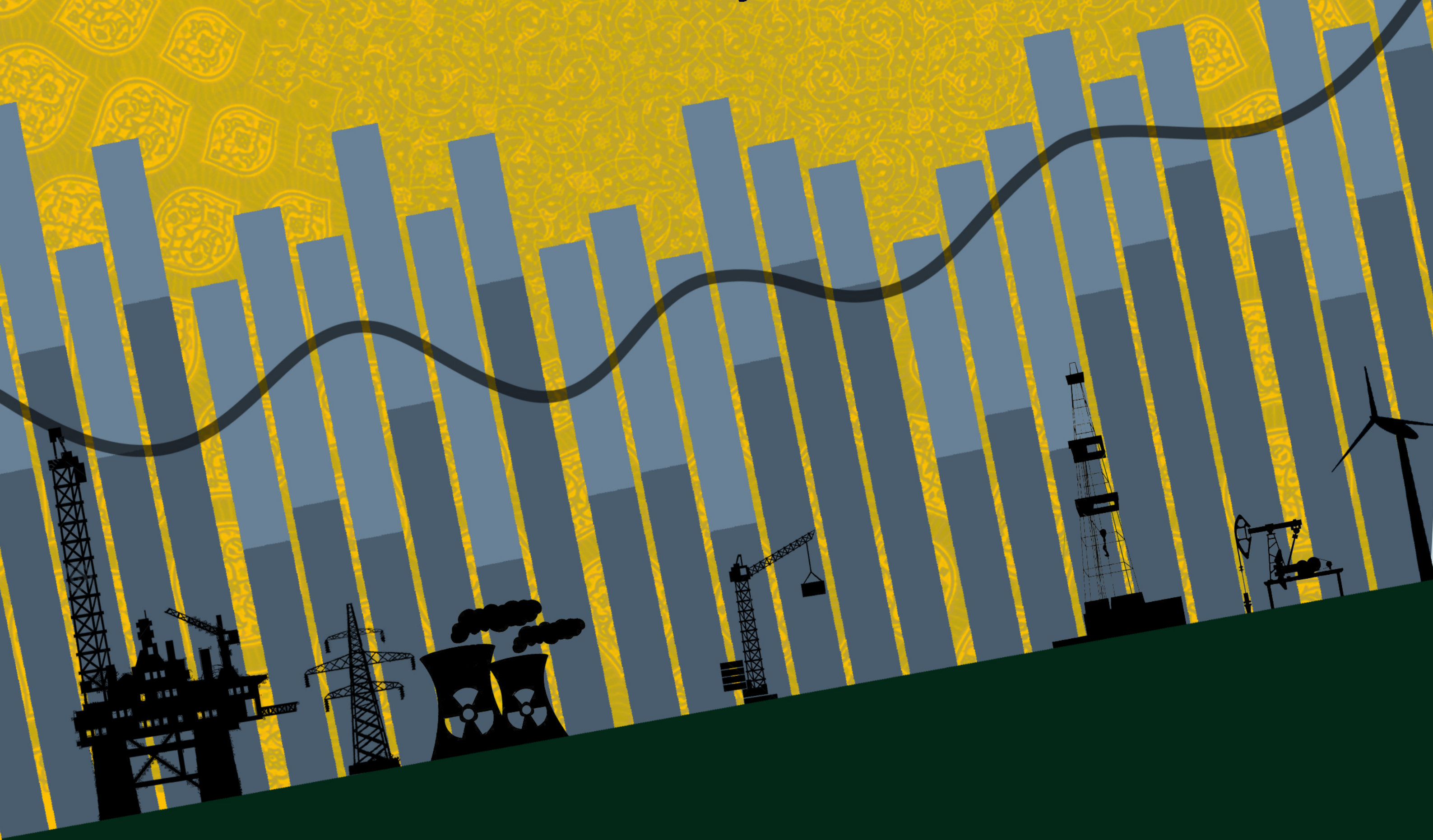
First Seminar on

Reliability Theory and its Appilcations

27-28, May 2015



Department of Statistics, University of Isfahan, Isfahan

http://osdce.um.ac.ir

In the Name of Allah

Proceeding of

The First Seminar on

Reliability Theory and its Applications

Department of Statistics
University of Isfahan, Isfahan, Iran

and

Ordered and Spatial Data Center of Excellence
Ferdowsi University of Mashhad, Iran

27-28 May, 2015

Preface

On behalf of the organizing and scientific committees, we would like to extend a very warm welcome to all the participants of the first Seminar on "**Reliability Theory and its Applications**".

Hope that this seminar provides an environment of useful discussions and would also exchange scientific ideas through opinions. We wish to express our gratitude to the numerous individuals and organizations that have contributed to the success of this seminar, in which more than 110 colleagues, researchers, and postgraduate students have participated.

Finally, we would like to extend our sincere gratitude to the students of the Department of Statistics at Isfahan for their kind cooperation. We wish them all the best.

Topics of the Seminar:

- Statistical inference based on reliability data
- Accelerated life testing
- System maintenance and repair policies
- Concepts of aging
- Stochastic ordering in reliability
- Stress- strength model
- Reliability of networks
- Survival analysis

Scientific Committee:

- Ahmadi, J. Ferdowsi University of Mashhad
- Arghami, N.R. Ferdowsi University of Mashhad
- Asadi, M. University of Isfahan (**Chairman**)
- Haghighi, F. University of Tehran
- Hamedani, A. Isfahan University of Technology
- Khaledi, B. Razi University
- Khanjari, S. M. University of Birjand
- Khodadadi, A. University of Shahid Beheshti

Organizing Committee:

- Ahmadi, J. Ferdowsi University of Mashhad
- Asadi, M. University of Isfahan
- Iranpana, N. University of Isfahan
- Jabbari Nooghabi, H. Ferdowsi University of Mashhad
- Tavangar, M. University of Isfahan
- Zamanzadeh, E. University of Isfahan

Contents

Estimating the Performance of Series System's Production Process

Ahmadi Nadi, A. [1] and Sadeghpour Gildeh, B. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

In former Lifetime performance index C_L studies, it is usually assume that quality characteristic is the lifetime of an electronic component, engine, camera or in special case lifetime of business. In this paper we suppose that the quality characteristic is the lifetime of a series system and under the assumption of exponential distribution for component lifetime, we provide a maximum likelihood estimator of C_L and then this estimate used to develop testing procedure of C_L. Finally, we give an example to illustrate the use of the testing procedure.

Keywords: Lifetime performance index , Series system, Capability analysis.

1 Introduction

Process capability indice (PCI) is an effective means for measure the ratio of the spread between the process specifications to the spread of the natural variation. Montgomery [1] proposed PCIs such as, C_p, C_{pk}, C_{pm} and C_{pmk} that measure the target-the-better type quality characteristics. Beside above PCIs, they also proposed the indices C_{PL} or C_L (lifetime performance index) for measure the larger-the-better type quality characteristics, where L is the lower specification limit. Clearly, a longer lifetime implies a better product quality, so the lifetime is a larger-the-better type quality characteristic. Recently, C_L have been an interesting subject for many researchers in capability analysis field, for example, Tong et al. [2], Hong et al. [3] and Ahmadi et al. [4] worked on statistical inference of C_L. In former C_L studies, it is usually assume that quality characteristic is the lifetime of an electronic component, engine, camera or in special case lifetime of business. For

[1] adelahmadinadi@um.ac.ir
[2] sadeghpour@um.ac.ir

example, Tong et al. [2] assume that the quality characteristic is the lifetime of electronic components or Hong et al. [3] suppose that the quality characteristic is the lifetime of businesses.
A series system is a system that work if and only if all of it components work. Let $X_1, X_2, ..., X_n$ be the lifetime of the system components. Since the failure time of the series system depends on the failure time of the weakest component, so the failure time of a series system can be modeled by $X_{1:n}$ (first order statistic).
In this paper we want to do a statistical inference about the performance of series system's production process. In Section 2, we introduce some properties of C_L for lifetime of series system. In Section 3, we discusses about the conforming rate. Sections 4 and 5 presents the ML estimator of C_L and a new hypothesis testing procedure for C_L based on lower confidence bound, respectively. Finally, in Section 6 we present a numerical example to illustrate proposed testing procedure.

2 The lifetime performance index

Lifetime performance indice C_L is defined as follows:

$$C_L = \frac{\mu - L}{\sigma}, \tag{1}$$

which μ denotes the process mean, σ represents the process standard deviation, and L is the known lower specication limit. Suppose that m series systems are placed independently in test at time zero. Also suppose that the components of each system are independent and identically distributed and comes from the one-parameter exponential distribution with below probability density function (p.d.f.):

$$f_{(X)}(x, \lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0, \tag{2}$$

where λ is the scale parameter. Let X_{ij} $\forall j = 1, 2, ..., n$ and $\forall i = 1, 2, ..., m$ be the lifetime of the j-th component of the i-th system and $T_i = X^i_{1:n}$ $\forall i = 1, 2, ..., m$ be the lifetime of the i-th series system, respectively. By a simple computation, it can be seen that the lifetime of systems (T) follow one-parameter exponential distribution with p.d.f (2) with scale parameter $n\lambda$. By substituting the mean and standard deviation of T in (1), the lifetime performance index can be obtained as follows:

$$C_L = \frac{\frac{1}{n\lambda} - L}{\frac{1}{n\lambda}} = 1 - nL\lambda, \quad -\infty < C_L < 1, \tag{3}$$

where $\frac{1}{n\lambda} = E(T) = \sigma(T)$. Obviously, when the mean lifetime of system exceed L (i.e $\frac{1}{n\lambda} > L$), then the lifetime performance index $C_L > 0$.

