1}}>t,\dots,X_{\pi_n}>t]$ and C_i is the set of all permutations $\{\pi_1,\pi_2,\dots,\pi_n\}$ of $\{1,2,\dots,n\}$ for which $1\le\pi_1<\dots<\pi_i\le n$ and $1\le\pi_{i+1}<\dots<\pi_n\le n$. It is obvious that in the exchangeable case, we have

$$\sum_{\boldsymbol{\pi}\in C_i} P\{A_i^{(t,\boldsymbol{\pi})}\}=\binom{n}{i}P_{i,n}(t),$$

where

$$P_{i,n}(t)=P\{X_1\le t, X_2\le t,\dots,X_i\le t, X_{i+1}>t,\dots,X_n>t\},$$

with $P_{0,n}(t) \equiv \bar{F}(t, t, \ldots, t)$.

First of all, we recall some definitions and concepts which are necessary for presenting the main results of the paper (see Shaked and Shanthikumar (2007)).

i) Let X and Y be two random variables with reliability functions $\bar{F}$ and $\bar{G}$, respectively. X is said to be smaller than Y in stochastic order (denoted by $X \leq_{\text{st}} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

ii) A random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is said to be smaller than another random vector $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ in the *usual multivariate stochastic order* (denoted by $\boldsymbol{X} \leq_{\text{st}} \boldsymbol{Y}$) if $E[\phi(\boldsymbol{X})] \leq E[\phi(\boldsymbol{Y})]$ holds for all increasing functions ϕ for which the expectations exist.

A density function $f : \mathbb{R}^n \to \mathbb{R}_+$ is said to be multivariate totally positive of order 2 (MTP_2) if $f(\boldsymbol{x})f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y})f(\boldsymbol{x} \vee \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Below, we give a useful result which will be required for our conclusions. Let $\{X_1, X_2, \ldots\}$ be a sequence of (not necessarily independent) random variables. Then $X_{i:m} \leq_{st} X_{j:n}$, whenever $i \leq j$ and $m - i \geq n - j$.

C Main results

Consider an $(n - m + 1)$-out-of-n system with arbitrary dependent components. Salehi and Tavangar (2019) obtained the reliability function of two interested conditional random variable $(X_{m:n} - t \mid X_{r:n} \leq t < X_{k:n})$ and $(t - X_{s:n} \mid X_{r:n} \leq t < X_{k:n})$ of the system as follow:

Let $X_1, X_2, \ldots, X_n$ be the arbitrary dependent lifetimes of components with joint reliability function $\bar{F}(x_1, x_2, \ldots, x_n)$.

i) For $x, t > 0$, and $0 \leq r < k \leq m \leq n$,

$$\begin{aligned} \psi_{m:n}^{r,k}(x|t) &= P\{X_{m:n} - t > x \mid X_{r:n} \leq t < X_{k:n}\} \\ &= \frac{\sum_{i=r}^{k-1} \sum_{\boldsymbol{\pi} \in C_i} P\{A_i^{(t,\boldsymbol{\pi})}\} \bar{F}_{m-i:n-i}^{(t,\boldsymbol{\pi})}(x)}{\sum_{i=r}^{k-1} \sum_{\boldsymbol{\pi} \in C_i} P\{A_i^{(t,\boldsymbol{\pi})}\}}, \end{aligned} \tag{C.1}$$

ii) For $0 \leq x \leq t$, and $1 \leq s \leq r < m \leq n$,

$$\begin{aligned} \phi_{s:n}^{r,m}(x|t) &= P\{t - X_{s:n} > x \mid X_{r:n} \leq t < X_{m:n}\} \\ &= \frac{\sum_{i=r}^{m-1} \sum_{\boldsymbol{\pi} \in C_i} P\{A_i^{(t,\boldsymbol{\pi})}\} \bar{H}_{i-s+1:i}^{(t,\boldsymbol{\pi})}(x)}{\sum_{i=r}^{m-1} \sum_{\boldsymbol{\pi} \in C_i} P\{A_i^{(t,\boldsymbol{\pi})}\}}, \end{aligned} \tag{C.2}$$

where $A_i^{(t,\boldsymbol{\pi})}$ and C_i are as defined in (B.1), $\bar{F}^{(t,\boldsymbol{\pi})}_{m-i:n-i}(x)$ is the reliability function of the $(m-i)$th order statistic according to the random vector $(X_{\pi_{i+1}} - t, \ldots, X_{\pi_n} - t \mid A_i^{(t,\boldsymbol{\pi})})$ and $\bar{H}^{(t,\boldsymbol{\pi})}_{i-s+1:i}(x)$ is the reliability function of the $(i-s+1)$th order statistic according to the random vector $(t - X_{\pi_1}, \ldots, t - X_{\pi_i} | A_i^{(t,\boldsymbol{\pi})})$.

Let $X_1, X_2, \ldots, X_n$ be exchangeable lifetimes of the components of a $(n-m+1)$-out-of-n system. Then representation (C.1) and (C.2) will be reduced to

i) for $x, t > 0$, and $0 \le r < k \le m \le n$,

$$\psi^{r,k}_{m:n}(x|t) = \frac{\sum_{i=r}^{k-1} \binom{n}{i} P\{A_i^{(t)}\} \bar{F}^{(i,t)}_{m-i:n-i}(x)}{\sum_{i=r}^{k-1} \binom{n}{i} P\{A_i^{(t)}\}} \tag{C.3}$$

ii) for $0 < x < t$ and $1 \le s \le r < k \le n$,

$$\phi^{r,k}_{s:n}(x|t) = \frac{\sum_{i=r}^{k-1} \binom{n}{i} P\{A_i^{(t)}\} \bar{H}^{(i,t)}_{i-s+1:i}(x)}{\sum_{i=r}^{k-1} \binom{n}{i} P\{A_i^{(t)}\}}, \tag{C.4}$$

where

$$A_i^{(t)} = [X_1 \le t, \ldots, X_i \le t, X_{i+1} > t, \ldots, X_n > t],$$

$\bar{F}^{(i,t)}_{m-i:n-i}(x)$ is the reliability function of the $(m-i)$th order statistic corresponding to the random vector $(X_{i+1} - t, \ldots, X_n - t \mid A_i^{(t)})$ and $\bar{H}^{(i,t)}_{i-s+1:i}(x)$ is the reliability function of the $(i-s+1)$th order statistic corresponding to the random vector $(t - X_1, \ldots, t - X_i | A_i^{(t)})$.

Now, we present some stochastic comparisons on the conditional residual lifetime and conditional inactivity time of the $(n-m+1)$-out-of-n system that is already derived by Salehi and Tavangar (2019). Let the joint probability density function of the exchangeable random vector $(X_1, X_2, \ldots, X_n)$ be MTP_2. Then,

i) for any $t \ge 0$ and $1 \le r < k+1 \le m \le n$,

$$(X_{m:n} - t \mid X_{r:n} \le t < X_{k+1:n}) \le_{st} (X_{m:n} - t \mid X_{r-1:n} \le t < X_{k:n}).$$

ii) for all $t \ge 0$ and $1 \le s \le r-1 < k-1 \le n-1$,

$$(t - X_{s:n} \mid X_{r-1:n} \le t < X_{k:n}) \le_{st} (t - X_{s:n} \mid X_{r:n} \le t < X_{k+1:n}).$$

Here, we obtain a new result that is extend the result of Theorem 4 in Salehi and Tavangar (2019) for exchangeable case. Let the joint probability density function of the exchangeable random vector $(X_1, X_2, \ldots, X_{n+1})$ satisfy the MTP$_2$ property. Furthermore, suppose that for each $t \geq 0$, $P\{X_{n+1} > t \mid A_i^{(t)}\}$ is increasing in i. Then, for any $t \geq 0$ and $1 \leq r < k \leq m \leq n$,

$$(X_{m:n+1} - t \mid X_{r:n+1} \leq t < X_{k:n+1}) \leq_{st} (X_{m:n} - t \mid X_{r:n} \leq t < X_{k:n}).$$

. Let $\psi_{m:n}^{r,k}(x|t)$ and $\psi_{m:n+1}^{r,k}(x|t)$ denote the reliability functions of $(X_{m:n}-t \mid X_{r:n} \leq t < X_{k:n})$ and $(X_{m:n+1} - t \mid X_{r:n+1} \leq t < X_{k:n+1})$, respectively. Define

$$A_j^{(n+1,t)} = [X_1 \leq t, X_2 \leq t, ..., X_j \leq t, X_{j+1} > t, \ldots, X_n > t, X_{n+1} > t].$$

Using Theorems 6.E.2, 6.E.4 and 6.E.8 in Shaked and Shanthikumar (2007), it can be shown that

$$(X_{j+1}, \ldots, X_{n+1} \mid A_j^{(t)}) \leq_{st} (X_{j+1}, \ldots, X_{n+1} \mid A_j^{(n+1,t)}).$$

Then, after simplifications and using Theorem B we have

$$\begin{aligned}
&\psi_{m:n}^{r,k}(x|t) - \psi_{m:n+1}^{r,k}(x|t) \\
&\quad \overset{\text{sgn}}{=} \sum_{i=r}^{k-1}\sum_{j=r}^{k-1} \binom{n}{i}\binom{n+1}{j} P\{A_i^{(t)}\}P\{A_j^{(n+1,t)}\}\left\{\bar{F}_{m-i:n-i}^{(i,t)}(x) - \bar{F}_{m-j:n+1-j}^{(j,t)}(x)\right\} \\
&\quad \geq \sum_{i=r}^{k-1}\sum_{j=r}^{k-1} \binom{n}{i}\binom{n+1}{j} P\{A_i^{(t)}\}P\{A_j^{(n+1,t)}\}\left\{\bar{F}_{m-i:n-i}^{(i,t)}(x) - \bar{F}_{m-j:n-j}^{(j,t)}(x)\right\}. \\
&\quad = \sum_{i=r}^{k-1}\sum_{j=i}^{k-1} \left\{\binom{n}{i}\binom{n+1}{j} P\{A_i^{(t)}\}P\{A_j^{(n+1,t)}\} - \binom{n+1}{i}\binom{n}{j} P\{A_i^{(n+1,t)}\}P\{A_j^{(t)}\}\right\} \\
&\qquad\qquad \times \left[\bar{F}_{m-i:n-i}^{(i,t)}(x) - \bar{F}_{m-j:n-j}^{(j,t)}(x)\right] \qquad \text{(C.5)}
\end{aligned}$$

If $P\{X_{n+1} > t \mid A_i^{(t)}\}$ is increasing in i, then so is

$$\frac{\binom{n+1}{i}P\{A_i^{(n+1,t)}\}}{\binom{n}{i}P\{A_i^{(t)}\}} = \frac{n+1}{n+1-i}P\{X_{n+1} > t \mid A_i^{(t)}\},$$

and hence the first braces in the right-hand side of (C.5) is non-negative. On the other hand, using Theorem 6.E.4 and then Theorem 6.E.8 in Shaked and Shanthikumar (2007) one can derive for $i < j$, that

$$\{X_1, X_2, \ldots, X_k | A_i^{(t)}\} \leq_{st} \{X_1, X_2, \ldots, X_k | A_j^{(t)}\}.$$

Therefore, it follows from Theorem B that for $i \leq j$, $\bar{F}_{m-i:n-i}^{(i,t)}(x) \geq \bar{F}_{m-j:n-j}^{(j,t)}(x)$. Hence, the desired result follows. $\square$

References

[1] Asadi, M. and Bayramoglu, I. (2006), On the mean residual life function of the k-out-of-n systems at the system level, *IEEE Transactions on Reliability*, **55**, 314-318.

[2] Khaledi, B-E. and Shaked, M. (2007), Ordering conditional lifetimes of coherent systems, *Journal of Statistical Planning and Inference*, **37**, 1173-1184.

[3] Li, X. and Zhao, P. (2008), Stochastic comparison on general inactivity time and general residual life of k-out-of-n: G systems, *Communications in Statistics: Simulation and Computation*, **37**, 1005-1019.

[4] Navarro, J. and Rubio, R. (2011), A note on necessary and sufficient conditions for ordering properties of coherent systems with exchangeable components, *Naval Research Logistics*, **58**(5), 478-489.

[5] Parvardeh, A. and Balakrishnan, N. (2013), Conditional residual lifetimes of coherent systems, *Statistics and Probability Letters*, **83**, 2664-2672.

[6] Salehi, E.T. and Asadi, M. (2012), Results on the past lifetime of $(n-k+1)$-out-of-n structures with nonidentical components, *Metrika*, **75**, 439-454.