3 The conforming rate

If the lifetime of a system, T, exceed the lower specification limit (i.e $T > L$) then the system defined as a conforming product. The ratio of conforming products is known as the conforming rate P_r which is defined as follows:

$$P_r = P(T > L) = e^{-nL\lambda} = e^{C_L - 1}.$$

Obviously, a strictly relationship exist between conforming rate P_r and lifetime performance index C_L.

4 MLE of lifetime performance index

The likelihood function corresponding observed sample $(t_1, t_2, ..., t_m)$ is given as follows:

$$L(\lambda) = \prod_{i=1}^{m} n\lambda e^{-n\lambda t_i} = (n\lambda)^m e^{-n\lambda \sum_{i=1}^{m} t_i}. \tag{4}$$

By setting the first partial derivatives of the natural logarithm of the likelihood function (Eq (4)) equal to zero with respect to λ, the MLE of λ obtained as $\widehat{\lambda} = \frac{m}{nW}$, where $W = \sum_{i=1}^{m} T_i$. From (4), it can be seen that $W = \sum_{i=1}^{m} T_i$ is a complete and sufficient statistic for λ and also $W \sim Gamma(m, n\lambda)$ therefore, $2n\lambda W \sim \chi^2_{(2m)}$. According to the invariance property of the MLE, the MLE of C_L can be written as:

$$\widehat{C_L} = 1 - \frac{mL}{W} \tag{5}$$

By taking expectation from $\widehat{C_L}$, it can be seen that when $m \longrightarrow \infty$ the MLE of C_L is a unbiased estimator for C_L.

5 Testing procedure for the lifetime performance index

At first suppose that the lifetime performance index target value is shown by c^*. Given the specified significance level α and the pivotal quantity $2n\lambda W$, which is distributed as $\chi^2_{(2m)}$, the level $100(1-\alpha)\%$ one-sided confidence interval for C_L can be derived as follows:

$$\begin{aligned} P\Big(2n\lambda W < CHIIN(1-\alpha, 2m))\Big) &= 1-\alpha, \\ \Rightarrow P\Big(C_L > 1 - \frac{(1-\widetilde{C_L})CHIIN(1-\alpha, 2m)}{2m}\Big) &= 1-\alpha. \end{aligned} \tag{6}$$

where CHIIN$(1-\alpha, 2m)$ function represents the lower $100(1-\alpha)$ percentile of $\chi^2_{(2m)}$. From (6), the level $100(1-\alpha)\%$ lower confidence bound for C_L can be derived as:

$$\underline{LB} = 1 - \frac{(1-\widetilde{C_L})CHIIN(1-\alpha, 2m)}{2m}, \tag{7}$$

where $\widetilde{C_L}$, α and m denote the MLE of C_L, the specified significance level and the observed number, respectively. So the testing procedure can be constructed with the one-sided confidence interval as follows:

step1: Determine the lower lifetime limit L for products and performance index target value c^*.

step2: Specify a significance level α.

step3: Calculate the value of lower confidence bound $\underline{LB}$ from (7).

step4: The decision rule of statistical test is "If performance index target value $c^* \notin [\underline{LB}, \infty)$, it is concluded that the lifetime performance index of products meets the required level".

6 Numerical example

Example 1. *Simulated data set*

A simulated data set of the failure times of m= 20 series systems with n=5 components from exponential distribution with p.d.f (2) and parameter λ= 0.2 ($n\lambda = 1$) are: 1.69, 0.98, 0.54, 0.16, 1.23, 3.92, 0.39, 5.11, 0.01, 0.08, 2.42, 0.42, 0.80, 1.18, 0.56, 0.18, 0.29, 0.41, 0.95, 2.29. **step1** the lower lifetime limit L and the lifetime performance target value c^* are assumed to be 0.1 and 0.8, respectively. In **step2** Specify a significance level α=0.05. Calculate the value of $\underline{LB} = 1 - \frac{((\frac{20*0.1}{23.61})*55.76)}{2*20} = 0.88$ in **step3**. In **step4**, because of $0.8 \notin [0.88, \infty)$, so it is concluded that the lifetime performance index of products meets the required level.

References

[1] Montgomery, D. C. (1985), Introduction to Statistical Quality Control, John Wiley and Sons, New York.

[2] Tong, L. I. Chen, K .S. Chen, H. T. (2002), Statistical testing for assessing the performance of lifetime index of electronic components with exponential distribution, *International Journal of Quality and Reliability Management*, **19**, 812-824.

[3] Hong, C. W. Wu, J. W. Cheng, C. H. (2009), Implementing lifetime performance index for the pareto lifetime businesses of the service industries, *Quality and Quantity*, **43**, 291-304.

[4] Ahmadi, M. V. Doostparast, M. Ahmadi, J. (2014), Estimating the lifetime performance index with Weibull distribution based on progressive first-failure censoring scheme, *Journal of Computational and Applied Mathematics*, **239**, 93-102.

A Preventive Maintenance Model for Periodically Inspected Deteriorating Systems

Ahmadi, R. [1]

School of Mathematics, Iran University of Science and Technology

Abstract

Using the repair alert model, this paper proposes a probabilistic model for the maintenance scheduling of periodically inspected systems whose state is described by a mean residual lifetime (MRL) process. Attention is restricted to the periodic inspection and perfect repair, but the model provides a framework for further developments.

Keywords: Maintenance, Inspection, Repair alert model, Scheduling function, Mean residual lifetime process.

1 Introduction

Benefiting from the repair alert model [3] and the joint modeling of the degradation phenomenon and the maintenance effect, we formulate a maintenance scheduling model with a general form for periodic inspection policy. The approach presented is appropriate for maintaining systems subject to failure due to aging and damage caused by operating environment factors. During operating, inspections at periodic times reveals the true state of the system and corrective maintenance (CM) and preventive maintenance (PM) actions are carried out in response to the observed system state. Since higher level of repair and maintenance incurs more costs, but on the other hand it would more likely prevent the system failure (CM avoided by a preceding PM), the maintenance procedure is faced with the dilemma of whether performing PM actions or experiencing critical failures (CM). In this case an appropriate PM policy is required which balances the amount of maintenance and resulting maintenance costs. Thus, the situation is a case of competing risk between CM and PM. The pleasant feature of the model making use of the so-called repair alert model and some other characteristics is to devise a scheduling function and a mean residual lifetime process.

[1] re_ahmadi_stat@yahoo.com

2 Modelling degradation

We consider a deteriorating system subject to failure due to aging and operating environment factors. It is assumed that the system state is described by the mean residual lifetime process

$$\mu(W_n) = \exp\left(\frac{V_n^2\psi(X_{V_n})}{2}\right)\sqrt{\frac{2\pi}{\psi(X_{V_n})}}\left[1-\Phi\left(V_n\sqrt{\psi(X_{V_n})}\right)\right], \tag{1}$$

incorporating a controllable bivariate process $W_n = (X_{V_n}, V_n)$ where V_n denotes the Kijima Type-I Virtual age (VA) model [4], X_{V_n} is a damage process reflecting the effect of operating environment factors, $\psi(\cdot)$ refers to the multiplicative factor of the proportional intensity model and $\Phi(\cdot)$ implies the cumulative distribution function of the standard normal distribution. One can note that the interrelation between the damage process and the maintenance effect reflected by a virtual age process is accommodated through the change of the time origin $X_t \mapsto X_{V_t}$. Based on some assumptions, the argument presented by Ahmadi [2] follows (1).

3 Features of the model

3.1 Model assumptions

a) PM actions are carried out in response to the observed MRL process (1); **b)** Inspections are perfect and instantaneous and reveal the true state of the system; **c)** The system is inspected at fixed intervals τ at cost C; **d)** The impact of PM is minor (minimal repair), or major (perfect repair); **e)** The preventive replacement rule is of threshold type. That means, the preventive replacement time T_p is defined as the first time the mean residual lifetime process (1) reaches or falls below a threshold k; **f)** Replacement after failure (PM) is instantaneous and incurs a cost $C_f(C_p)$ $(C_f > C_p)$; **g)** The pair $(T_f; T_p)$ of life variables satisfies the requirements of the repair alert model. The last assumption implies that **(i)** T_p is a random signs censoring of the catastrophic failure time T_f. That means, the event $\{T_p < T_f\}$ is stochastically independent of T_f; **(ii)** there exists an increasing function G with G(0) = 0 such that for all $t > 0$,

$$\mathbb{P}(T_p \le u | T_p < T_f, T_f = v) = \frac{G(u)}{G(v)}, \quad 0 < u \le v,$$

where G is called the cumulative repair alert function. For details see Lindqvist et al. [3].

3.2 The framework

At inspection time $n\tau$ and before any maintenance action, a decision is made based on the mean residual lifetime process $\mu(W_n^-)$ $n \ge 1$, and the replacement threshold k where $W_n^- = (X_{V_{n-1}+\tau}, V_{n-1}+\tau)$. There are two states at $n\tau$:

(i) Operating state: A minimal repair is performed and it is left to continue until the next planned inspection in τ units of time at $(n+1)\tau$ with inspection cost C, if the mean residual lifetime process $\mu(W_n^-)$ is observed in the non-critical region (working perfectly). That means, $\mu(W_n^-) > k, \quad n \ge 1$. At inspection time $(n+1)\tau$ and before any maintenance action, a decision is made based on the updated mean residual lifetime

$\mu\big(W_n\big)$ where $W_n = (X_{V_n+\tau}, V_n + \tau)$;
(ii) Failure state: there are two possibilities: **a**) the system either experiences a critical failure at time T_f, or; **b**) it undergoes a perfect repair (replacement) at time $T_p = n\tau$, if the mean residual lifetime process is found in the critical region. In other words, $\mu\big(W_n^-\big) \leq k, \quad n \geq 1$. At time $T = \min(T_f, T_p)$ the system is instantaneously replaced with the random cost of $\mathbb{C}$. Theses failure times form a renewal process.

3.3 Optimizing the model: average cost criterion

3.3.1 Expected cycle length

Let the starting state of the system at initial time be $w_0 = (x, v)$ and L_τ^x denote the length of a cycle with the expected cycle length l_τ^x. A renewal reward argument yields the length of a cycle:

$$L_\tau^x = \underbrace{\big(\tau + L_\tau^{X_\tau}\big) I\left(T > \tau^-\right)}_{\textbf{Operating state}} + \underbrace{TI(T < \tau)}_{\textbf{Failure state}} \tag{2}$$

Taking expectations:

$$l_\tau^x = A(x) + \int_{\bar{\mathcal{C}}} l_\tau^y K_\tau^x(y) dy, \tag{3}$$

where $A(x) = \int_0^\tau R^{w_0}(t)dt$ with the survival function $R^{w_0}(t)$ given the starting state $w_0 = (x, v)$, $K_\tau^x(y) = \mathbb{P}\big(X_{\tau^-} = y | X_0 = x\big)$ denotes the kernel function and $\bar{\mathcal{C}} = \{y : \mu(y, \tau) > k\}$ is the non-critical region with the mean residual lifetime $\mu(y, \tau)$ given in (1).

3.3.2 Expected cost per cycle

The expected cost of a cycle, c_τ^x, is obtained similarly. The cost of a cycle C_τ^x given the starting state $w_0 = (x, v)$ is

$$C_\tau^x = \underbrace{\big(C + C_\tau^{X_\tau}\big) I\left(T > \tau^-\right)}_{\textbf{Operating state}} + \underbrace{\mathbb{C}I(T < \tau)}_{\textbf{Failure state}} \tag{4}$$

Taking expectations:

$$c_\tau^x = B(x) + \int_{\bar{\mathcal{C}}} c_\tau^y K_\tau^x(y) dy, \tag{5}$$

where $B(x) = C + C_1 F^{w_0}(\tau) + qC_2 G(\tau) \int_\tau^\infty \frac{dF^{w_0}(t)}{G(t)}$ with $q = \mathbb{P}(T_p < T_f)$, which is the probability that the CM is avoided by a preceding PM, the catastrophic failure time distribution $F^{w_0}(t)$ given the starting state w_0, and

$$C_1 = \big(C_f - C\big) - q\big(C_f - C_p\big), \quad C_2 = C_p - C.$$

3.3.3 Expected cost per unit time

The structure described above allows a renewal reward argument to be used. A standard renewal reward argument gives the cost per unit time $\mathcal{C}_\tau^x = c_\tau^x / l_\tau^x$ with expressions for l_τ^x and c_τ^x given in (3) and (5).

3.4 Obtaining solutions

As noted the equations (3) and (5) refer to Fredhom equations

$$l_\tau^x = A(x) + \int_{\bar{\mathcal{C}}} K_\tau^x(y) l_\tau^y dy \qquad v_\tau^x = B(x) + \int_{\bar{\mathcal{C}}} K_\tau^x(y) v_\tau^y dy.$$

They are solved numerically using the Nystrom routine with a Gauss-Legendre rule [1]. The optimal period of inspection τ^* and the optimal replacement threshold k^* can be obtained as

$$(\tau^*, k^*) = \underset{(\tau,k)\in\mathbb{R}^+\times\mathbb{R}^+}{\arg\min}\ \mathcal{C}_\tau^x. \tag{6}$$

References

[1] Ahamdi, R. (2014), A new approach to modelling condition based maintenance for stochastically deteriorating systems, *International Journal of Reliability, Quality and Safety Engineering*, **21**(5), 25 pages.

[2] Ahmadi, R. (2014), Optimal maintenance scheduling for a complex manufacturing system subject to deterioration, *Annals of Operations Research*, **217**, 1-29.

[3] Lindqvist, B., Langseth, H. and Stove, B. (2006), Modelling of dependence between critical failure and preventive maintenance: The repair alert model, *Advances in Applied Probability*, **136**(5), 1701-1717.

[4] Kijima, M. (1989), Some results for repairable systems with general repair, *J. Appl. Prob*, **26**, 89-102.

Estimation of Stress-Strength Reliability for Stable Distributions

Alizadeh Noughabi, R. [1] and Mohammadpour, A. [2]

Department of Statistics, Amirkabir University of Technology

Abstract

This paper deal with the estimation of Stress-Strength reliability parameter, $R = P(X < Y)$, when stress and strength are two independent stable distributions. The maximum likelihood estimator of stable distribution studied. Furthermore, we investigate the $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$ for Lévy distribution as a member of stable family. Using a Monte Carlo simulation, the MSE and Bayes risk estimators are computed and compared.

Keywords: Stable distributions, Stress-Strength, Maximum likelihood estimator, Lindley approximation.

1 Introduction

Stable distributions are a class of probability distributions that specified by four parameters, an index of stability $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$ and finally a location parameter $\delta \in \Re$. A stable distribution determined by its characteristic function, that is $X \sim S(\alpha, \beta, \gamma, \delta)$ if and only if $\varphi_X(t)$ as follows

$$\varphi_X(t) = \begin{cases} \exp\left\{-\gamma^\alpha |t|^\alpha \left[1 - i\beta \left(\tan \frac{\pi\alpha}{2}\right)(sign\, t)\right] + i\delta t\right\} & \alpha \neq 1, \\ \exp\left\{-\gamma |t| \left[1 + i\beta \frac{2}{\pi} (sign\, t) \log |t|\right] + i\delta t\right\} & \alpha = 1. \end{cases}$$

where $sign\, t$ is sign function, see Nolan [6].

Many authors discussed inference on R in reliability context. But there has not been much work on the estimation of R for Lévy distribution, the only paper, we are study is Ali and Woo [1]. In 2010 Eryilmaz [3] studied stress-strength reliability for a general coherent system. the probability $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$ discussed by Pakdaman and Ahmadi [5].

---
[1] al@aut.ac.ir
[2] adel@aut.ac.ir

2 General case

To estimate Stress-Strength reliability for stable distributions we consider two cases:

I : X and Y have same and know skewness parameter, i.e. $X \sim S(\alpha_1, \beta)$ and $Y \sim S(\alpha_2, \beta)$ be independent random variables.

II : Skewness parameter be unknown, that is $X \sim S(\alpha_1, \beta_1)$ and $Y \sim S(\alpha_2, \beta_2)$ be independent random variables.

2.1 MLE

In case I we have

$$R = P(X < Y) = \int_{-\infty}^{+\infty} [1 - S_Y(z|\alpha_2, \beta)]\, s_X(z|\alpha_1, \beta)\, dz. \tag{1}$$

The S_Y and s_X are used to show the distribution function and density function of stable distributions, respectively. By computing the ML estimators of α_1 and α_2 we can calculate the (1) by a numerical method. Case II is same as I but we must obtain the ML estimators of α_1, β_1, α_2 and β_2. Simulation results are shown in Table 1.

3 Lévy distribution

Let X and Y be two independent random variables. In other word, $X \sim Lev(\alpha)$ and $Y \sim Lev(\beta)$ respectively. Ali and Woo [1] study $P(X < Y)$ for the Lévy distribution. In this paper, we investigate the case of $R_{r,k} = P(X_{r:n_1} < Y_{k:n_2})$.

Thus, the $R_{r,k}$ follows

$$R_{r,k} = P(X_{r:n_1} < Y_{k:n_2}) = \int_{-\infty}^{+\infty} F_{X_{r:n_1}}(z)\, f_{Y_{k:n_2}}(z)\, dz. \tag{2}$$

By formulas of pdf and cdf of the ith order statistic (See David and Nagaraja [2]) and writing the binomial expansion for $F_X^j(z)$ and $F_Y^{k-1}(z)$ we simplify $R_{r,k}$ as follows

$$\begin{aligned} R_{r,k} &= P(X_{r:n_1} < Y_{k:n_2}) \\ &= k \begin{pmatrix} k \\ n_2 \end{pmatrix} \sum_{j=r}^{n_1} \sum_{t=0}^{j} \sum_{l=0}^{k-1} \begin{pmatrix} n_1 \\ j \end{pmatrix} \begin{pmatrix} j \\ t \end{pmatrix} \begin{pmatrix} k-1 \\ l \end{pmatrix} (-1)^{t+l} \\ &* \textstyle\int_{-\infty}^{+\infty} [1 - F_X(z)]^{n_1 - t} [1 - F_Y(z)]^{n_2 - l - 1} f_Y(z)\, dz. \end{aligned} \tag{3}$$

3.1 MLE

Suppose $X_1, X_2, \cdots, X_{n_1}$ is a sample from $Lev(\alpha)$ and $Y_1, Y_2 \cdots, Y_{n_2}$ is a sample from $Lev(\beta)$, Calculate the likelihood function and by taking logarithm and derivative to α and β we obtain the maximum likelihood of parameters as follows

$$\hat{\alpha} = \frac{n_1}{\sum_{i=1}^{n_1} \frac{1}{x_i}}, \hat{\beta} = \frac{n_2}{\sum_{i=1}^{n_2} \frac{1}{y_i}}.$$

Since the maximum likelihood estimators have invariance property, we can calculate $R_{r,k}$ by numerical methods.