[7] Salehi, E. and Hashemi-Bosra, S.S. (2017), The mean general residual lifetime of $(n-k+1)$-out-of-n systems with exchangeable components, *Communications in Statistics: Theory and Methods*, **46**(13), 6382-6400.

[8] Salehi, E. and Tavangar, M. (2019), Stochastic comparisons on conditional residual lifetime and inactivity time of coherent systems with exchangeable components, *Statistics and Probability Letters*, **145**, 327-337.

[9] Shaked, M. and Shanthikumar, J.G. (2007), *Stochastic Orders*, Springer, New York.

[10] Tavangar, M. (2016), Conditional inactivity time of components in a coherent operating system, *IEEE Transactions on Reliability*, **65**(1), 359-369.

[11] Tavangar, M. and Asadi, M. (2015), The inactivity time of exchangeable components of k-out-of-n structures, *Sankhya B*, **136**, 1197-1206.

[12] Tavangar, M. and Bairamov, I. (2015), On conditional residual lifetime and conditional inactivity time of k-out-of-n systems, *Reliability Engineering & System Safety*, **144**, 225-233.

[13] Zhao, P., Li, X. and Balakrishnan, N. (2008), Conditional ordering of k-out-of-n systems with independent but non-identical components, *Journal of Applied Probability*, **45**, 1113-1125.

Optimal Design for Step-Stress Accelerated Degradation tests under Inverse Gaussian Process

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Abstract: In this study a step-stress accelerated degradation test (SSADT) is considered when the degradation follows an Inverse Gaussian process (IG). Under constraint that the total experimental cost does not exceed a pre-specified budget the optimal setting such as sample size, measurement frequency and number of measurement at each stress level are obtained. Finally an example is presented to illustrate the proposed method.

Keywords Inverse Gaussian process, step-stress accelerated degradation test, mean-time-to-failure.
Mathematics Subject Classification (2010) : 90B25, 97K60**.**

A Introduction

For highly-reliable products, it is not an easy task to assess the lifetime distribution of the products by using the traditional life-testing procedures which record only time to failure data. Even using the accelerating techniques, the information about the lifetime distribution is still very limited. Under this situation, an alternative approach is to collect the degradation data at higher levels of stress for predicting a products lifetime at a certain use stress level. Such an experiment is called an accelerated degradation tests(ADT). Although ADT is an efficient life-test method, it is usually very expensive to conduct. In addition, the selection of suitable levels of stress is not straightforward. In this situation, a constant-stress ADT is not suitable. Tseng and Wen (4) proposed a SSADT to handle this problem. In the SSADT experiment, an item is first tested, subject to a predetermined stress for a specied length of time. If it does not fail, it is tested again subject to a higher stress level for another specied length of time. The stress on a specimen is thus increased step by step until an appropriate termination time is reached. Obviously, the advantage of the SSADT is that only a few test units are needed

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to conduct a life test. Moreover is useful to record degradation measure of a product over the time. Degradation behavior can describe by stochastic process. Wiener ((1),(3)), Gamma (5) and Inverse Gaussian process (IG) (8) are most popular stochastic process in these studies. Recent studies have shown that in real-world the Wiener and Gamma processes cannot handle all degradation data. As we can mention ((6)) has been shown that both Wiener and Gamma processes can not fit some GaAs laser degradation data, while the IG process model performs well in fitting these data because it has many nice properties as like as monotone increasing behavior and flexible dealing with covariates and random effects. In this paper, we first use an Inverse Gaussian process to model a typical SSADT problem. Next, under the constraint that the total experimental cost does not exceed a predetermined budget, the optimal settings of decision variables are obtained by minimizing the asymptotic variance of the estimated mean time to failure of the lifetime distribution of the product. The rest of this paper is organized as follow s. SectionB, discribes the problem of a SSADT test and general assumptions. Section C, introduces this model under an Inverse Gaussian degradation process. Section D, presentes the optimal design.Section E, presentes an example to describe the proposed method.

B Problem description

Inverse Gaussian process (IG) utilized to describe the degradation behavior of a product and $L(t, S_0)$ denote the degradation path at time t under normal stress (S_0). This process has the following properties.

1. X(0)=0, whit probability one.
2. Inverse gaussian process has independent increment.
3. Each increment follow Inverse gaussian distribution

$$\Delta L(t, S_i) \sim IG(\Lambda_i(t), \lambda \Lambda_i(t)^2)$$

.

Where

$$\Lambda_i(t) = \mu_i t$$

$$\mu_i = \exp\{a + bS_i\} \tag{B.1}$$

and $\Delta L(t, S_i) = L(\Lambda_i(t+\Delta t), S_i) - L(\Lambda_i(t), S_i)$ with probability density function (PDF) is defined by:

$$f_{\Delta L}(x) = \sqrt{\frac{\lambda\Lambda_i(t)^2}{2\pi x^3}} \exp\left\{ -\frac{\lambda\Lambda_i(t)^2(x-\Lambda_i(t))^2}{2\Lambda_i(t)^2 x} \right\} \tag{B.2}$$

$\Lambda_i(t)$ and $\lambda\Lambda_i(t)^2$ are scale and shape parameters of Inverse gaussian distribution.

4. The first passage time of this process defined when $L(t, S_0)$ crosses a critical threshold (D) and we can write:

$$T_D = \inf\{t | L(t, S_0) \geq D\}$$

5. The formula for the mean time- to-failure (MTTF) under normal stress S_0 is given by:

$$MTTF = E[T_D|S_0] = (\frac{D}{\mu_0} + \frac{1}{\mu_0\lambda})\Phi(\sqrt{\lambda D}) + \frac{1}{\mu_0}\sqrt{\frac{D}{\lambda}}\phi(\sqrt{\lambda D}) - \frac{1}{2\mu_0\lambda} \tag{B.3}$$

For high reliable products which are not fail in a short period time we are interested to calculate MTTF with design an efficient Step-stress accelerated degradation testing (SSADT) experiment. Let $S_0, S_1, \ldots, S_m$ are different stress levels such as

$$S_0 \leq S_1 \leq \cdots \leq S_m$$

There are n test units subject to a degradation test with a measurement frequency per f units time under stress S_1 up to t_1. In this time stress level increase to S_2 and the product stay in this situation until t_2. This process continue until the stress is up to S_m. Generally we can say:

$$S = \begin{cases} S_1, & if:\ 0 \leq t < t_1 \\ S_2, & if:\ t_1 \leq t < t_2 \\ \vdots & \\ S_m, & if:\ t_{m-1} \leq t < t_m \end{cases}$$

Let l_i denote the total number of measurements under stress S_i and $\widehat{MTTF}$ denote the estimated MTTF. To minimize the total cost of a SSADT experiment ($TC(n, f, l_1, l_2, \ldots, l_m)$) we try to solve

$$AVar(\widehat{MTTF}|n, f, l_1, l_2, \ldots, l_m)$$

by determine the optimal value of $n, f, l_1, l_2, \ldots, l_m$, under confined cost

$$TC(n, f, l_1, l_2, \ldots, l_m) < C_b$$

In which

$$TC(n, f, l_1, l_2, \ldots, l_m) = C_{op} f \sum_{i=1}^{m} l_i + C_{mea} n \sum_{i=1}^{m} l_i + C_{it} n \tag{B.4}$$

Where C_{op},C_{mea} and C_{it} are the unit cost of operation, the unit cost of measurement, and the unit cost of an item.

C SSADT subject Inverse Gaussian process

Let $L_{ss}(t)$ denote the degradation path of a SSADT with an Inverse Gaussian degradation model. Therefor

$$L_{ss}(t) = L(t, S_1) \sim IG(\Lambda_1(t), \lambda\Lambda_1(t)^2), \qquad t \in [0, t_1)$$

and when the stress level up to S_2 at time t_1

$$\begin{aligned} L_{ss}(t) &= L(t_1, S_1) + L((t - t_1), S_2) \\ &\sim IG(\Lambda_1(t_1) + \Lambda_1(t - t_1), \lambda(\Lambda_1(t_1) + \Lambda_2(t - t_1))^2), \qquad t \in [t_1, t_2) \end{aligned}$$

Similarly in the stress level S_m at time t_{m-1}

$$\begin{aligned} L_{ss}(t) &= L(t_1, S_1) + L((t_2 - t_1), S_2) + \cdots + +L((t - t_{m-1}), S_m) \\ &\sim IG(\sum_{i=1}^{m-1} \Lambda_1(t_i - t_{i-1}), \lambda(\sum_{i=1}^{m-1} \Lambda_2(t_i - t_{i-1}))^2) \end{aligned}$$

If $L_{ss}^{(k)}(t)$ denote the SSADT degradation path of the k^{th} test sample at time t so that $\tau_{i-1} < t_{i-1} < t_i < \tau_i$. Let $Y_{ijk} = L_{ss}^{(k)}(tj|S_i) - L_{ss}^{(k)}(t_{j-1}|S_i)$. So by using independent increment property of the inverse gaussian process we can say:

$$Y_{ijk} \sim IG(\Lambda_i(t_j) - \Lambda_i(t_{j-1}), \lambda(\Lambda_i(t_j) - \Lambda_i(t_{j-1}))^2)$$

The likelihood function of the SSADT model for an inverse gaussian process is given by

$$L(a, b, \lambda|y) = \prod_{k=1}^{n} \prod_{i=1}^{m} \prod_{j=\xi_{i-1}+1}^{\xi_i} \sqrt{\frac{\lambda(\Lambda_i(t_j) - \Lambda_i(t_{j-1}))^2}{2\pi y_{ijk}^3}} \exp\left\{\frac{\lambda(y_{ijk} - \Lambda_i(t_j) - \Lambda_i(t_{j-1}))^2}{2y_{ijk}}\right\} \tag{C.1}$$

In above equation $\xi_i = l_1 + l_2 + \cdots + l_i$ and $\xi_0 = 0$.

D Optimization

To minimize the total cost of an SSADT experiment at first we should compute the approximate variance $\widehat{MTTF}$. The asymptotic variance of $\widehat{MTTF}$ can then be derived using the delta method as:

$$AVar(\widehat{MTTF}) = h\acute{(\Theta)} I(\Theta)^{-1} h(\Theta) \tag{D.1}$$

Where $\Theta = (a, b, \lambda)$ and matrix $I(\Theta)$ is the Fisher information matrix which can be computed by:

$$I(\Theta) = \begin{pmatrix} E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial a^2} \right) & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial a \partial b} \right) & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial a \partial \lambda} \right) \\ E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial b \partial a} \right) & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial b^2} \right) & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial b \partial \lambda} \right) \\ E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial \lambda \partial a} \right) & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial \lambda \partial b} \right) & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial \lambda^2} \right) \end{pmatrix} \tag{D.2}$$

and $h\acute{(\Theta)}$

$$h\acute{(\Theta)} = \left(\frac{\partial MTTF}{\partial a}, \frac{\partial MTTF}{\partial b}, \frac{\partial MTTF}{\partial \lambda} \right) \tag{D.3}$$

Vector $h\acute{(\Theta)}$ denotes the transpose of $h(\Theta)$ and it can be expressed as

$$\begin{aligned} \frac{\partial MTTF}{\partial a} &= \left(- \frac{D}{\mu_0} - \frac{1}{\mu_0 \lambda} \right) \Phi(\sqrt{\lambda D}) - \frac{\sqrt{\frac{D}{\lambda}} \phi(\sqrt{\lambda D})}{\mu_0} + \frac{1}{2\mu_0 \lambda} \\ \frac{\partial MTTF}{\partial b} &= \left(- \frac{D S_0}{\mu_0} - \frac{S_0}{\mu_0 \lambda} \right) \Phi(\sqrt{\lambda D}) - \frac{\sqrt{\frac{D}{\lambda}} \phi(\sqrt{\lambda D}) S_0}{\mu_0} + \frac{S_0}{2\mu_0 \lambda} \\ \frac{\partial MTTF}{\partial \lambda} &= - \frac{\Phi(\sqrt{\lambda D})}{\lambda^2 \mu_0} + \frac{1}{2\mu_0 \lambda} \end{aligned} \tag{D.4}$$

and the elements of the Fisher information matrix $I(\Theta)$ in Eq. (D.2) can be expressed as:

$$\begin{aligned} E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial a^2} \right) &= nf\lambda \sum_{i=1}^m \mu_i l_i + 2n \sum_{i=1}^m l_i, & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial a \partial \lambda} \right) &= \frac{n}{\lambda} \sum_{i=1}^m l_i \\ E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial a \partial b} \right) &= nf\lambda \sum_{i=1}^m \mu_i S_i l_i + 2n \sum_{i=1}^m l_i S_i, & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial b \partial \lambda} \right) &= \frac{n}{\lambda} \sum_{i=1}^m l_i S_i \\ E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial b^2} \right) &= nf\lambda \sum_{i=1}^m \mu_i S_i^2 l_i + 2n \sum_{i=1}^m l_i S_i^2, & E\left(- \frac{\partial^2 \ln L(\Theta|y)}{\partial \lambda^2} \right) &= \frac{1}{2} \frac{1}{\lambda^2} n \sum_{i=1}^m l_i \end{aligned}$$

The main objective of this paper is minimizing the total cost $TC(n, f, l_1, l_2, \ldots, l_m)$ by detemine the optimal value of $n, f, l_1, l_2, \ldots, l_m$ through the following algorithm.