3.2 Bayes estimator

Suppose the parameters, α and β, have the gamma priors, with following parameters

$$\alpha \sim GAM\left(k, \theta\right), \quad and \;\; \beta \sim GAM\left(\mu, \sigma\right).$$

Posterior pdfs of α and β are

$$\alpha|\underline{x} \sim GAM\left(\frac{n_1}{2}+k,\, \frac{1}{2}\sum_{i=1}^{n_1}\frac{1}{x_i}+\theta\right), \, and \;\beta|\underline{y} \sim GAM\left(\frac{n_2}{2}+\mu,\, \frac{1}{2}\sum_{i=1}^{n_1}\frac{1}{y_i}+\sigma\right).$$

3.3 Lindley's approximation

We consider Lindley's approximation (See Lindley [4]) from expanding about the posterior mode. Lindley's approximation leads to

$$\hat{U}_{Lindley} = \left(U(\theta) + \frac{1}{2}[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}]\right)|_{(\theta_1,\theta_2)=(\tilde{\theta_1},\tilde{\theta_2})}, \qquad (4)$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2} U_{ij}\tau_{ij}$ and $Q_{\eta\xi} = \frac{\partial^{\eta+\xi}}{\partial^{\eta}\theta_1\partial^{\xi}\theta_2}$ that $\eta, \xi = 0, 1, 2, 3$. Furthermore, $i, j = 1, 2$, $U_i = \frac{\partial U}{\partial\theta_i}$ and for $i \neq j$, $U_{ij} = \frac{\partial^2 U}{\partial\theta_i\partial\theta_j}$, $B_{ij} = (U_i\tau_{ii} + U_j\tau_{ij})\tau_{ii}$, $C_{ij} = 3U_i\tau_{ii}\tau_{ij} + U_j(\tau_{ii}\tau_{ij} + 2\tau_{ij}^2)$.
Is the τ_{ij} (i,j)th element in the inverse of matrix $Q^* = (-Q^*_{ij}), i, j = 1, 2$ so that $Q^*_{ij} = \frac{\partial^2 Q}{\partial\theta_i\partial\theta_j}$.

It is not difficult to obtain the above terms.

4 Simulation study

Table 1 shows the bias and MSE of R in general case of a stable distribution. Furthermore,

Table 1: Bias and MSE of R for stable law with unknown β

	Symmetric				Positive				Asymmetric			
$n_1 = n_2$	α	β	Bias	MSE	α	β	Bias	MSE	α	β	Bias	MSE
5	0.3	0	-0.0396	0.0973	0.2	1	0.0738	0.0288	1.2	1	0.0793	0.1223
	0.8	0	-0.0094	0.1427	0.4	1	-0.0120	0.0472	1.4	1	0.0028	0.0496
	1.2	0	-0.0801	0.0529	0.7	1	-0.0226	0.0060	1.7	1	-0.0119	0.0695
	2	0	-0.0025	0.0151	0.9	1	-0.0041	0.0025	2	1	0.0174	0.0171
10	0.3	0	-0.0335	0.827	0.2	1	-0.0153	0.0203	1.2	1	0.0019	0.1091
	0.8	0	0.0999	0.0951	0.4	1	0.0451	0.0020	1.4	1	0.0013	0.0765
	1.2	0	0.1120	0.0970	0.7	1	0.0117	0.0142	1.7	1	-0.0036	0.0370
	2	0	-0.0108	0.0101	0.9	1	0.1056	0.0401	2	1	0.0391	0.0167

we simulate the Estimated Risks (ER) of Bayes and approximation Bayes estimators with respect to prior parameters.

Conclusion

We have used the symmetric, positive asymmetric and asymmetric stable laws for simulating bias and MSE. We have observed that the Bayes estimator has the smallest estimated risk. The estimated risk decreases as the priors become more "informative".

Table 2: Bias and ER for Bayes Estimators of $R_{n_1,1}$

Informative	k	θ	μ	σ	$n_1 = n_2$	ER(R^B_{Lin})	Bias(R^B_{Lin})	ER(R^B)	Bias(R^B)
Least informative	1	2	1	2	5	0.320	0.562	0.358	0.583
	1	2	1	2	10	0.264	-0.513	0.273	0.508
Informative	5	10	5	10	5	0.273	0.503	0.289	0.519
	5	10	5	10	10	0.203	-0.467	0.229	-0.471
Most informative	10	20	10	20	5	0.241	0.488	0.251	0.497
	10	20	10	20	10	0.191	-0.438	0.197	-0.439

References

[1] Ali, M.M., Woo, J.S. (2005). Inference on reliability $P(X < Y)$ in the Lévy distribution, *Mathematical and Computer Modeling*, **41**, 965–971.

[2] David, H. A. and Nagaraja, H. N. (2003), *Order Statistics*, John Wiley and Sons, New York.

[3] Eryilmaz, S. (2010), On system reliability in stress-strength setup, *Statistics and Probability Letters*, **80**, 834–839.

[4] Lindley, D. V. (1980). Approximation Bayesian methods, *Trabajos de Estadistica*, **21**, 223–237.

[5] Pakdaman, Z., Ahmadi, J. (2013). Stress-strength reliability for $P(X_{r:n_1} < Y_{k:n_2})$ in the exponential case.

[6] Nolan, J. (2003). *Stable distributions: models for heavy-tailed data.* Birkhauser. Statistical Planning and Inference, 140, pp. 480–494.

Prediction of Times to Failure of Censored Units in Progressive Hybrid Censored Samples from Exponential Distribution

Ameli, S. [1] , Rezaie, M. [2] and Ahmadi, J. [3]

[1,2] Department of Statistics, University of Birjand, Birjand
[3] Department of Statistics, Ferdowsi University of Mashhad

Abstract

Survival data often come in a form called "censoring". When exact survival times are known only for a portion of the individuals or units under study censoring occurs. In this paper, we consider a progressive hybrid censoring. The problem of predicting times to failure of units censored in multiple stages of progressively hybrid censored from exponentil distribution is discused. The best unbiased predictor (BUP), best linear unbiased predictor (BLUP) and maximum likelihood predictor are derived.

Keywords: Best linear unbiased predictor, Conditional median predictor, Maximum likelihood predictor, Order statistics, Progressive hybrid censoring.

1 Introduction

Kundu and Joarder (2006) and Childs et al. (2008) proposed respectively type I and type II progressive hybrid censoring procedures by introducing a stopping time T^* to a progressive type-II censored experiment. The termination times are defined by a given (fixed) threshold time T as follow:

(i) $T_1^* = min\{X_{m:m:n}, T\}$, this procedure is called type-I progressive hybrid censoring scheme; the life-testing experiment is stopped when either m failures have been observed or the threshold time T has been exceeded. The number of observations may be zero (for the case when $X_{1:m:n} > T$).(Kundu and Joarder (2006))

[1] s.ameli@birjand.ac.ir
[2] mjrezaei@birjand.ac.ir
[3] ahmadi-j@um.ac.ir

(ii) $T_1^* = max\{X_{m:m:n}, T\}$, this procedure is called type-II progressive hybrid censoring scheme. The number of observations is between m and $R_m + m$, where $X_{i:m:n}$ is ith progressively type II censored order statistic.(Childs et al. (2008))

Let $X_1, X_2, \cdots, X_n$, denote the ordered failure times of n-independent units placed on a life testing experiment simultaneously. Assume these Xs come from a common exponential distribution with the density function

$$f(x;\theta) = \theta e^{-\theta x} \qquad x > 0, \theta > 0. \quad (1.1)$$

. For simplicity we show k progressively hybrid right censored order statistics by $Y_1, Y_2, \cdots, Y_k$. Our purpose is to discuss the prediction of life lengths $Y_{j:r_i}(j = 1, 2, \cdots, k)$ that denotes the jth-order statistic out of r_i removed units at stage $i = 1, 2, \cdots, k, T$. Prediction of times to failure of $Y_{j:r_i}$ at progressive censored data has been discussed earlier by Balakrishnan and Basak (2006 and 2009) and Asgharzadeh and Valliollahi (2010). Since F is continuous, the conditional distribution of $Y_{j:r_i}$ given $\underline{Y} = (Y_1, Y_2, \cdots, Y_k)$ is just the distribution of $Y_{j:r_i}$ given Y_i due to the well-known Markovian property of progressively hybrid right censored order statistics. Hence the best unbiased predictor (BUP) of $Y_{j:r_i}(j = 1, 2, \cdots, r_i; i = 1, 2, \cdots, k)$, $E_\theta(Y_{j:r_i}|\underline{Y})$ is nothing but $E_\theta(Y_{j:r_i}|Y_i)$ and hence it depends only on Y_i. In progressive hybrid censoring type-I

(i) if $Y_m \leq T$ then censoring method is similar to the ordinary progressive censoring with censoring scheme $(r_1, r_2, \cdots, r_m)$,

(ii) if $Y_k \leq T < Y_{k+1}$ then r_i units are randomly withdrawn at i th stage ; $i = 1, 2, \cdots, k$ and r_T units at time T. Censoring scheme change to $(r_1, r_2, \cdots, r_k, r_T)$; $r_T = n - k - \Sigma_{i=1}^{k} r_i$

In progressive hybrid censoring type-II

(iii) If $Y_k < T < Y_{k+1}$, $k \geq m$ then r_i units are randomly withdrawn at i th stage ; $i = 1, 2, \cdots, m-1$ and r_T units which $r_T = n - k - \Sigma_{i=1}^{m-1} r_i$ at time T.