[h!] Optimization of SSADT plan [1] Giving degradation parameters, stress levels, C_{op},C_{mea}, C_{it} ,n and m $n_{max} = \lfloor \frac{C_b - C_{op} m}{C_{mea} m + C_{it}} \rfloor$ n = 1 to n_{max} $f_{max} = \lfloor \frac{C_b - C_{mea} nm - C_{it} n}{C_{op} m} \rfloor$ f= 1 to f_{max} Find $l_1, l_2, \ldots, l_m \in \mathbb{N}$ which $TC(n, f, l_1, l_2, \ldots, l_m) < C_b$ Calculate $AVar(\widehat{MTTF}|n, f, l_1, l_2, \ldots, l_m)$ Find optimal value of $n^*, f^*, l_1^*, l_2^*, \ldots, l_m^*$ then be obtained as
$\min AVar(\widehat{MTTF}|n, f, l_1, l_2, \ldots, l_m)$

E Numerical example

In this section the proposed method describe with a numerical example based on the data from the stress relaxation problem description by Yang (2007), Example 8.7, p.351. We consider Inverse gaussian process to describe degradation process by shape parameter $\lambda = 0.10019014$ and scale parameter $\mu_i = \exp\{1.9274 - 0.088718 S_i\}$

In this study we consider only two stress levels ($S_0 = 65, S_1 = 85$,$S_2 = 100$) and critical threshold is $D = 20$. The cost configuration of C_{op},C_{mea}, and C_{it} are respectively

$$C_{op} = \$2.9/\text{per unit cost of operation}$$

$$C_{mea} = \$2.3/\text{per unit cost of measurement}$$

$$C_{it} = \$60/\text{per unit cost of device}$$

Table 1 has shown the optimal test plan for various budgets C_b based on the proposed method.

Table 1: Optimal value under different C_b

C_b	n^*	f^*	l_1^*	l_2^*	Total test cost	$Std(\widehat{MTTF}|n, f, l_1, l_2, \ldots, l_m)$
1500	8	5	21	10	545.8	5284.471
2000	10	5	25	12	675	4326.12
2500	11	6	29	14	745.4	3686.212
3000	14	7	27	14	945	3243.728

For example, when $C_b = 1500$, the optimal test plan is $(n^*, f^*, l1^*, l2^*) = (8, 5, 21, 10)$; that is the optimal sample size is 11, the optimal measurement frequency is 8 and the corresponding optimal numbers of measurements for stress S_1, S_2 are 21 and 10, respectively.

References

[1] DeGroot, M.H. and Goel, P.K. (1979), Bayesian estimation and optimal designs in partially accelerated life testing, *Naval Research Logistics*, **26**(2), 223-235

[2] Peng, C.Y., Tseng, S.T. (2010), Progressive-stress accelerated degradation test for highly-reliable products, *IEEE Transactions on Reliability*, **59**(1), 30-37

[3] Sung, S.L. and Yum, B.J. (2016), Optimal design of step-stress accelerated degradation tests based on the Wiener degradation process, *Quality Technology and Quantitative Management*, **13**(4), 367-393.

[4] Tseng, S.T and Wen, Z.C. (2000), Step-stress accelerated degradation analysis of highly-reliable products, *Journal of Quality Technology*, **32**, 209-216.

[5] Tsai, C.C., Tseng, S.T. and Balakrishnan, N. (2012), Optimal Design for Degradation Tests Based on Gamma Processes With Random Effects, *IEEE Transactions on Reliability*, **61**(2), 604-613.

[6] Wang, X. and Xu, D. (2010), An inverse Gaussian process model for degradation data, *Technometrics*, **52**(2), 188197.

[7] Yang, G. (2007), *Life Cycle Reliability Engineering*, Hoboken, NJ, USA: Wiley

[8] Ye, Z.S., Chen, L.P., Tang, L.C. and Xie, M. (2014), Accelerated degradation test planning using the inverse Gaussian process, *IEEE Trans Reliab*, **63**(3),750-63.

[9] Zhang, C., Lu, X., Tan, Y. and Wang, Y. (2015), Reliability demonstration methodology for products with Gamma Process by optimal accelerated degradation testing, *Reliability Engineering and System Safety*, **142**, 369-377.

On a New Bivariate Survival Model for the Analysis of Dependent Lives and Its Generalization

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Abstract: In this paper, a new bivariate model based on the model of dependent lives is introduced. This new bivariate distribution has natural interpretations, and it can be applied in fatal shock models or in competing risks models. Then, the proposed bivariate model is generalized. We call these new distributions as the bivariate Gompertz (BGP) distribution and bivariate Gompertz-geometric (BGPG) distribution, respectively. Then, we present various properties of the new bivariate models. Also, the ageing properties and the bivariate hazard gradient are discussed. We propose to use the EM algorithm to compute the maximum likelihood estimators of the unknown parameters. Finally, we analyze one real data set.

Keywords Bivariate model, Competing risks model, Expectation-Maximization algorithm, Gompertz distribution, Shock model.

Mathematics Subject Classification (2010) : 62H10, 62H12, 62E15.

A Introduction

The modeling of a lifetime is an important problem in a variety of scientific and technological fields. Also, models that consider the dependency feature are very interesting and are applied in many various areas such as reliability, survival analysis, insurance risk analysis, life insurance (see, e.g., Iyer and Manjunath(9)).

Traditionally in statistics and also the actuarial theory of multiple life insurance is based on the assumption of independence for the remaining lifetimes. However, in many situations, this assumption is not valid. Intuitively, pairs of individuals exhibit dependence in mortality because they share common risk factors, which may be purely genetic, as in the case of twins, or environments, as in the case of a married couple. So, we propose that to use the model of dependent lives. For an example, Carriere(5)) showed that there is a very high positive correlation between the times of deaths of coupled lives. Also, Jagger and Sutton (10) addressed that after the marital bereavement, the risk of mortality is significantly increased.

One classical model of dependent lives that captured our attention is called the "common shock" model,

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This model assumes that the lifetimes of two persons, say T_1 and T_2, are independent unless a common shock causes the death of both. For example, a contagious deadly disease, a natural catastrophe or a car accident may affect the lives of the two spouses. Thus, if T_0 denotes the time until the common disaster, the actual ages-at-death are modeled by

$$X_1 = \min(T_1, T_0), \qquad and \qquad X_2 = \min(T_2, T_0).$$

Then, the joint survival function of (X_1, X_2) is given by

$$\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = P(T_1 > x_1)P(T_2 > x_2)P(T_0 > \max(x_1, x_2)).$$

in view of the mutual independence of T_0, T_1 and T_2. This model is also called the bivariate survival model of the Marshall-Olkin type. Several other bivariate distributions of Marshall-Olkin type have been proposed. For more details, the readers can refer to (18), (2), (12) and (16).

Also, for modeling survival data, Marshall and Olkin (14) introduced a class of univariate distributions which can be obtained by minimum and maximum of independent and identically distributed continuous random variables, where the sample size follows the geometric distribution. In fact, their method induces an extra parameter to a model, hence affords more flexibility. Also, extensive work has been done on their method. For more details, the readers can refer to (17), (7), (8), (15), (6), (4), (3) and the references cited therein. In fact, it can be seen that due to the lack of analyzes for bivariate distributions, see for example (13). In the following, we assume that the failure times follow the Gompertz (GP) distribution. Therefore, we introduce two new bivariate distributions.

In the first case, a new bivariate distribution based on the model of dependent lives is obtained. This new bivariate distribution is called the bivariate Gompertz (BGP) distribution and the marginal distributions have GP distributions.

In the second case, we generalize the BGP distribution. So, a new bivariate distribution by compounding geometric distribution and Gompertz model is introduced. We call this new bivariate distribution as the bivariate Gompertz-geometric (BGPG) distribution. In fact, this method produces a new bivariate distribution which is analytically quite tractable. Also, the marginals and conditionals are univariate Gompertz-geometric distributions (UGPG), and it is also very flexible.

The maximum likelihood estimators (MLEs) of the unknown parameters of the BGPG distribution cannot be obtained in closed forms. The Newton-Raphson or Gauss-Newton type algorithm iterative procedure is needed to solve these non-linear equations. Moreover, the choice of initial values and the convergence of the iterative algorithm are important subjects. To avoid these problems, we investigate it as a missing value problem and propose to use the EM algorithm.

The paper is organized as follows. In Section 2, we introduce a new bivariate distribution based on the model of dependent lives and discuss various properties of the new bivariate distribution. A new bivariate distribution by compounding geometric distribution and Gompertz model are introduced and also, different properties of this model are investigated in Section 3. The maximum likelihood estimators are discussed in Section 4. The analysis of one real data set has been presented in Section 5 and finally we conclude the paper in Section 6.

B Model Formulation

In this section, a new bivariate distribution is obtained based on the model of dependent lives. We use the Gompertz distribution. This distribution is often applied to describe the distribution of adult lifespans by demographers and actuaries. Also, the bivariate distributions are introduced in this structure has popular descriptions and it can be used in shock models or in competing risks models.

B.1 Bivariate Gompertz Distribution

Suppose $T_i \sim GP(\alpha, \lambda_i)$ for $i = 0, 1, 2$ and they are independent. Define $X_i = \min\{T_0, T_i\}$ for $i = 1, 2$, then the bivariate vector (X_1, X_2) is a bivariate Gompertz distribution and it will be denoted from now on as $BGP(\alpha, \lambda_0, \lambda_1, \lambda_2)$. If $(X_1, X_2) \sim BGP(\alpha, \lambda_0, \lambda_1, \lambda_2)$, for $z = \max\{x_1, x_2\}$,

$$\bar{F}_{X_1,X_2}(x_1, x_2) = \begin{cases} \bar{F}_{GP}(x_1, \alpha, \lambda_1 + \lambda_0)\bar{F}_{GP}(x_2, \alpha, \lambda_2) & \text{if} \quad x_2 < x_1 \\ \bar{F}_{GP}(x_1, \alpha, \lambda_1)\bar{F}_{GP}(x_2, \alpha, \lambda_2 + \lambda_0) & \text{if} \quad x_1 < x_2 \\ \bar{F}_{GP}(x, \alpha, \lambda_0 + \lambda_1 + \lambda_2, \lambda) & \text{if} \quad x_1 = x_2 = x. \end{cases}$$

The joint PDF of (X_1, X_2) is:

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \alpha^2\lambda_2(\lambda_0 + \lambda_1)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_1)(e^{\alpha y_1}-1)}e^{-\lambda_2(e^{\alpha y_2}-1)} & \text{if} \quad x_2 < x_1 \\ \alpha^2\lambda_1(\lambda_0 + \lambda_2)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_2)(e^{\alpha y_2}-1)}e^{-\lambda_1(e^{\alpha y_1}-1)} & \text{if} \quad x_1 < x_2 \\ \alpha\lambda_0 e^{\alpha y}e^{-(\lambda_0+\lambda_1+\lambda_2)(e^{\alpha y}-1)} & \text{if} \quad x_1 = x_2 = x, \end{cases} \tag{B.1}$$

The joint PDF of (X_1, X_2), for $z = \max\{x_1, x_2\}$ is

$$f_{X_1,X_2}(x_1, x_2) = \frac{\lambda_1 + \lambda_2}{\lambda_0 + \lambda_1 + \lambda_2} f_a(x_1, x_2) + \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} f_s(z), \tag{B.2}$$

where

$$f_a(x_1, x_2) = \frac{\lambda_1 + \lambda_2}{\lambda_0 + \lambda_1 + \lambda_2} \begin{cases} f_{GP}(x_1, \alpha, \lambda_0 + \lambda_1)f_{GP}(x_2, \alpha, \lambda_2) & \text{if} \quad x_2 < x_1 \\ f_{GP}(x_2, \alpha, \lambda_1)f_{GP}(x_2, \alpha, \lambda_0 + \lambda_2) & \text{if} \quad x_1 < x_2, \end{cases}$$

and

$$f_s(z) = f_{GP}(z, \alpha, \lambda_0 + \lambda_1 + \lambda_2).$$

where $f_a(x_1, x_2)$ and $f_s(z)$ are the absolutely continuous and singular part. If $(X_1, X_2) \sim BGP(\alpha, \lambda_0, \lambda_1, \lambda_2)$. Then,

1) $\bar{F}_{X_1|X_2\geq x_2}(x_1)$ is an absolute continuous survival function as follows

$$\bar{F}_{X_1|X_2\geq x_2}(x_1) = \begin{cases} e^{-(\lambda_0+\lambda_1)(e^{\alpha x_1}-1)}e^{\lambda_0(e^{\alpha x_2}-1)} & \text{if} \quad x_2 < x_1 \\ e^{-\lambda_1(e^{\alpha x_1}-1)} & \text{if} \quad x_1 < x_2. \end{cases} \tag{B.3}$$

2) The conditional survival function in (B.3) has a representation

$$\bar{F}_{X_1|X_2\geq x_2}(x_1) = pG(x_1) + (1-p)H(x_1),$$

where,

$$G(x_1) = \frac{1}{p}\begin{cases} e^{-(\lambda_0+\lambda_1)(e^{\alpha x_1}-1)}e^{\lambda_0(e^{\alpha x_2}-1)} & \text{if} \quad x_2 < x_1 \\ e^{-\lambda_1(e^{\alpha x_1}-1)} - \frac{\lambda_0}{\lambda_0+\lambda_2}e^{-\lambda_1(e^{\alpha x_2}-1)} & \text{if} \quad x_1 < x_2, \end{cases} \qquad H(x) = \begin{cases} 1 & \text{if} \quad x < x_2 \\ 0 & \text{if} \quad x > x_2, \end{cases}$$

and

$$p = 1 - \frac{\lambda_0}{\lambda_0+\lambda_2}e^{-\lambda_1(e^{\alpha x_2}-1)}.$$

If $(X_1, X_2) \sim BGP(\alpha, \lambda_0, \lambda_1, \lambda_2)$. Then,

1) $X_1 \sim GP(\alpha, \lambda_0 + \lambda_1)$ and $X_2 \sim GP(\alpha, \lambda_0 + \lambda_2)$.

2) $P(X_1 < X_2) = \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}$.

3) $\min\{X_1, X_2\} \sim BGP(\alpha, \lambda_0 + \lambda_1 + \lambda_2)$.

If $(X_1, X_2) \sim BGP(\alpha, \lambda_0, \lambda_1, \lambda_2)$. Then,

a) The multivariate increasing failure rate (MIFR) property for $\alpha \geq 1$ and has the multivariate decreasing failure rate (MDFR) property for $0 < \alpha < 1$.

b) For all value of $x_1, x_2 > 0$, both the components of $h_{X_1,X_2}(x_1, x_2) = (-\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}) \ln P(X_1 > x_1, X_2 > x_2)$ are increasing functions of x_1 and x_2.

c) The positive upper orthant dependent (PUOD) property.

d) The right tail increasing (RTI) property.

e) The right corner set increasing (RCSI) property.

C Generalization

C.1 Bivariate Gompertz-Geometric Distribution

Suppose $\{(X_{1n}, X_{2n}); n = 1, 2, \ldots\}$ is a sequence of i.i.d. non-negative bivariate random variables with common joint distribution function $F_X(.,.)$ where $X = (X_1, X_2)$ and N is a geometric random variable independent of $\{(X_{1n}, X_{2n}), n = 1, 2 \ldots\}$. Consider the following bivariate random variable $Y = (Y_1, Y_2)$, so that, $Y_i = \min\{X_{i1}, \ldots, X_{iN}\}$ for $i = 1, 2$. The joint survival function of $Y = (Y_1, Y_2)$ becomes

$$\bar{G}(y_1, y_2) = P(Y_1 > y_1, Y_2 > y_2) = \frac{\theta \bar{F}_X(y_1, y_2)}{1-(1-\theta)\bar{F}_X(y_1, y_2)}. \tag{C.1}$$

Therefore, the random variable $Y = (Y_1, Y_2)$ is said to have the bivariate Gompertz-geometric distribution with parameters $\theta, \alpha, \lambda_0, \lambda_1, \lambda_2$, if the distribution F in (C.1) is $BGP(\alpha, \lambda_0, \lambda_1, \lambda_2)$.Therefore, the joint survival function of (Y_1, Y_2) becomes

$$\bar{G}(y_1, y_2) = \begin{cases} \frac{\theta e^{-(\lambda_0+\lambda_1)(e^{\alpha y_1}-1)} e^{-\lambda_2(e^{\alpha y_2}-1)}}{1-(1-\theta)e^{-(\lambda_0+\lambda_1)(e^{\alpha y_1}-1)} e^{-\lambda_2(e^{\alpha y_2}-1)}} & \text{if} \quad y_2 \leq y_1 \\ \frac{\theta e^{-(\lambda_0+\lambda_2)(e^{\alpha y_2}-1)} e^{-\lambda_1(e^{\alpha y_1}-1)}}{1-(1-\theta)e^{-(\lambda_0+\lambda_2)(e^{\alpha y_2}-1)} e^{-\lambda_1(e^{\alpha y_1}-1)}} & \text{if} \quad y_1 < y_2. \end{cases} \tag{C.2}$$

It will be denoted by $(Y_1, Y_2) \sim BGPG(\theta, \alpha, \lambda_0, \lambda_1, \lambda_2)$.

Let $(Y_1, Y_2) \sim BGPG(\theta, \alpha, \lambda_0, \lambda_1, \lambda_2)$, the joint PDF of (Y_1, Y_1) is

$$g_{Y_1,Y_2}(y_1, y_2) = \begin{cases} g_1(y_1, y_2) & \text{if} \quad y_2 < y_1 \\ g_2(y_1, y_2) & \text{if} \quad y_1 < y_2 \\ g_0(y_1, y_2) & \text{if} \quad y_1 = y_2 = y. \end{cases}$$

where

$$g_1(y_1, y_2) = \frac{\theta\alpha^2\lambda_2(\lambda_0+\lambda_1)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_1)(e^{\alpha y_1}-1)}e^{-\lambda_2(e^{\alpha y_2}-1)}}{[1-(1-\theta)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_1)(e^{\alpha y_1}-1)}e^{-\lambda_2(e^{\alpha y_2}-1)}]^3}$$
$$\times [1+(1-\theta)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_1)(e^{\alpha y_1}-1)}e^{-\lambda_2(e^{\alpha y_2}-1)}].$$
$$g_2(y_1, y_2) = \frac{\theta\alpha^2\lambda_1(\lambda_0+\lambda_2)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_2)(e^{\alpha y_2}-1)}e^{-\lambda_1(e^{\alpha y_1}-1)}}{[1-(1-\theta)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_2)(e^{\alpha y_2}-1)}e^{-\lambda_1(e^{\alpha y_1}-1)}]^3}$$
$$\times [1+(1-\theta)e^{\alpha(y_1+y_2)}e^{-(\lambda_0+\lambda_2)(e^{\alpha y_2}-1)}e^{-\lambda_1(e^{\alpha y_1}-1)}].$$
$$g_0(y) = \frac{\theta\alpha\lambda_0 e^{\alpha y}e^{-(\lambda_0+\lambda_1+\lambda_2)(e^{\alpha y}-1)}}{[1-(1-\theta)e^{-(\lambda_0+\lambda_1+\lambda_2)(e^{\alpha y}-1)}]^2}.$$

Let $(y_1, y_2) \sim BGPG(\theta, \alpha, \lambda_0, \lambda_1, \lambda_2)$. Then

(I) Each Y_i has a univariate Gompertz-geometric distribution (UGPG) with parameters $\alpha, \lambda_0 + \lambda_i$ and θ.

(II) The random variable $Y = \min(Y_1, Y_2)$ has an UGPG distribution with parameters $\lambda_0 + \lambda_1 + \lambda_2$, α and θ.

(III) $P(Y_1 < Y_2) = \frac{\lambda_1}{\lambda_0+\lambda_1+\lambda_2}$.

Using the conditional probability mass function of N given $Y_1 = y_1$ and $Y_2 = y_2$, we can compute

$$E(N|y_1, y_2) = \begin{cases} \frac{(1-\xi_1(y_1,y_2,\theta,\gamma))^2-6(1-\xi_1(y_1,y_2,\theta,\gamma))+6}{(1-\xi_1(y_1,y_2,\theta,\gamma))^2} & \text{if} \quad y_2 < y_1 \\ \frac{(1-\xi_2(y_1,y_2,\theta,\gamma))^2-6(1-\xi_2(y_1,y_2,\theta,\gamma))+6}{(1-\xi_2(y_1,y_2,\theta,\gamma))^2} & \text{if} \quad y_1 < y_2 \\ \frac{1+\xi_0(y_1,y_2,\theta,\gamma)}{1-\xi_0(y_1,y_2,\theta,\gamma)} & \text{if} \quad y_1 = y_2 = y. \end{cases}$$

D Estimation

In this section, we describe the problem of computing the MLEs of the unknown parameters of the BGPG distributions using the EM algorithm.

D.1 EM Algorithm

Suppose $\{(y_{11}, y_{21}), \ldots, (y_{1m}, y_{2m})\}$ is a random sample from BGPG$(\theta, \alpha, \lambda_0, \lambda_1, \lambda_2)$. Therefore, $I_0 = \{i : y_{1i} = y_{2i} = y_i\}$, $I_1 = \{i : y_{1i} > y_{2i}\}$ and $I_2 = \{i : y_{1i} < y_{2i}\}$. Also, $|I_0| = m_0$, $|I_1| = m_1$, $|I_2| = m_2$ and $m = m_0 + m_1 + m_2$. Therefore, the log-likelihood function can be written as

$$\ell(\Theta) = \sum_{i \in I_0} \ln g_0(y_i) + \sum_{i \in I_1} \ln g_1(y_{1i}, y_{2i}) + \sum_{i \in I_2} \ln g_2(y_{1i}, y_{2i}), \tag{D.1}$$

where g_0, g_1 and g_2 are defined in Theorem C.1. We can obtain the MLEs of the parameters by maximizing $\ell(\Theta)$ in (D.1) with respect to the unknown parameters. Clearly, it is difficult to compute the MLEs of the unknown parameters directly. We propose to use the EM algorithm and treat this as a missing value problem.

For given n, consider that independent random variables

$$\{U_i | N = n\} \sim GP(\alpha, n\lambda_i), \qquad i = 0, 1, 2. \tag{D.2}$$

Also, It is well known that $\{Y_i | N = n\} = \min\{U_0, U_i\} | N = n$, for $i = 1, 2$.

Assumed that for the bivariate random vector (Y_1, Y_2), there is an associated random vectors

$$(\Delta_1, \Delta_2) = \begin{cases} (0,0) & \text{if} \quad Y_1 = U_0,\, Y_2 = U_0 \\ (0,1) & \text{if} \quad Y_1 = U_0,\, Y_2 = U_2 \\ (1,0) & \text{if} \quad Y_1 = U_1,\, Y_2 = U_0 \\ (1,1) & \text{if} \quad Y_1 = U_1,\, Y_2 = U_2\ . \end{cases} \tag{D.3}$$

Here Y_i's are same as defined above. Therefore, a sample is obtained from $(Y_1, Y_2, \Delta_1, \Delta_2, N)$ which is the complete observation. It is clear that, if we know (Y_1, Y_2), the associated (Δ_1, Δ_2) may not always be known. We compute the pseudo log-likelihood function. The conditional 'pseudo' log-likelihood function is formed by conditioning on N, and then N is replaced by $E(N | Y_1, Y_2)$.

In the 'E' step, we kept the log-likelihood contribution of all the observations belonging to I_0 intact, as in this case the corresponding (Δ_1, Δ_2) are known completely. The observations are treated as missing observations, if they belong to I_1 or I_2.