(iv) If $Y_m > T$ and $Y_k \leq T < Y_{k+1}$, censoring method is similar to the ordinary progressive censoring which censoring scheme changes to $(r_1, r_2, \cdots, r_k, 0^{m-k-1}, r_m^*)$; $r_m^* = n - m - \Sigma_{i=1}^{k} r_i$.

2 Best linear unbiased predictor

For exponential distribution it can be shown that $Y_{j:r_i} - Y_i$ for $i = 1, 2, \cdots, k$ is distributed as $\dfrac{Z_{j:r_i}}{\theta}$ where $Z_{j:r_i}$ denotes the jth order statistic out of r_i units from a standard exponential distribution. Therefore the distribution of $Y_{j:r_i} - Y_i$ is independent of Y_i.
Let $\pi_1(j, r_i)$, $\pi_2(j, r_i)$, $\pi_0(j, r_i)$ respectively denote the mean, variance and mode of $Z_{j:r_i}$ which are given by:
$\pi_1(j, r_i) = \sum_{l=r_i-j+1}^{n} \dfrac{1}{l}$, $\pi_2(r_i, r_i) = \sum_{l=r_i-j+1}^{n} \dfrac{1}{l^2}$
and $\pi_0(j, r_i) = \log(\dfrac{r_i}{r_i - j + 1})$. (2.1)

In cases (i), (iv) and (ii), (iii) for $i = 1, 2, \cdots, k$ predicting problem is the same as the one which studied by Balakrishnan and Basak (2006). According to it
If θ is known: The best unbiased predictor (BUP) $Y_{j:r_i}^*$ of $Y_{j:r_i}$ is simply given by

$E[Y_{j:r_i}|Y_i]$. Since $(Y_{j:r_i}-Y_i)$ and Y_i are independent and $Y_{j:r_i}-Y_i \stackrel{d}{=} \frac{1}{\theta}Z_{j:r_i}$ the BUP $Y^*_{j:r_i}$ is given by:

$$Y^*_{j:r_i}=E[Y_{j:r_i}|Y_i]=Y_i+\frac{1}{\theta}\pi_1(j,r_i). \quad (2.2)$$

According to (2.2) BUP is a linear combination of Y_i so the best linear unbiased predictor (BLUP) is:

$$Y^B_{j:r_i}=Y_i+\frac{1}{\theta}\pi_1(j,r_i)\ ;i=1,2,\cdots,k. \quad (2.3)$$

If θ is unknown: The BLUP of $Y_{j:r_i}$; is given by $Y^B_{j:r_i}=Y_i+\frac{1}{\theta^*}\pi_1(j,r_i)$ where $\frac{1}{\theta^*}=\frac{1}{m}\sum_{l=1}^{m}(r_l+1)Y_l$ is BLUE of $\frac{1}{\theta}$.

In cases (ii) and (iii) BUP for $Y_{j:r_T}$ is like Asgharzadeh and Valiollahi (2012). Based on Markovian property $Y^*_{j:r_T}=\int_T^\infty yF(y|y>T)dy$. By substituting $f_\theta(y|\underline{y})$ and using the binomial expansion BUP is:

$$Y^*_{j:r_T}=j\begin{pmatrix} r_T \\ j\end{pmatrix}\Sigma_{i=0}^{j-1}\begin{pmatrix} j-1 \\ i\end{pmatrix}(-1)^{j-i-1}\frac{(r_T-j)T+\theta}{(r_T-j)^2} \quad (2.4)$$

When the parameter θ is unknown, we can use MLE of θ.

3 Maximum likelihood predictor

In cases (i) and (iv) predictive likelihood function of $Y_{j:r_i}$ and θ is:

$$L=c\theta^{k+1}e^{-\theta y}[e^{-\theta y_i}-e^{-\theta y}]^{j-1}[e^{-\theta y}]^{r_i-j}\prod_{l=1}^{k}e^{-\theta y_l}\prod_{l=1;l\neq i}^{k}e^{-\theta y_l r_l}\ ;\ y\geq y_i.$$

θ is known: Maximum likelihood predictor(MLP) $Y^{MLP}_{j:r_i}$ of $Y_{j:r_i}$ is:

$$Y^{MLP}_{j:r_i}=Y_i+\frac{1}{\theta}\pi_0(j,r_i) \quad (3.1)$$

In cases (ii) and (iii) predictive likelihood function of $Y_{j:r_i}$ and θ is:

$$L=c\theta^{k+1}e^{-\theta\Sigma_{l=1;l\neq i}^{k}(r_l+1)y_l+(r_i-j+1)y+Tr_T-r_iy_i}[e^{-\theta y_i}-e^{-\theta y}]^{j-1}$$

$$\Longrightarrow \qquad Y^{MLP}_{j:r_i}=Y_i+\frac{1}{\theta}\log\frac{r_i}{r_i-j+1} \quad (3.2)$$

predictive likelihood function of $Y_{j:r_T}$ and θ is:

$$L=c\theta^{k+1}e^{-\theta\Sigma_{l=1}^{k}(r_l+1)y_l+(r_T-j+1)y}[e^{-\theta T}-e^{-\theta y}]^{j-1}$$

$$\Longrightarrow \qquad Y^{MLP}_{j:r_T}=T+\frac{1}{\theta}\log\frac{r_T}{r_T-j+1} \quad (3.3)$$

θ is unknown: in cases (i) and (iv)

$$Y^{MLP}_{j:r_i}=Y_i+\frac{1}{\theta^{**}}\pi_0(j,r_i);\frac{1}{\theta^{**}}=\frac{1}{k+1}\Sigma_{l=1}^{k}(r_l+1)y_l \quad (3.4)$$

In a similar way for (ii), (iii)

$$Y_{j:r_i}^{MLP} = Y_i + \frac{1}{\theta^{**}} \log \frac{r_i}{r_i - j + 1}; \frac{1}{\theta^{**}} = \frac{1}{k+1} \Sigma_{l=1;l\neq i}^{k} (r_l + 1) y_l + T r_T \quad (3.5)$$

$$Y_{j:r_T}^{MLP} = T + \frac{1}{\theta^{**}} log \frac{r_T}{r_T - j + 1}; \frac{1}{\theta^{**}} = \frac{1}{k+1} \Sigma_{l=1;l\neq i}^{k} (r_l + 1) y_l + T r_T \quad (3.6)$$

References

[1] Asgharzadeh. A and Valiollahi. R (2012) Prediction of Times to Failure of Censored Units in Hybrid Censored Samples from Exponential Distribution, J. Statist. Res. Iran, 9, 11–30.

[2] Balakrishnan, N. Cramer, E (2014). The Art of Progressive Censoring: Application to Reliability and Quality. Springer, Boston.

[3] Basak, I. AND Balakrishnan, N. (2006) On some predictors of times to failure of censored items in progressively censored samples, Computational Statistics & Data Analysis, 50, 1313–1337.

Stochastic Comparisons of Parallel and Series Systems from Heterogeneous Scale Populations

Amini-Seresht, E. [1]

Department of Statistics, Bu-Ali Sina University

Abstract

In this paper, we investigate stochastic comparisons of lifetimes of parallel and series systems with the multiple-outlier independent components of the scale models with respect to likelihood ratio and dispersive orders.

Keywords: Likelihood ratio order, Dispersive order, p-larger order, Parallel system, Series system.

1 Introduction

Random variable X be said to belong to the scale family of distributions if it has the distribution function $F(\lambda x)$ and the density function $\lambda f(\lambda x)$, where λ is a scale parameter and F is an absolutely continuous distribution function with density function f and called the base line distribution. Tow special cases of this model are gamma and exponential distributions. There are many papers in literature about stochastic comparisons of the order statistics when the random variables are independent and identically distributed(i.i.d). For more details the reader refer, Balakrishnan and Rao (1998a, 1998b), Kochar (1996), Bapat and kochar (1994), Khaledi and Kochar (2000), However, in the literature, some interesting new results on order statistics when random variables are non-i.i.d. have been obtained by Pledger and Proschan (1971), Proschan and Sethuraman (1976) and Zhao, Li and Balakrishnan (2009).

Many researchers have investigated the effect of two different vectors of parameters in some parametric models like exponential, gamma and Weibull families of distributions on the survival function, the hazard rate function and other characteristics of the time to failure of parallel and series systems.