If $(y_1, y_2) \in I_1$, the 'pseudo observation' is formed, by fractioning (y_1, y_2) to two partially complete 'pseudo observations' of the form $(y_1, y_2, u_1(\Theta))$ and $(y_1, y_2, u_2(\Theta))$. The fractional mass $u_1(\Theta)$ and $u_2(\Theta)$ assigned to the 'pseudo observation' are the conditional probabilities that (Δ_1, Δ_2) takes values (0,1) or (1,1), respectively, given that $(Y_1, Y_2) \in I_1$. Similarly, if $(Y_1, Y_2) \in I_2$, 'pseudo observations' are formed. Therefore,

$$v_1(\Theta) = \frac{\lambda_0}{\lambda_0 + \lambda_2}, \qquad v_2(\Theta) = \frac{\lambda_2}{\lambda_0 + \lambda_2}, \qquad u_1(\Theta) = \frac{\lambda_0}{\lambda_0 + \lambda_1}, \qquad u_2(\Theta) = \frac{\lambda_1}{\lambda_0 + \lambda_1}.$$

Therefore, we will use the following notations in the k-th step of the EM algorithm for the estimates of the parameters.

- $\Theta^{(k)} = (\alpha^{(k)}, \lambda_0^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)})$ is defined for the estimates of the parametersin the k-th step.
- $E(N|y_{1i}, y_{2i}, \Theta) = a_i$, and $E(N|y_{1i}, y_{2i}, \Theta^{(k)}) = a_i^{(k)}$.
- $u_1(\Theta^{(k)}) = u_1^{(k)}, \quad u_2(\Theta^{(k)}) = u_2^{(k)}, \;\; v_1(\Theta^{(k)}) = u_1^{(k)}$ and $v_2(\Theta^{(k)}) = u_2^{(k)}$.

E-Step: At the k-step of the EM algorithm, the 'pseudo' log-likelihood function without the additive constant can be written as follows:

$$\begin{aligned}
\ell_{pseudo}(\Theta) = {} & (m_0 + 2m_1 + 2m_2)\ln\lambda_0 + (m_2 + m_1 u_2^{(k)})\ln\lambda_1 + (m_2 v_2^{(k)} + m_1)\ln\lambda_2 \\
& - \lambda_0\{\sum_{i\in I_0} a_i^{(k)}(e^{\alpha y_i} - 1) + \sum_{i\in I_2} a_i^{(k)}(e^{\alpha y_{2i}} - 1) + \sum_{i\in I_1} a_i^{(k)}(e^{\alpha y_{1i}} - 1)\} \\
& - \lambda_1\{\sum_{i\in I_0} a_i^{(k)}(e^{\alpha y_i} - 1) + \sum_{i\in I_2} a_i^{(k)}(e^{\alpha y_{1i}} - 1) + \sum_{i\in I_1} a_i^{(k)}(e^{\alpha y_{1i}} - 1)\} \\
& - \lambda_2\{\sum_{i\in I_0} a_i^{(k)}(e^{\alpha y_i} - 1) + \sum_{i\in I_2} a_i^{(k)}(e^{\alpha y_{2i}} - 1) + \sum_{i\in I_1} a_i^{(k)}(e^{\alpha y_{2i}} - 1)\} \\
& + \alpha\{\sum_{i\in I_0} y_i + \sum_{i\in I_2} y_{1i} + \sum_{i\in I_2} y_{2i} + \sum_{i\in I_1} y_{2i} + \sum_{i\in I_1} y_{1i}\} \\
& + (m_0 + 2m_1 + 2m_2)\ln\alpha + m\ln\frac{\theta}{1-\theta} + \ln(1-\theta)\sum_{i=1}^{m} a_i^{(k)}.
\end{aligned} \tag{D.4}$$

M-Step: The 'M'-step involves maximizing $\ell_{pseudo}(\Theta)$ with respect to the unknown parameters.

ALGORITHM

- Step 1: Take some initial value of Θ, say $\Theta^{(0)} = (\theta^{(0)}, \alpha^{(0)}, \lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)})$.
- Step 2: Compute $a_i^{(0)} = E(N|y_{1i}, y_{2i}; \Theta^{(0)})$.
- Step 3: Compute u_1, u_2, v_1, and v_2.
- Step 4: Find $\hat{\alpha}$ and say $\hat{\alpha}^{(1)}$.
- Step 5: Compute $\hat{\lambda}_i^{(1)} = \hat{\lambda}_i(\hat{\alpha}^{(1)})$, $i = 0, 1, 2$.
- Step 6: Find $\hat{\theta}$.
- Step 7: Replace $\Theta^{(0)}$ by $\Theta^{(1)} = (\theta^{(1)}, \alpha^{(1)}, \lambda_0^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)})$, go back to step 1 and continue the process until convergence take place.

E Data analysis and comparison study

For illustrative purposes, we have analyzed one data set to see how the proposed model and the EM algorithm works in practice. This data set has been reported in Johnson and Wiechern (11). It represents the two diefferent measurements of stiffness, "Shock" and "Vibration" of each of 30 boards.

For illustrative purposes, first we plot the scaled TTT plots, see Aarset (1) for details, of the marginals in Figure 1 for real data. Since both are concave functions, it can be assumed that the hazard function of the marginals is increasing functions. Moreover, the correlation between the two marginals

Figure 1: The scaled TTT plots of the marginals for data set.

is positive.

Before going to analyze the data using BGPG distribution, we fit the GP distribution to Y_1, Y_2 and $\min\{Y_1, Y_2\}$ separately. The MLEs of parameters, the corresponding Kolmogorov-Smirnov distances and the associated p-values are calculated and the results are presented in Table 1. Based on the p-values the Gompertz distribution cannot be rejected for the marginals and for the minimum also.

Table 1: The MLEs of parameters, the K-S distances and the associated p-values.

Data Set	Variable	α	λ	K-S	p-value
Shock and Vibration data	Y_1	2.3598	0.0075	0.1420	0.9032
	Y_2	2.4296	0.0097	0.1553	0.8660
	$\min\{Y_1, Y_2\}$	2.4301	0.0098	0.1977	0.5775

Now, we will fit the BGPG model. Using the proposed EM algorithm, the MLE's and their corresponding log-likelihood values are calculated. For the fitted model, the Akaike Information Criterion (AIC) and the Bayesian information criterion (BIC) are calculated. The results are given in Table 2.

Table 2: The MLEs of parameters, the corresponding log-likelihood, AIC and BIC.

Data Set	$\hat{\alpha}$	$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\theta}$	$log(\ell)$	AIC	BIC
Shock and Vibration data	2.3833	0.0019	3.1077×10^{-4}	0.0028	0.5941	-34.5795	79.1590	86.1650

We also obtain the Kolmogorov-Smirnov (K-S) distances with the corresponding p-values between the fitted distribution and the empirical CDF for three random variables Y_1, Y_2 and $\min(Y_1, Y_2)$. The results are given in Table 3.

Finally, the likelihood ratio test (LRT) and the corresponding p-values are obtained for testing the BGP model against the BGPG model. On the other hand, our goal is to test the null hypothesis $H_0 : BGP$ against the alternative hypothesis $H_1 : BGPG$. The statistics and the corresponding p-values are given in Table 4. Hence, for any usual significance level, we reject the model in H_0 (BGP) in favor of the alternative model (BGPG).

F Conclusions

In this paper, we have proposed two new bivariate distributions using the model of dependent lives. This new model was called as the bivariate Gompertz distribution (BGP). Then, this model has been generalized. We called this distribution as the bivariate Gompertz-geometric distribution (BGPG). Several properties of this distribution have been established. The estimation of unknown parameters by

Table 3: The K-S distances and the corresponding p-values for three random variables Y_1, Y_2 and $\min(Y_1, Y_2)$.

Data Set	Y_1		Y_2		$\min\{Y_1, Y_2\}$	
	K-S	p-value	K-S	p-value	K-S	p-value
Shock and Vibration data	0.1260	0.9591	0.1776	0.7442	0.1877	0.7255

Table 4: The log-likelihood, AIC, BIC, LRT and the corresponding p-values for different models.

Data Set	Models	Test				
		AIC	BIC	$log\ell$	LRT	p-value
Shock and Vibration data	BGPG	79.1590	86.1650	-34.5795	20.1320	7.2277×10^{-6}
	BGP	99.2910	106.2970	-44.6455		

the method maximum likelihood is acquired. However, it is not directly easy to solve the associated log likelihood equations. Therefore, we have suggested using the EM algorithm to compute the MLEs of the unknown parameters, and it is observed that the proposed EM algorithm works quite well in practice. As shown, the proposed models work quite well for data analysis purposes. Finally, we compare BGPG model with BGP model.

References

[1] Aarset, M. (1987), How to identify a bathtub hazard rate, *IEEE Transactions on Reliability*, **36**, 106-108.

[2] Al-Khedhairi, A. and El-Gohary, A. (2008), A new class of bivariate gompertz distributions and its mixture, *International Journal of Mathematical Analysis*, **2**, 235-253.

[3] Bagheri, S.F., Bahrami Samani, E. and Ganjali, M. (2016), The generalized modied weibull power series distribution: Theory and applications, *Computational Statistics and Data Analysis*, **94**, 136-160.

[4] Barreto-Souza, W. (2012), Bivariate gamma-geometric law and its induce levy process, *Journal of Multivariate Analysis*, **109**, 130-145.

[5] Carriere, J.F. (2000), Bivariate survival models for coupled lives, *Scandinavian Actuarial Journal*, **2000**, 17-32.

[6] Chahkandi, M. and Ganjali, M. (2009), On some lifetime distributions with decreasing failure rate, *Computational Statistics and Data Analysis*,**53**, 4433-4440.

[7] Ghitany, Al-Hussaini, M.E. and Al-Jarallah, R. (2005), Marshall-Olkin extended weibull distribution and its application to censored data, *Journal of Applied Statistics*, **32**, 1025-1034.

[8] Ghitany, M. Al-Awadhi, F. and Alkhalfan, L. (2007), Marshall-Olkin extended lomax distribution and its application to censored data, *Communications in Statistics-Theory and Methods*, **36**, 1855-1866.

[9] Iyer, S.K. and Manjunath D. (2004), Correlated bivariate sequences for queueing and reliability applications, *Communications in Statistics-Theory and Methods*, **33**, 331-350.

[10] Jagger C. and Sutton C.J. (1991), Death after marital bereavement is the risk increased? *Statistics in Medicine*, **10**, 395-404.

[11] Johnson, R.A. and Wiechern D.W. (1992), *Applied Multivariate Statistical Analysis*, Prentice Hall, New Jersey.

[12] Kundu, D. and Gupta, R.D. (2010), Modied sarhan-balakrishnan singular bivariate distribution, *Journal of Statistical Planning and Inference*, **140**, 526-538.

[13] Kundu, D. and Gupta, A.K. (2014), On bivariate Weibull-geometric distribution, *Journal of Multivariate Analysis*, **123**, 19-29.

[14] Marshall, A.W. and Olkin, I. (1997), A new method of adding a parameter to a family of distributions with application to the exponential and weibull families, *Biometrika*, **84**, 641-652.

[15] Pham, H. and Lai, C.D. (2007), On recent generalizations of the weibull distribution, *IEEE Transactions on Reliability*, **56**, 454-458.

[16] Sarhan, A.M. and Balakrishnan, N. (2007), A new class of bivariate distributions and its mixture, *Journal of Multivariate Analysis*, **98**, 1508-1527.

[17] Silva, R.B., Bourguignon, M., Dias, C.R.B. and Cordeiro, G.M. (2013), The compound class of extended weibull power series distributions, *Computational Statistics and Data Analysis*, **58**, 352-367.

[18] Veenus, P. and Nair, K.R.M. (1994), Characterization of a bivariate pareto distribution, *Journal of Indian Statistical Association*, **32**, 15-20.

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Defining Stochastic Orderings and Ageing Classes of Life Distributions: A Unified Approach

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Abstract: In this talk, we present some new characterizations of the well-known reliability classes of life distributions such as IFR, DFR, NBU, HNBUE, NBUC, etc. For this purpose, a unified approach based on a weighted average of the failure rate of the equilibrium distribution is utilized. Different properties of the proposed measure are also considered.

Keywords Equilibrium distribution, Failure rate, Stochastic orders.

Mathematics Subject Classification (2010) : 62N05, 60K10**.**

A Introduction

In the literature, many attempts have been made to classify various categories of life distributions. A number of classes of life distributions have been studied in reliability theory. Usually, the fact that a distribution belongs to a particular class can be justified by a physical understanding of the failure mechanism. In these circumstances, one can utilize an appropriate method of data analysis among many statistical procedures developed for the cited classes.

Let F be a cumulative distribution function (cdf) with the corresponding reliability function $\bar{F} = 1 - F$. The function $-\log \bar{F}(t)$ is called the hazard function of the cdf F. The concept of monotone hazard rate has played a crucial role in reliability engineering. In the following, we give formal definitions of some basic reliability classes.

- F is said to have an increasing (decreasing) failure rate [IFR (DFR)] if its hazard function is concave (convex).
- A cdf F with $F(0-) = 0$ is said to have an increasing failure rate average (IFRA) if its hazard function is star-shaped (that is, $-\log \bar{F}(t)/t$ is increasing in $t \geq 0$). Similarly, F is said to have a decreasing failure rate average (DFRA) if $-\log \bar{F}(t)/t$ is decreasing in $t \geq 0$.
- F is said to be new better (worse) than used [NBU (NWU)] if $\bar{F}(s+t) \leq (\geq)\bar{F}(s)\bar{F}(t)$, for $s, t \geq 0$.
- F is said to be new better (worse) than used in expectation [NBUE (NWUE)] if $\int_t^\infty \bar{F}(x)dx \leq (\geq)\mu\bar{F}(t)$, for $t \geq 0$.

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- F is said to be harmonically new better (worse) than used in expectation [HNBUE (HNWUE)] if $\int_t^\infty \bar{F}(x)dx \leq (\geq) \mu \exp(-t/\mu)$, for $t \geq 0$.

- F is said to be new better than used in failure rate average [NBUFRA (NWUFRA)] if $r(0) \leq (\geq) -\frac{1}{t} \log \bar{F}(t)$ for $t \geq 0$.

- F is said to be new better than used in convex ordering (NBUC) if $\int_{s+t}^\infty \bar{F}(x)dx \leq \bar{F}(t) \int_s^\infty \bar{F}(x)dx$, for all $s, t \geq 0$.

Various aspects of the classes of life distributions mentioned above and many others have been extensively studied in the literature. These classes possess a number of very interesting properties; see Barlow and Proschan (1975), Lai and Xie (2006) and Marshall and Olkin (2007). For example, it is known that a mixture of DFR distributions is DFR. Another key result is that the class of IFRA is closed under the formation of coherent systems; it is the smallest class containing the exponential distributions which is closed under weak limits.

Throughout the article, we also need the following concepts of stochastic orders. Let X and Y be two non-negative random variables with reliability functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. X is said to be less than Y in the

- usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{G}(t) \geq \bar{F}(t)$, for all $t \geq 0$.

- hazard rate order (denoted by $X \leq_{hr} Y$) if $\frac{\bar{G}(t)}{\bar{F}(t)}$ is increasing in $t \geq 0$.

- mean residual life order (denoted by $X \leq_{mrl} Y$) if $\int_t^\infty \bar{G}(x)dx / \int_t^\infty \bar{G}(x)dx$ is increasing in $t \geq 0$.

The following implications hold among the stochastic orders:

$$X \leq_{hr} Y \quad \Rightarrow \quad X \leq_{st} Y, \qquad X \leq_{hr} Y \quad \Rightarrow \quad X \leq_{mrl} Y.$$

For more details on properties of stochastic orders, we refer the reader to Shaked and Shanthikumar (2007).

The main purpose of this work is to study various properties of the MEFR. We also aim to unify the definitions of all mentioned ageing classes based on the notion of MEFR. Precisely, It is shown that all the above mentioned classes of life distributions can be characterized by considering different properties of the MEFR. We will also introduce new ageing classes of life distributions and discuss about their interpretations in the field of reliability engineering. Also, by utilizing the concept of MEFR, the definitions of some stochastic orders (such as usual, hazard rate and mean residual life orders) will be unified.

B Characterization of the reliability classes based on MEFR

Suppose that F is the cdf of a nonnegative random variable X with finite mean μ. A distribution related to F is the *equilibrium distribution* with the probability density function

$$f_e(t) = \begin{cases} \frac{\bar{F}(t)}{\mu}, & t \geq 0; \\ 0, & t < 0. \end{cases}$$

The equilibrium distribution arises as the limiting distribution of the forward recurrence time in a renewal process. It is well-known that the hazard rate of the equilibrium distribution is the reciprocal of the mean residual life of the original distribution.

One can define higher-order equilibrium distributions. For $n = 1, 2, ...$, the equilibrium distribution of order n is given by

$$\bar{F}_n(t) = \frac{\int_t^\infty \bar{F}_{n-1}(x)dx}{\mu_{n-1}},$$

where μ_{n-1} is the mean of the distribution F_{n-1} and $\bar{F}_0 = \bar{F}$ is the baseline reliability function.

In the present research, we introduce the concept of *mean equilibrium failure rate* (MEFR). This new measure is defined as

$$\xi_n(s, s+t) = -\frac{1}{t} \log\left(\frac{\bar{F}_n(s+t)}{\bar{F}_n(s)}\right), \quad s, t \geq 0;\ n = 0, 1, ...$$

It may be rephrased as

$$\xi_n(s, s+t) = E\left[w_{s,t}(X) h_n(X)\right],$$

where X denotes the original random variable and h_n is the failure rate of the nth order equilibrium distribution. The last expression is, in fact, a weighted average of h_n with the weight function

$$w_{s,t}(x) = \frac{1}{tf(x)} I_{[s,s+t]}(x),$$

where $I_A(\cdot)$ denotes the indicator function on the set A. This explains the terminology of the definition of MEFR. Using a result of Gupta (2007), the MEFR can also be written as

$$\xi_n(s, s+t) = -\frac{1}{t} \log\left(\frac{E\left[(X-s-t)_+^n\right]}{E\left[(X-s)_+^n\right]}\right),$$

where $E\left[(X-s)_+^n\right]$ is known as the nth order stop-loss transform.

In the following results, we briefly describe the applications of the MEFR in unification of the reliability classes, and stochastic orderings. The proofs together with more results including characterizations of the reliability classes based on reversed hazard rate, mean inactivity time, etc., can be seen in Tavangar (2019). In the cited reference, some new classes of life distributions are also introduced.

Let F be an arbitrary cdf.

a F is IFR (DFR) if and only if $\xi_0(s, s+t)$ is increasing (decreasing) in s for all $t \geq 0$.

b F is IFRA (DFRA) if and only if $\xi_0(0, t)$ is increasing (decreasing) in t.

c F is NBU (NWU) if and only if $\xi_0(s, s+t) \geq (\leq)\xi_0(0, t)$ for $s, t \geq 0$.

d F is NBUE (NWUE) if and only if $\xi_1(0, t) \geq (\leq)\xi_0(0, t)$ for $t \geq 0$.

e F is HNBUE (HNWUE) if and only if $\xi_1(0, t) \geq (\leq)\xi_1(0, 0)$ for $t \geq 0$.

f F is NBUFRA (NWUFRA) if and only if $\xi_0(0, t) \geq \xi_0(0, 0)$ for $s, t \geq 0$.

g F is NBUC if and only if $\xi_1(s, s+t) \geq \xi_0(0, t)$ for $s, t \geq 0$.

Let X and Y be two random variables with the reliability functions $\bar{F}$ and $\bar{G}$, and the MEFR $\xi_n(s, s+t)$ and $\zeta_n(s, s+t)$, respectively. In the next result, some characterizations of the usual stochastic order, the hazard rate order and the mean residual order, based on the notion of MEFR, are provided. Let X and Y be two random variables with the corresponding MEFR $\xi_n(s, s+t)$ and $\zeta_n(s, s+t)$. Then

a $X \leq_{st} Y$ if and only if $\xi_0(0, t) \leq \zeta_0(0, t)$ for all $t \geq 0$.

b $X \leq_{hr} Y$ if and only if $\xi_0(s, s+t) \leq \zeta_0(s, s+t)$ for all $s, t \geq 0$.

c $X \leq_{mrl} Y$ if and only if $\xi_1(0, t) \geq \zeta_1(0, t)$ for all $t \geq 0$.

References

[0] Barlow, R.E. and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing*, New York, Holt, Rinehart and Winston.

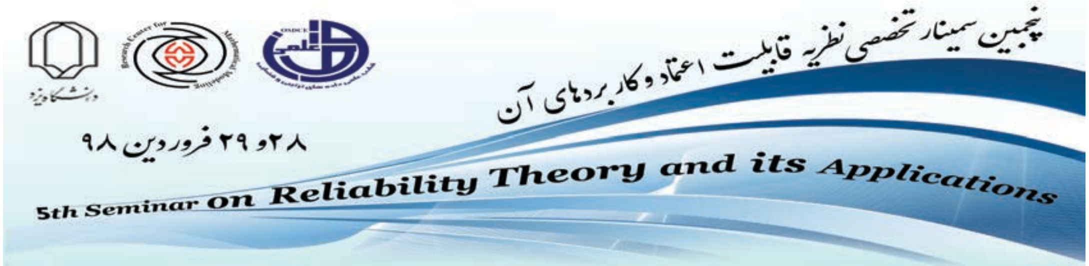
[0] Gupta, R.C. (2007), Role of equilibrium distribution in reliability studies, *Probability in the Engineering and Informational Sciences*, **21**, 315-334.

[0] Lai, C.D., Xie, M. (2006), *Stochastic ageing and dependence for reliability,* Springer, New York.

[0] Marshall, A.W. and Olkin, I. (2007), *Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families,* Springer, New York.

[0] Shaked, M. and Shanthikumar, J.G. (2007), *Stochastic Orders*, Springer, New York.

[0] Tavangar, M. (2019), New characterizations of ageing classes of life distributions, *Submitted*.



Survival Function of Generalized δ-Shock Model Based on Polya Process

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Abstract: The shock models have attracted great deal of attention because of their important role in the engineering systems. A shock model is called when a system fails if the interval time between two consecutive shocks is less than a fixed threshold δ. In this paper, the generalized δ-shock model by assuming that the system is subject to two types of shocks under a Polya process of shock arrival which has dependent interarrival times is studied. The survival function of the system are obtained and also, the illustrative examples is presented. **Keywords** δ-shock model, Interarrival times, Polya process, Survival function.

Mathematics Subject Classification (2010) : 62N05, 90B25**.**

A Introduction

The δ-shock model proposed by Li et al. (3; 4) is a special type of shock model, which if the interarrival time between two shocks is shorter than a prespecified threshold δ, the system fails. this shock model is useful for systems that needs a period of time to recover from the shock. They studied lifetime properties of the model that the shocks arrive according to a Poisson process. The δ-shock model is widely utilized in many areas such as electrical systems, inventory theory, earthquake modeling, insurance mathematics. Also, some generalizations were provided for the δ-shock model.

Eryilmaz (1) presented a generalization of the δ-shock model by using the concept of runs and obtained the survival function and the mean value of the failure time of the system. Recently, Wang and Peng (5) generalized the δ-shock model by assuming that the system is subject to two types of shocks under homogeneous Poisson process of shock arrivals which the system fails, if a shock occurs while the system still has not recovered from the consequence of the previous shock. Eryilmaz (2) investigated the δ-shock model under a Polya process of shock arrival which has dependent interarrival times and obtained survival function and mean lifetime of the system.

The rest of paper is organized as follows. In Section 2, some notations and the assumption of models is provided. The survival function of the system are obtained in Section 3. In Section 4, the illustrative examples is presented to evaluate the results. Finally, Concluding remarks are given in Section 5.

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B Notations and model assumption

In this section, some notations and the assumption of models is provided for generalized δ-shock model subject to two types of shocks.

B.1 Notations

The following notations are needed.

$N(t)$	Polya process
α, β	Parameters of Gamma distribution
$N_i(t)$	Number of type i shocks occurred in time interval $(0, t], i = 1, 2$
n_i	Realization of $N_i(t), i = 1, 2$
p, q	Probability of a shock belongs to type 1 or 2, with $p + q = 1$
X_n	Interval time between the $(n-1)$th and nth shocks, $n = 1, 2, ...$
$F(t)$	Common cumulative distribution function (CDF) of
	the interval time $X_n, n = 1, 2, ...$
δ_i	System recovery time for a type i shock, $i = 1, 2$
Z_n	Type of the nth shock, equal to 1 or 2, $n = 1, 2, ...$
T	Lifetime of the δ-shock model with two types of shocks

B.2 Assumption

Consider a generalized δ-shock model for a single component system with two types of shocks by the following assumption.