[1] e.amini64@yahoo.com

Let us now review some well known results concerning stochastic comparisons of parallel systems. Let X_i, $i = 1, \ldots, n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$ and let X_i^*, $i = 1, \ldots, n$ be another independent exponential random variables with X_i^* having hazard rate λ_i^*. Pledger and Proschan (1971), for the first time proved the following interesting result, for $1 \leq k \leq n$,

$$(\lambda_1, \ldots, \lambda_n) \succeq^m (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{k:n} \geq_{st} X_{k:n}^*, \tag{1}$$

Khaledi and Kochar (2000) further improved the result (1) under a weaker condition as

$$(\lambda_1, \ldots, \lambda_n) \succeq^p (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{k:n} \geq_{st} X_{k:n}^*. \tag{2}$$

Khaledi et al. (2011) considered the problem of comparing of lifetimes of parallel and series systems with heterogeneous components from the scale model. They showed that under some conditions series and parallel systems with component lifetimes from the scale model of distributions are ordered in terms of the hazard rate order and the reversed hazard rate order.

In this paper specifically, we will study stochastic comparisons of parallel and series systems in the scale model in terms of the likelihood ratio and dispersive orders but shift our attention to multiple-outlier scale model.

The definitions and study of all the stochastic orders that are mentioned in this paper can be found in Müller and Stoyan (2002) or in Shaked and Shanthikumar (2007). Specifically, $\geq_{lr}$ stands for the *likelihood ratio order*, $\geq_{disp}$, $\geq_{hr}$ and $\geq_{rh}$ stand for the *dispersive order*, the *hazard rate order* and the *reversed hazard rate order*, respectively, when two univariate random variables are compared (cf. Shaked and Shanthikumar, 2007, pages 16 and 36).

The rest of the paper is organized as follows: Stochastically comparing parallel and series systems in terms of the likelihood ratio order and dispersive order are investigated, respectively, in section 2 and section 3.

2 Parallel systems

In this section, we compare parallel systems from multiple-outlier scale model according to likelihood ratio and dispersive orders. In the first theorem we consider the likelihood ratio order in order to compare the lifetimes of parallel systems arising from two sets of independent heterogeneous random variables in the scale models.

Theorem 1. *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier scale model with parameters $(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q})$ and let $X_1^*, \ldots, X_n^*$ be another set of independent random variables following the multiple-outlier scale model with parameters $(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q})$, where $p \geq 1$ and $p + q = n \geq 2$. If $\lambda \geq \lambda_1^* \geq \lambda_1$,*

$$xr(x) \text{ and } \frac{xr'(x)}{r(x)} \text{ are decreasing in } x,$$

then

$$X_{n:n}(p, q) \geq_{lr} X_{n:n}^*(p, q), \tag{3}$$

where $r(x) = \frac{f(x)}{F(x)}$ is the reversed hazard rate function.

Theorem 2. *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier scale model with parameters* $(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q})$ *and let $X_1^*, \ldots, X_n^*$ be another set of independent random variables following the multiple-outlier scale model with parameters* $(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q})$*, where $p \geq 1$ and $p + q = n \geq 2$. Suppose that $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$ and $p \geq q$. If*

$$xr(x) \text{ and } \frac{xr'(x)}{r(x)} \quad \text{are decreasing in } x,$$

then

$$(\lambda_1, \lambda_2) \succeq^p (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n}(p, q) \geq_{disp} X_{n:n}^*(p, q). \tag{4}$$

where, $r(x) = \frac{f(x)}{F(x)}$ is the reversed hazard rate function.

3 Series systems

In this section we give some new results involving series systems with multiple-outlier components from scale model.

Theorem 3. *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier scale model with parameters* $(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q})$ *and let $X_1^*, \ldots, X_n^*$ be another set of independent random variables following the multiple-outlier scale model with parameters* $(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q})$*, where $p \geq 1$ and $p + q = n \geq 2$. If $\lambda \geq \lambda_1^* \geq \lambda_1$,*

$$xh(x) \text{ and } \frac{xh'(x)}{h(x)} \quad \text{are decreasing in } x,$$

then

$$X_{1:n}(p, q) \geq_{lr} X_{1:n}^*(p, q),$$

where $h(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate function.

References

[1] Balakrishnan, N. and Rao, C. R. (1998b). Handbook of Statistics. **17**. Order Statistics: Applications. Amsterdam: Elsevier.

[2] Balakrishnan, N. and Rao, C. R. (1998a) Handbook of Statistics. **16**. Order Statistics: Theory and Methods. Amsterdam: Elsevier.

[3] Bapat, R. B. and Kochar, S. C. (1994). On likelihood-ratio ordering of order statistics. *Linear Algebra and Its Applications.* **199**, 281-291.

[4] Ding, W. and Li, X. (2012). The optimal allocation of active redundancies to k-out-of-n systems with respect to hazard rate ordering. *Journal of Statistical Planning and Inference.* **142**, 1878-1887.

[5] Khaledi, B.-E. and Kochar, S. C. (2000). Some new results on stochastic comparisons of parallel systems. *Journal of Applied Probability.* **37**, 1123-1128.

[6] Khaledi, B-E., Farsinezhad, S. and Kochar, S. C. (2011). Stochastic comparisons of order statistics in the scale model. *Journal of Statistical Planning and Inference.* **141**, 276-286.

[7] Misra, N., and van der Meulen, E.C.. (2003). On stochastic properties of m-spacings. *Journal of Statistical Planning and Inference.* **115**, 683-697.

[8] Müller A. and Stoyan, D. (2002). Comparison methods for stochastic models and risks. Wiley, New York.

[9] Pledger, P. and Proschan, F. (1971). Comparisons of order statistics and of spacings from heterogeneous distributions, in Optimizing Methods in Statistics, J. S. Rustagi, Ed., 89-113, Academic Press, New York, NY, USA.

[10] Proschan F. and Sethuraman, J. (1976). Stochastic comparisons of order statistics from heterogeneous populations, with applications in reliability. *Journal of Multivariate Analysis.* **6**, 608- 616.

[11] Zhao, P., Li, X. and Balakrishnan, N. (2009). Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables. *Journal of Multivariate Analysis.* **100**, 952-962.

Baysian Estimation of Lifetime Performance Index Based on RSS Sample

Asghari, S. [1] and Sadeghpour Gildeh, B. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

Lifetime performance index (C_L) is a flexible and effective tool for evaluating product quality and conforming rate. Ranked set sampling (RSS) scheme is applied for Baysian estimator of C_L based on square error loss. We assume that lifetimes of products follow a one-parameter exponential distribution. The simulation result for this scheme is compared with simple random sample (SRS) scheme based on bias, risk, pitman nearness, relative efficiency.

Keywords: Ranked set sampling, Lifetime performance index, Baysian estimation

1 Introduction

In lifetime testing experiments, the experimenter because of time limitation or other restrictions such as lack of funds, lack of material resources, mechanical or experimental difficulties, etc on data collection, may not always be in a position to observe the lifetimes of all the products on test. In this paper, we propose sampling scheme known as ranked-set sampling (RSS), introduced by McIntyre [3], instead of simple random sample (SRS) for estimating and testing a lifetime performance index C_L, since this method requires fewer observations to provide the same information[1] . C_L index, proposed by Montgomery [4], has many applications in health care and public health monitoring and surveillance and used to measure the larger-the-better type quality characteristics. This index defined as: $C_L = \frac{\mu - L}{\sigma}$, where μ is the process mean, σ is the process standard deviation and L is the lower specification limit. The ratio of conforming products is known as the conforming rate and can be defined as $p = P(X \geq L) = e^{C_L - 1}, -\infty < C_L < 1$. Obviously, a strictly

[1] asghari.samaneh@stu.um.ac.i
[2] Sadeghpour@um.ac.ir

increasing relationship exists between the conforming rate p and the lifetime performance index C_L. C_L and θ can be derived as $C_L = 1 + \ln p$ and $\theta = -\frac{L}{\ln p}$ respectively. The RSS scheme can be used for hospital monitoring with respect to patient infection rates, patient falls or accidents, emergency waiting room times, and so on. The data from patients can be obtained via RSS schemes using expert's knowledge or using auxiliary variables [2]. In this paper we assume that the lifetime data follow a one-parameter exponential distribution, $\varepsilon(\theta)$, with pdf $f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$. In this case, the capability index C_L reduces to $C_L = 1 - \frac{L}{\theta}$. To obtain a ranked set sampling, suppose $X_1, X_2, ..., X_n$ be a random sample of size n with pdf $f(x)$ and we have n set of such sample. Ranked set sampling is sourced by the sample selection which is based on two stages, involves an initial ranking samples of size n as follows:

Table 1: Ranking the samples

1	$X_{1(1)}$	$X_{1(2)}$	$\cdots$	$X_{1(n-1)}$	$X_{1(n)}$
2	$X_{2(1)}$	$X_{2(2)}$	$\cdots$	$X_{2(n-1)}$	$X_{2(n)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$X_{n(1)}$	$X_{n(2)}$	$\cdots$	$X_{n(n-1)}$	$X_{n(n)}$

Here, $X_{i(j)}$,$(i, j = 1 : n)$ denotes the jth order statistic of the ith random sample. RSS sample is formed by selecting the diagonal elements in Table 1. Element of new sample RSS are independent but not identically distributed. In certain situations, the whole procedure to generate an RSS of size n can be repeated m times. Success of RSS depends very much on our ability to rank the units without any error.

2 Baysian estimatin for C_L based on RSS

In this section, based on two different prior, $IG(a, b)$ with known parameters a, b and Jeffrey's prior, we obtain Bayes estimators for C_L based on RSS samples and study performance of these estimators.

2.1 Inverse Gamma prior

Suppose $\theta \sim IG(a, b)$, which probability density function is defined as $\pi(\theta) = \frac{b^a}{\Gamma(a)}(\frac{1}{\theta})^{a+1}e^{-\frac{b}{\theta}}$. By Bayes' theorem, the posterior distribution of θ, $\pi(\theta|X_{srs})$, is $IG(n+a, b+\sum x_i)$. Therefore the Bayes estimators for θ under square error loss is equal to $E(\theta|X_{srs}) = \frac{b+n\bar{X}}{n+a-1}$, and Bayes estimator for C_L will be obtained $\hat{C}^{srs}_{Lbayes-IG} = 1 - LE(\frac{1}{\theta}|X_{srs}) = 1 - L(\frac{n+a}{b+n\bar{X}})$. Let $X_{rss} = \{X_{(1,1)}, X_{(2,2)}, \ldots, X_{(m,m)}\}$ is one-cycle RSS sample from $\varepsilon(\theta)$. The joint probablity density function of the RSS, due to the independence of element X_{rss}, is given by Sadek et al. [5] such as

$$g_\theta(X_{rss}) = \sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m}(\frac{1}{\theta})^m e^{-\frac{1}{\theta}\sum_{i=1}^{m}(n+k-i+1)}x(i, i). \tag{1}$$

So the posterior density can be written as

$$\pi(\theta|X_{rss}) = \frac{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m} C_{j_i}(i)(\frac{1}{\theta})^{m+a+1}e^{-\frac{1}{\theta}(b+\sum_{i=1}^{m}(n+j_i-i+1)x_{(i,i)})}}{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_m=0}^{m-1}\prod_{i=1}^{m} C_{j_k}(i)(b+\sum_{i=1}^{m}(m+j_i-i+1)(x_{(i,i)})^{-(m+a)}\Gamma(m+a)}$$

Then the Bayes estimator of C_L is $\hat{C}^{rss}_{Lbayes_{IG}} = 1-LE(\frac{1}{\theta}|X_{rss}) = 1-L\int_0^\infty \frac{1}{\theta}\pi(\theta|X_{rss})d\theta$

2.2 Jeffry's prior

If θ has the Jeffrey prior, $\pi(\theta) \propto \frac{1}{\theta}$, then $\frac{1}{\theta}|X_{srs} \sim G(n, \frac{1}{\sum X_i})$. Therefore, the Bayes estimator of C_L is $E(C_L|X_{srs}) = 1 - L\frac{n}{\sum X_i} = 1 - \frac{L}{\bar{X}_{srs}}$.
In the case of RSS scheme

$$\pi(\theta|X_{rss}) = \frac{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m} C_{j_i}(i)(\frac{1}{\theta})^{m+1}e^{-\frac{1}{\theta}(\sum_{i=1}^{m}(m+j_i-i+1)x_{(i,i)})}}{\sum_{j_1=0}^{0}\sum_{j_2=0}^{1}\cdots\sum_{j_n=0}^{m-1}\prod_{i=1}^{m} C_{j_k}(i)(\sum_{i=1}^{m}(m+j_i-i+1)(x_{(i,i)})^{-(m)}\Gamma(m)},$$

therefore $\hat{C}^{rss}_{Lbayes_J} = 1 - LE(\frac{1}{\theta}|X_{rss}) = 1 - \int_0^\infty \frac{1}{\theta}\pi(\theta|X_{rss})d\theta$

2.3 Simulation study

For studying performance of discussed estimators, we carry out Monte-Carlo simulations as follows:

1. Determine lower specification limit L, hyper parameter (a, b), sample size n. Generate θ_0 from distribution $IG(a, b)$ and calculate C_L. 10000 times repeat steps 2, 3.

2. Generate SRS and RSS samples of size n from $\varepsilon(\theta_0)$ and derived $\hat{C}^{srs}_{Lbayes_{IG}}$, $\hat{C}^{srs}_{Lbayes_J}$, $\hat{C}^{rss}_{Lbayes_{IG}}$, $\hat{C}^{rss}_{Lbayes_J}$.

3. For each samples and estimators in the step (2), calculate, $di = (\hat{C}_{Li} - C_L), i = 1, ..., 10000$. In each times, for calculating the Pitman nearness criteria between two estimators, we investigate if $|\hat{C}_{L1} - C_L| < |\hat{C}_{L2} - C_L|$.

4. The risk values of $\hat{C}_{Li}$ is the mean of di^2. Relative Efficiency between $\hat{C}_{L1}$ and $\hat{C}_{L2}$ is $RE(\hat{C}_{L1}, \hat{C}_{L2}) = \frac{MSE(\hat{C}_{L1})}{MSE(\hat{C}_{L2})}$. The bias of $\hat{C}_L$ is $\frac{1}{10000}\left(\sum_{i=1}^{10000}\hat{C}_{Li}\right) - C_L$. The pitman nearness between $\hat{C}_{L1}$ and $\hat{C}_{L2}$ is $\frac{1}{10000}\#|\hat{C}_{L1} - C_L| < |\hat{C}_{L2} - C_L|$.

Example 1. *We select values of hyper parameters, (a, b), in prior distribution, such that the mean of prior distribution, $IG(a, b)$, is fixed at 0.5 and for it's variance we consider three state: small (0.0357), moderate (0.0833) and large (0.25). With this strategy select $(a, b) = (9, 4), (5, 2), (3, 1)$. Let $L = 1.04$ and $n = 4, 5, 6$. Table 2 shows the values of Bias, Risk, Relative Efficiency (RE) and Pitman Nearness criterion (PN).*

Table 2: Observed values of Bias, Risk, RE, PN

				IGestimator		jeffrys estimator			pitman-nearness		Relative efficiency	
4	3	1	Bias	0.0347	0.0187	-1.0689	-0.3888	PN	0.4103	0.3779	0.6346	0.5873
			Risk	1.6624	1.0309	15.718	2.429	RE	1.6126	6.3284	0.1081	0.4244
	5	2	Bias	-0.0145	-0.0099	-0.8825	-0.3292	PN	0.4115	0.3727	0.6676	0.6083
			Risk	0.8147	0.5279	8.512	1.4876	RE	1.5434	5.7221	0.0957	0.3549
	9	4	Bias	0.0085	0.0012	-0.7505	-0.2963	PN	0.4306	0.3876	0.7057	0.6446
			Risk	0.437	0.3231	5.525	1.1639	RE	1.3525	4.747	0.0791	0.2776
5	3	1	Bias	0.0157	0.0013	-0.7643	-0.2641	PN	0.3906	0.359	0.6144	0.5795
			Risk	1.4285	0.7502	7.3862	1.4537	RE	1.9042	0.1016	5.0810	0.7634
	5	2	Bias	-0.0088	-0.0001	-0.6944	-0.2202	PN	0.3998	0.3517	0.6564	0.602
			Risk	0.7473	0.411	4.8278	0.9213	RE	1.8182	5.2402	0.1548	0.4461
	9	4	Bias	0.0055	0.0019	-0.5689	-0.1889	PN	0.4113	0.3575	0.691	0.6207
			Risk	0.4117	0.2602	3.3125	0.6376	RE	1.5822	5.1953	0.1243	0.4081
6	3	1	Bias	-0.0115	0.0046	-0.69	-0.1802	PN	0.3624	0.3349	0.6117	0.5558
			Risk	1.3246	0.5905	6.2285	0.93	RE	2.2431	6.6974	0.2127	0.6350
	5	2	Bias	0.0047	0.0041	-0.5264	-0.1544	PN	0.3823	0.3356	0.6404	0.5796
			Risk	0.6953	0.3356	3.1049	0.5839	RE	2.0718	5.3175	0.2239	0.5748
	9	4	Bias	0.0007	0	-0.4829	-0.1407	PN	0.3898	0.331	0.6772	0.6049
			Risk	0.3849	0.2114	2.489	0.4472	RE	1.8211	5.5656	0.1547	0.4727

Table 2 shows that absolute values of bias and also risk for $\hat{C}^{rss}_{Lbayes-IG}$ and $\hat{C}^{rss}_{Lbayes-J}$ are smaller than similar estimators in SRS scheme.

Moreover the RE and PN probability criteria indicates the efficiency of RSS estimators with respect to SRS estimators. Because of reducing the cost of data collection and better performance estimators in simulation for RSS scheme, we suggest that $\hat{C}^{rss}_{Lbayes-IG}$ and $\hat{C}^{rss}_{Lbayes-J}$ estimators as long as there are no ranking errors caused by a large set of size m.

References

[1] Barabesia,L., El-Sharaawi,A. (2001). The effciency of ranked set sampling for parameter estimation, *Statistics & Probability Letters*, **53**,189 -199.

[2] Haq, A., Brown, J. and Moltchanova, E. (2015), New Exponentially Weighted Moving Average Control Charts for Monitoring Process Mean and Process Dispersion, *Quality and Reliability Engineering International* DOI:10.1002/qre.1694.

[3] McIntyre GA., (1952), A method for unbiased selective,sampling, using ranked sets. *Australian Journal of Agriculture Research*,**3**(4):385-390.

[4] Montgomery DC., (2009), *Introduction to Statistical Quality Control 6th edn*, John Wiley & Sons, NewYork.

[5] Sadek,A., Sultan, K.S., Balakrishnan, N., (2013). Bayesian Estimation Based on Ranked Set Sampling Using Asymmetric Loss Function, *Bull. Malays. Math. Sci. Soc.*, DOI 10.1007/s40840-014-0045-5.