Assumption. A new installed system at time $t = 0$, is deal with external shocks that includes two types of shocks. Each shock is of type 1 with probability p, type 2 with probability $q = 1 - p$, independently that based on a Polya process $\{N_t, t \geq 0\}$. Polya process is a special case of mixed Poisson process when the structure distribution H is Gamma with density

$$dH(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda},$$

and hence its one dimensional distribution can be written as

$$\begin{aligned} P(N(t) = n) &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} dH(\lambda) && \text{(B.1)} \\ &= \binom{\alpha + n - 1}{n} \left(\frac{t}{t+\beta}\right)^n \left(\frac{\beta}{\beta+t}\right)^\alpha, \qquad \forall\, n = 0, 1, \ldots. && \text{(B.2)} \end{aligned}$$

The interarrival times $X_i, i = 1, 2, \ldots, n$ are dependent, exchangeable and the marginal distribution of X_i is Pareto with cummulative distribution function

$$P(X_i \leq t) = 1 - \left(\frac{\beta}{\beta+t}\right)^\alpha, \qquad \forall\, t \geq 0.$$

C Survival function of the generalized δ-shock Model

To obtain the survival function of the generalized δ-shock model with two types of shocks, the following lemma is requested.

Let $N(t), t \geq 0$ be a homogeneous Poisson process (HPP) with rate λ, and denote $X_1, X_2, \cdots, X_n$ the interarrival times of the process. Given $N(t) = n$, then for any fixed constant $a > 0$,

$$P(X_2 > a, X_3 > a, , X_n > a | N(t) = n) = \left(1 - \frac{(n-1)a}{t}\right)_+^n, \tag{C.1}$$

where $y_+ = max(y, 0)$; Li et al. (3; 4) and Eryilmaz (1).

If a shock occurs while the system has not recovered from a previous shock, the system breaks down. So, it is important to obtain the survival function. The survival function of δ-shock model can be indicated as

$$P(T > t) = \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} P(T > t, N_1(t) = n_1, N_2(t) = n_2). \tag{C.2}$$

To get the survival function (C.2), there are three cases. Firstly, there is no shock occurred in $[0, t]$, the system will survive this time interval. So,

$$P(T > t, N_1(t) = 0, N_2(t) = 0) = \left(\frac{\beta}{\beta + t}\right)^\alpha. \tag{C.3}$$

Secondly, we assume that there is only one type of shocks occurred in time interval $[0, t]$. If the first type of shock occured, the survival function is

$$\begin{aligned}
&P(T > t, N_1(t) = n_1, N_2(t) = 0) \\
&\quad = P(X_1 > \delta_1, ..., X_{n_1} > \delta_1 | N_1(t) = n_1, N_2(t) = 0) P(N_1(t) = n_1, N_2(t) = 0) \\
&\quad = P(X_1 > \delta_1, ..., X_{n_1} > \delta_1 | N_1(t) = n_1, N_2(t) = 0) P(N_1(t) = n_1) P(N_2(t) = 0) \\
&\quad = p \left(\frac{\beta}{\beta + t}\right)^{2\alpha} \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \binom{\alpha + n_1 - 1}{n_1} \left(\frac{t - n_1\delta_1}{t + \beta}\right)_+^{n_1}, \qquad \forall\, n_1 = 1, 2, \cdots,
\end{aligned} \tag{C.4}$$

where the last statement holds by using Lemma C. Likely for the second type of shock, we have

$$\begin{aligned}
&P(T > t, N_1(t) = 0, N_2(t) = n_2) \\
&\qquad = (1 - p) \left(\frac{\beta}{\beta + t}\right)^{2\alpha} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} \binom{\alpha + n_2 - 1}{n_1} \left(\frac{t - n_2\delta_2}{t + \beta}\right)_+^{n_2},
\end{aligned} \tag{C.5}$$

for $n_2 = 1, 2, \cdots$.

Finally, for the general case that both two types of shocks have occurred in time interval $[0, t]$. By investigating the probability $P(T > t, N_1(t) = n_1, N_2(t) = n_2)$, the survival function can be obtained as following. The survival function of the generalized δ-shock model is

$$
\begin{aligned}
P(T>t) =& \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} P(T>t, N_1(t)=n_1, N_2(t)=n_2) \\
=& P(T>t, N_1(t)=0, N_2(t)=0) \\
&+ p \sum_{n_1=1}^{[\frac{t}{\delta_1}]} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=1) \\
&+ (1-p) \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \sum_{n_2=1}^{[\frac{t}{\delta_2}]} P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=2) \\
=& \left(\frac{\beta}{\beta+t}\right)^{\alpha} + p \sum_{n_1=1}^{[\frac{t}{\delta_1}]} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} \binom{\alpha+n_1-1}{n_1} \binom{\alpha+n_2-1}{n_2} \left(\frac{t'}{\beta+t'}\right)^{n} \left(\frac{\beta}{\beta+t'}\right)^{2\alpha} \\
&\times \left(\frac{\beta}{\beta+(n_1-1)\delta_1+n_2\delta_2}\right)^{\alpha} + (1-p) \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \sum_{n_2=1}^{[\frac{t}{\delta_2}]} \binom{\alpha+n_1-1}{n_1} \\
&\times \binom{\alpha+n_2-1}{n_2} \left(\frac{t_1'}{\beta+t_1'}\right)^{n} \left(\frac{\beta}{\beta+t_1'}\right)^{2\alpha} \left(\frac{\beta}{\beta+n_1\delta_1+(n_2-1)\delta_2}\right)^{\alpha},
\end{aligned} \tag{C.6}
$$

where $t' = [t-(n_1-1)\delta_1 - n_2\delta_2]_+$, $t_1' = [t-n_1\delta_1-(n_2-1)\delta_2]_+$ and $[x]$ denotes the integer part of x.

D Example

In the following, the survival function is plotted for different values of parameters, $\alpha > 1, \beta, \delta_1, \delta_2$ and p. First, setting $\delta_1 = 2, \delta_2 = 1, \alpha = 1, \beta = 0.5$ and varying the values of p, the survival function of the generalized δ-shock model based on Polya process can be calculated. From 1, we can see that the survival function decreases gradually with the increase of p for any $t > 0$. This is because that the system needs a longer time to recover from a type 1 shock than from a type 2 shock, and the increase of p means the proportion of type 1 shock increases in the shock process and the system is more easily destroyed by the shocks.

The next, setting $\delta_1 = 2, \delta_2 = 1, \alpha = 1, p = 0.5$ and varying the values of β, the survival function of the generalized δ-shock model based on Polya process can be calculated. From 2, we can see that the survival function decreases gradually with the decrease of β for any $t > 0$. Because both types of shocks occur with the same probability, then the system needs a longer time to recover from both types of shocks, and the decrease of β means the proportion of both types of shocks increase in the shock process and the system is more easily destroyed by the shocks.

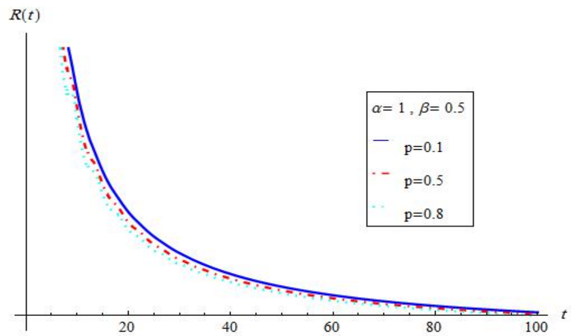

Figure 1: The survival function $R(t)$ with $\delta_1 = 2, \delta_2 = 1, \alpha = 1, \beta = 0.5$, and for different p.

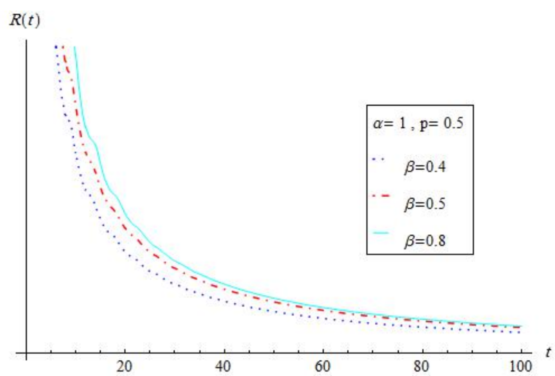

Figure 2: The survival function $R(t)$ with $\delta_1 = 2, \delta_2 = 1, \alpha = 1, p = 0.5$, and for different β.

E Conclusion

In this paper, we have discussed a generalized δ-shock model with two types of shocks. By assuming that the shocks are generated by Polya process, and the recovery times for the two types shocks are δ_1 and δ_2, respectively. We have derived explicit expressions for the survival function of the system. This paper assumes that the two types of shocks occure independently but the interarrival times between shocks are exchangeable and dependent.

References

[1] Eryilmaz, S. (2012), Generalized δ-shock model via runs, *Statistics and Probability Letters,* **82**, 326-331.

[2] Eryilmaz, S. (2017), δ-shock model based on Polya process and its optimal replacement policy, *European Journal of Operational Research,* **263**(2), 690-697.

[3] Li, Z.H., Chan, L.Y. and Yuan, Z.X. (1999), Failure time distribution under a δ-shock model and its application to economic design of system, *International Journal of Reliability, Quality and Safety Engineering,* **3**, 237-247.

[4] Li, Z.H., Huang, B.S. and Wang, G.J. (1999), Life distribution and its properties of shock models under random shocks, *Journal of Lanzhou University,* **35**, 1-7.

[5] Wang, G.J. and Peng, R. (2017), A generalised δ-shock model with two types of shocks, *International Journal of Systems Science: Operations & Logistics,* **4**(4), 372-383.

پنجمین سمینار تخصصی نظریه قابلیت اعتماد و کاربردهای آن

۲۸ و ۲۹ فروردین ۹۸

5th Seminar on Reliability Theory and its Applications

A Study on Methods for Estimating the Parameters of the Exponentiated Weibull Distribution Under Randomly Right Censored Data Based on Misspesification of Model

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Abstract: Exponentiated Weibull distribution introduced as an extention of the weibull distribution is derived usefull applications in reliability and survival studies. In this paper, we compared the maximum likelihood estimator(MLE), the approximate maximum likelihood estimator(AMLE) and the approximate maximum likelihood Jackniffe estimator(AMLJE) of the parameters of the exponentiated weibull distribution in case of the randomly right censored data. The performance of the MLE, AMLE and AMLJE are compared by the simulation study. Simulation study show that, AMLE and AMLJE be have better than MLE when the proposed model is misspecified and thay are not better when not so.

Keywords Exponentiated Weibull distribution, approximate maximum likelihood , Right censord data.

Mathematics Subject Classification (2010) : 62N02, 62N01**.**

A Introduction

The exponentiated Weibull(EW) distribution introduced by Mudholkar and Srivastava (1993) as an extension of the Weibull distribution, is characterized by unimodal failure rates besides a broader class of monotone failure rates. The applications of the EW distribution in reliability and survival studies were illustrated by Mudholkar et al. (1995). Its properties were studied in detail by Mudholkar and Hutson (1996) and Nassar and Eissa (2003, 2004).They denoted useful applications of the distribution in the modeling of flood data and in reliability. Singh et al. (2005a, 2005b) obtained Bayes estimators of the parameters, reliability function and hazard function for EW distribution type II censored data under squared error. Jaheen and Al Harbi (2011) discussed Bayesian estimation based on dual generalized order statistics from the EW distribution. Ashour and Afify (2007) analyzed EW distribution life time data observed under type I progressive interval censoring with random removals. Ashour and Afify (2008) derived MLEs of the parameters of the EW distribution and their asymptotic variance for type II progressive interval censoring with random removals. A random variable X is said to have a three parameter EW distribution if its probability density function and cumulative distribution function are

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$$h(x) = \alpha\beta \frac{x^{\beta-1}}{\theta^{\beta}} e^{-\frac{x^{\beta}}{\theta^{\beta}}} (1 - e^{-\frac{x^{\beta}}{\theta^{\beta}}})^{\alpha-1}, x > 0, \alpha > 0, \beta > 0, \theta > 0,$$

and

$$H(x) = (1 - e^{-\frac{x^{\beta}}{\theta^{\beta}}})^{\alpha}, x > 0, \alpha > 0, \beta > 0, \theta > 0,$$

respectively, where α and β are the shape parameters and θ is the scale parameter. Because it was not always facing with complete data, it is also important to study the parameters estimation for incomplete data. Including incomplete data, we can refer the randomly right censored data that are of high standing because of saving time, money and also having vast applications in time-life tests, survival analysis and reliability theory. Although a considerable number of studies have been made on parametric estimation for censored data, little attention has been given to the misspecification of the parametric model. Our main concern are to consider the parametric estimation of the EW distribution under the misspecification. This idea of parametric estimation based on censored data was first proposed by Oakes(1986), and which is reffered to as approximate maximum likelihood procedured In parametric estimation, the Kullback-Leibler information is used as a measure of the divergence a true distribution relative to the proposed parametric model.

B Main results

Suppose that $X_1, X_2, ..., X_n$ are i.i.d. random variables from an unknown distribution $H(x)$ with probability density $h(x)$. Parametric inference is done within an assumed parametric family of densities $A = \{f(x, \theta), \theta \in \Theta\}$. If A contains h, there exists $\theta_0 \in \Theta$ such that, $h(x) = f(x, \theta_0)$, and θ_0 is called the true parameter value and the proposed model is wellspecified, otherwise, the proposed model is misspecified, On the other hand, if $h(x)$ is not contained in A, we can obtained nearest $f(x, \theta)$ to the true density $h(x)$ by the Kullback-Leibler information. This means that a purpose of the MLE is to find a parameter θ which minimizes the Kullback-Leibler information

$$KL(h(.), f(., \theta)) = \int h(x) \log \frac{h(x)}{f(x, \theta)} dx, \tag{B.1}$$

which is a measure of the divergence of $h(x)$ relative to $f(x, \theta)$. Under suitable regularity conditions, the maximum likelihood estimators(MLE), which is defined as a value of $\theta \in \Theta$ is obtained from derivation of logarithm of the likelihood function. So, MLE is converged to θ_0 which is true parameter of data, and data is generated from it, which is a parameter value minimizing(2.1).

In the analysis of lifetime data, an important problem is censorship of observations. For $i = 1, ..., n$, suppose that X_i and Y_i for be random variables which represent a lifetime and a censoring time of the i-th individual, respectively. In lifetime data analysis, X_i and Y_i are not observed. We can observe

$$(Z_i, \delta_i) = (min(X_i, Y_i), I(X_i \leq Y_i)),$$

where $I(B)$ denotes the indicator function of the set B. The set of observations (Z_i, δ_i), $i = 1, .., n$ is called randomly right censored data in survival and reliability theory. Note that X_i' s are independent of Y_i's. $G(y)$ where $g(y)$ and are an unknown distribution and the probability density function. respectively. Let $Y_1, Y_2, ..., Y_n$ are i.i.d. from $G(y)$.

$$\hat{F}_n(x) = 1 - \prod_{i=1}^{n} [1 - \frac{\delta_i}{n - i + 1}]^{I(Z_{(i)} \leq x)},$$

where $Z_{(1)} \leq Z_{(2)} \leq ... \leq Z_{(n)}$ are the order values of Z_i and δ_i denotes the concomitaint associated with $Z_{(i)}$. In the uncensored case the Kaplan-Meier estimator $\hat{F}_n(x)$ coincides with the empirical distribution. The parametric model A is assumed for the distribution of X_i, the log likelihood function is given by

$$Lf_n(\theta) = \sum_{i=1}^{n} \{\delta_i \log f(Z_i, \theta) + (1 - \delta_i) \log \bar{F}(Z_i, \theta)\}, \tag{B.2}$$

where $\bar{F}(Z_i, \theta) = \int I(u > z) f(u, \theta) du$. The maximum likelihood estimator is an element $\hat{\theta}_n \in \Theta$ which attains the maximum likelihood value if $l_n(\theta)$ in Θ. When data are complete , the MLE is a consistent estimator of minimizing (2.1). Under random censorship, $\hat{\theta}_n$ is not suitable estimator when A does not contain h. Oakes (1986) introduced the approximate maximum likelihood estimator to parametic estimation based on censored data. Therefore, we consider another estimator $\hat{\theta}_n^*$, which is defined as an element in Θ which maximizes

$$Lf_n^*(\theta) = n \int \log f(x, \theta) d\hat{F}_n(X), \tag{B.3}$$

When all X_i's are observable, the log-likelihood function can be expressed as

$$\sum_{i=1}^{n} \log f(x_i, \theta) = n \int \log f(x, \theta) dF_n(X),$$

Thus $Lf_n^*(\theta)$ is a natural extension to the censored data in the sense that the empirical distribution F_n is replaced by the Kaplan-Meier estimator $\hat{F}_n$. In case of complete data, $Lf_n^*(\theta) = Lf_n(\theta)$ and therefore $\hat{\theta}_n^* = \hat{\theta}_n$.

Stute and Wang (1993) proved the law of large numbers of the Kaplan-Meier integral. The following theorem, using Suzukawa' assumptions(Suzukawa et al. (2001)) is shown.

Under the conditions $(A1)$-$(A7)$, the AMLE $(\hat{\theta}_n^*)$ converges to θ_0^* in probability as $n \longrightarrow \infty$. where θ_0^* gives the neast density function in A to the true density function. Suzukawa et al. (2001) consistency and asymptotic normality of the AMLE under the misspecification of the proposed model and it converges to θ_0^* in probability that which is a parameters value minimizingis (2.1). So if the proposed model is wellspecified and in case uncensored $\theta_0 = \theta_0^*$.

Mauro (1985) and Stute(1994) pointed out, for every integrable φ, $\int \varphi(x) d\hat{F}_n(x)$ has a nonnegligible bias as an estimator of $\int \varphi(x) h(x) dx$.

$$\int \varphi d\hat{F}_n(x) + K_n \varphi(Z_{(n)})$$

as an estimator of $\int \varphi(x)h(x)dx$, where

$$K_n = \frac{n-1}{n}\delta_{(n)}(1-\delta_{(n)})\prod_{j=1}^{n-2}(\frac{n-1-j}{n-j})^{\delta_{(j)}}.$$

Thay showed that this estimator has smaller bias than $\int \varphi(x)d\hat{F}_n(x)$ for $\varphi(x)=x$.

We consider an estimator θ_n^{*JK} which attains the maximum of $Lf_n^{*JK}(\theta)$ in Θ, where

$$Lf_n^{*JK}(\theta) = Lf_n^{*}(\theta) + nK_n \log f(Z_{(n)}, \theta). \tag{B.4}$$

So, we discuss comparision of the mentioned estimators based on a simulation study for the EW distribution. We assume that X_i's denote the EW distribution as follow:

$$h(x) = \alpha\beta\frac{x^{\beta-1}}{\theta^{\beta}}e^{-\frac{x^\beta}{\theta^\beta}}(1-e^{-\frac{x^\beta}{\theta^\beta}})^{\alpha-1},$$

and the censoring time Y_i's draw independently of the X_i's are as follow:

$$g(y) = \beta\frac{x^{\beta-1}}{\theta^{\beta}}e^{-\frac{x^\beta}{\theta^\beta}},$$

and the proposed model is $A = \{f(x,\theta) = \frac{1}{\theta}exp(-\frac{x}{\theta}), \theta > 0\}$.

Therefore, the MLE, AMLE and AMLJE are obtained as (2.2), (2.3) and (2.4), respectively, as follow:

$$\hat{\theta}_n = \frac{\sum_{i=1}^n Z_i}{\sum_{i=1}^n \delta_i},$$

$$\hat{\theta}_n^* = \sum_{i=1}^n W_i Z_{(i)},$$

$$\theta_n^{*JK} = \frac{K_n Z_{(n)} + \sum_{i=1}^n W_i Z_{(i)}}{1+K_n},$$

where $W_i = \frac{\delta_{(i)}}{n-i+1}\prod_{j=1}^{i-1}(\frac{n-j}{n-j+1})^{\delta_{(i)}}$. Based on Monte Carlo simulation a comparision these estimators for the EW distribution. We derive mean square errors(MSEs) of these three estimators under right censorship. Based on generating one thousand replications of samples of size $n = 50, 100, 150, 200$ from the EW distribution with parameters (α ,β)=(1,1/3),(1,1), (3/2,1) and (5/4,3/2), the MSEs are computed. For a fixed $\theta = 3.00$ the MSEs are observed in Table 1. Its worth to mention that the inversion method is used to generate samples from the EW distribution. So the sample is generated by solving following equation

$$(1-e^{-\frac{x^\beta}{\theta^\beta}})^{\alpha} - u = 0,$$

where $u \sim U(0,1)$ with "uniroot" function in R statistical program. The smallest MSE is written in blod script.

From Table 1, we can see the MSEs for $\alpha = 1$ and $\beta = 1$(denote the proposed model is wellspecified) the MLE is better than AMLE and AMLJE and when α and β are far from one, the propsed model is misspecified therfore the result show that the smallest value of MSE for AMLE and AMLJE , so thay are better than the MLE. Also, AMLJE is better than AMLE in case of heavy censorship. On the other hand, if the censoring probability is large, AMLJE is best.

In this paper we discussed that the proposed model must be checked carefully in analysis of censored data. the result of this study can be shown that if the value of MLE and AMLE are significantly different for large n, the possibility of misspecification is strong.

Table 1: Simulation results for MSEs of the MLE, the AMLE and the AMLJE , $\theta = 3.00$

	(α,β)	(1,1/3)	(1,1)	(3/2,1)	(5/4,3/2)
	P(d=0)	0.42	0.31	0.38	0.26
n=50	$\hat{\theta}_n$	1.925	**1.409**	2.897	5.712
	$\hat{\theta}_n^*$	**1.422**	1.898	**0.613**	1.176
	$\hat{\theta}_n^{*JK}$	1.663	1.896	0.642	**1.022**
n=100	$\hat{\theta}_n$	1.925	**0.409**	2.6603	4.699
	$\hat{\theta}_n^*$	1.421	1.598	**0.513**	**1.124**
	$\hat{\theta}_n^{*JK}$	**1.401**	1.896	0.628	1.330
n=150	$\hat{\theta}_n$	1.273	**0.258**	5.002	5.021
	$\hat{\theta}_n^*$	1.241	1.8985	**1.301**	**0.421**
	$\hat{\theta}_n^{*JK}$	**1.236**	1.896	1.802	0.047
n=200	$\hat{\theta}_n$	0.889	**0.210**	4.663	4.052
	$\hat{\theta}_n^*$	0.8741	1.496	**1.020**	**1.211**
	$\hat{\theta}_n^{*JK}$	**0.798**	1.520	1.108	1.211

References

[1] Ashour, S.K. and Afify, W.M. (2007), *Statistical analysis of exponemtiated Weibull family under type I progressive interval with random removals, Applied Sciences Research*, **3**, 1851-1863.

[2] Ashour, S.K. and Afify, W.M. (2008), *Estimation of the parameters of exponentiated Weibull family with type II progressive interval censoring with random removals, Applied Sciences Research*, **4**, 1428-1442..

[3] Jaheen, Z.F. and Al Harbi, M.M. (2011), *Baysian estimation based on dual generalized order statistics from the exponentiated Weibull model, Jurnal of Statistical Theory and Applications*, **10**, 591-602.

[4] Mudholkar, G.S. and Hutson, A.D. (1996), *The exponentiated Weibull family; Some properties and a flood data application, Communications Statistics-Theory and Methods*, **25**, 3059-3083.

[5] Mudholkar, G.S. and Srivastava, D.K. (1993), *Exponentiated Weibull family for analyzing bathtub failure-real data, Biometric*, **42**, 177-182.

[6] Mudholkar, G.S., Srivastava, D.K. and Freimes, M (1995), *The exponentiated Weibull family: A reanalysis of the bus-motor-failure data, Technometrics*, **37**, 436-445.

[7] Mauro, D. (1985), *A combinatoric approach to the Kaplan-Meier estimation, Ann. Statist.*, **13**, 142-149.

[8] Nasser, M.M. and Eissa, F.H. (2003), *On the exponentiated Weibull distribution, Communications Statistics-Theory and Methods*, **32**, 1317-1333.

[9] Nasser, M.M. and Eissa, F.H. (2004), *Bayesian estimation for the exponentiated Weibull model. Communications Statistics-Theory and Methods*, **33**, 2343-2362.

[10] Oakes, D. (1986), *An approximate likelihood procedure for censored data, Journal of Statistical Planning and Inference*, **134**, 350-372.

[11] Singh, U.S., Gupta, P.K. and Upadhaya, S.K. (2005a), *Estimation of parameters for exponentiated Weibull distribution under type II censoring scheme, Computational Statistics and DataAnalysis* , **48**, 509-523.

[12] Singh, U.S. Gupta, P.K. and Upadhaya, S.K. (2005b), *Estimation of three parameters exponentiated Weibull distribution under type II censoring, Journal of Statistical Planning and Inference*, **134**, 350-372.

[13] Stute, W. and Wang, J.L. (1993), *The strong law under random censorship, The Annals of Statistics*, **21**, 1591-1607.

[14] Stute, W. and Wang, J.L. (1994), *The jackniffe estimate of a Kaplan-Meier integral, Biometrika*, **81**, 602-606.

[15] Suzukawa, A., Imai, H. and Sato, Y. (2001), *Kullback-Leibler information consistent estimation for censored data, The Annals of Statistics*, **53**, 262-276.

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