On the Number of Failed Links in a Three-State Network

Ashrafi, S. [1]

Department of Statistics, University of Isfahan

Abstract

In this paper, we consider a single-step network consists of n links and assume that the links are subject to failure. It is assumed that the network can be in three states, *up* ($K = 2$), *partial performance* ($K = 1$) and *down* ($K = 0$). Under different scenarios on the states of the network and using the concept of two-dimensional signature, we obtain the probabilities that i links fail at time t_1 and j links fail at time t . Several stochastic and aging properties of the proposed probabilities are studied.

Keywords: Signature matrix, Bivariate increasing failure rate, Total positive of order 2, Stochastic order.

1 Introduction

In this paper, we consider a three-state network consisting of n i.i.d. binary links. We assume that the network can be in three states, *up* ($K = 2$), *partial performance* ($K = 1$) and *down* ($K = 0$). Let the network start to function at time $t = 0$ in state $K = 2$. Denote by T_1 the lifetime of the network which remains in state $K = 2$. Also, denote by T the network lifetime i.e. the entrance time into state $K = 0$. Using these notations, the two-dimensional signature of the network is defined to be a probability matrix S with elements defined by

$$s_{i,j} = \frac{n_{i,j}}{n!}, \quad 1 \leq i < j \leq n,$$

where $n_{i,j}$ is the number of ways that the ith and the jth links failure cause the state of the network changes from $K = 2$ to $K = 1$ and from $K = 1$ to $K = 0$, respectively.

Recently Erylmaz (2010) studied the distribution and expected value of the number of working components at time t in a consecutive k-out-of-n system under the condition that it is working at time t. Asadi and Berred (2012) studied the number of failed components in a binary coherent system. In this paper, we assume that at time t_1 the network is

[1] s.ashrafi@ui.ac.ir

in state $K=2$ and at time t, it is in state $K=1$ or it is functioning. Then, we present a model for the probabilities that k and l, $0\leq k<l\leq n-1$ links have failed at times t_1 and t, respectively. Based on the notion two-dimensional signature, we obtain some stochastic and aging properties of the proposed probabilities.

2 Main results

Consider a network consisting of n links. Suppose that $X_1, ..., X_n$ denote the links lifetimes, where we assume that X_i's are i.i.d with a common continuous distribution function $F(x)$. Suppose that, we have some information about the states of the network at times t_1 and t, $t_1<t$, for example, we know $T_1\in A_1$ and $T\in A$ where $A_1, A\subseteq[0,\infty)$. Denote by $N(t)$ the number of failed links in $[0,t]$. In such a situation, we are interested in the conditional probability

$$p_{A_1,A}(k,l)=P(N(t_1)=k, N(t)=l|T_1\in A_1, T\in A),\quad 0\leq k\leq l\leq n.$$

In this paper, we consider two following cases:

(I) Suppose that at time t_1 the network is in state $K=2$ and at time t, $t>t_1$, it is in state $K=1$. In such a situation $A_1=(t_1,t)$ and $A=(t,\infty)$. In this case, $p_{A_1,A}(k,l)$, which we denote it by $p_{t_1,t}(k,l)$, is

$$p_{t_1,t}(k,l)=P(N(t_1)=k, N(t)=l|t_1<T_1<t, T>t),\quad 0\leq k<l\leq n-1.$$

(II) Suppose that at time t_1 network is in state $k=2$, and at time t, it is functioning. In such a situation, $A_1=(t_1,\infty)$ and $A=(t,\infty)$. In this case, $p_{A_1,A}(k,l)$, which we denote it by $q_{t_1,t}(k,l)$, is

$$q_{t_1,t}(k,l)=P(N(t_1)=k, N(t)=l|T_1>t_1, T>t),\quad 0\leq k\leq l\leq n-1.$$

In the following theorem, $p_{t_1,t}(k,l)$ and $q_{t_1,t}(k,l)$ are computed.

Theorem 1. *Consider a network consists of n links with i.i.d. lifetimes. Suppose that $F(x)$ denotes the common distribution of the links lifetimes and T_1 and T are the lifetime in state $K=2$ and the lifetime of the network, respectively. Assume that S is the signature matrix of the network.*

(a) If $\beta_{k,l}=\sum_{i=k+1}^{l}\sum_{j=l+1}^{n}s_{i,j}$ then

$$p_{t_1,t}(k,l)=\frac{\beta_{k,l}c_{k,l,n}F^k(t_1)(F(t)-F(t_1))^{l-k}\bar{F}^{n-l}(t)}{\sum_{k=0}^{n-2}\sum_{l=k+1}^{n-1}\beta_{k,l}c_{k,l,n}F^k(t_1)(F(t)-F(t_1))^{l-k}\bar{F}^{n-l}(t)},\quad 0\leq k<l\leq n-1$$

where $c_{k,l}=\frac{n!}{k!(l-k)!(n-l)!}$.

(b) If $\bar{S}_{k,l}=\sum_{i=k+1}^{l}\sum_{j=\max\{i,l\}+1}^{n}s_{i,j}$ then

$$q_{t_1,t}(k,l)=\frac{c_{k,l,n}\bar{S}_{k,l}F^k(t_1)(F(t)-F(t_1))^{l-k}\bar{F}^{n-l}(t)}{\sum_{i=1}^{n-1}\sum_{j=i}^{n}c_{i,j,n}\bar{S}_{i,j}F^i(t_1)(F(t)-F(t_1))^{j-i}\bar{F}^{n-j}(t)},\quad 0\leq k\leq l\leq n-1.$$

In the following, we present results that compare the probabilities of the number of failed links of two networks. Before it, we need the following definition.

Definition 1. Let $f_1(x,y)$ and $f_2(x,y)$ be two nonnegative functions. $f_1(x,y)$ is said to be smaller than $f_2(x,y)$ in the total positive order (denoted by $f_1 \leq_{TP_2} f_2$) if $f_1(\mathbf{x})f_2(\mathbf{y}) \leq f_1(\mathbf{x}\wedge\mathbf{y})f_2(\mathbf{x}\vee\mathbf{y})$ for every $\mathbf{x},\mathbf{y} \in R^2$, where $\mathbf{x}\wedge\mathbf{y} = (\min\{x_1,y_1\},\min\{x_2,y_2\})$ and $\mathbf{x}\vee\mathbf{y} = (\max\{x_1,y_1\},\max\{x_2,y_2\})$.

Theorem 2. *Consider two networks each consists of n i.i.d. links. Suppose that the links lifetimes of two networks have the same distribution. Let S_1 and S_2 be the corresponding signature matrices and $\beta^{(r)}_{k,l} = \sum_{i=k+1}^{l}\sum_{j=l+1}^{n} s_{r,i,j}$ and $\bar{S}^{(r)}_{k,l} = \sum_{i=k+1}^{l}\sum_{j=\max\{i,l\}+1}^{n} s_{r,i,j}, r = 1,2$. Suppose that $p^{(r)}_{t_1,t}(k,l)$ and $q^{(r)}_{t_1,t}(k,l)$ are the probability functions corresponding to $\beta^{(r)}_{k,l}$ and $\bar{S}^{(r)}_{k,l}$, $r = 1,2$, respectively.*

(a) If $\beta^{(1)}_{k,l} \leq_{TP_2} \beta^{(2)}_{k,l}$ then $p^{(1)}_{t_1,t}(k,l) \leq_{TP_2} p^{(2)}_{t_1,t}(k,l)$.

(b) If $\bar{S}^{(1)}_{k,l} \leq_{TP_2} \bar{S}^{(2)}_{k,l}$ then $q^{(1)}_{t_1,t}(k,l) \leq_{TP_2} q^{(2)}_{t_1,t}(k,l)$.

Recall that if in Definition 1, f_1 and f_2 are probability mass functions of (X_1,X_2) and (Y_1,Y_2), respectively, then TP_2 order is called likelihood ratio order and denoted by $(X_1,X_2) \leq_{lr} (Y_1,Y_2)$.

In the following theorem, under some stochastic comparisons between links lifetimes of two networks, we compare the probabilities of the number of failed links of two networks.

Theorem 3. *Consider two networks each consists of n i.i.d. links. Assume that two networks have the same structure and F_1 and F_2 are the corresponding distributions of the link lifetimes. Suppose that $p^{(i)}_{t_1,t}(k,l)$ and $q^{(i)}_{t_1,t}(k,l)$ are the probability functions corresponding to F_i, $i = 1,2$. Let $(I^{(i)}_1, I^{(i)}_2)$ and $(J^{(i)}_1, J^{(i)}_2)$ have joint probability mass functions $p^{(i)}_{t_1,t}(k,l)$ and $q^{(i)}_{t_1,t}(k,l)$, $i = 1,2$, respectively. If $F_1 \leq_{rh} F_2$, $F_1 \leq_{hr} F_2$ and*

(a) $\beta_{k,l}$ is TP_2 in k and l then $(I^{(1)}_1, I^{(1)}_2) \geq_{lr} (I^{(2)}_1, I^{(2)}_2)$.

(b) $\bar{S}_{k,l}$ is TP_2 in k and l then $(J^{(1)}_1, J^{(1)}_2) \geq_{lr} (J^{(2)}_1, J^{(2)}_2)$.

The following definition is an analogue to that of Harris (1970) in the continuous set up.

Definition 2. The bivariate mass function $p_{i,j}$ with survival function $\bar{P}_{i,j}$ is said to be $BIFR$ if $\bar{P}_{i,j}$ is TP_2 and $\frac{\bar{P}_{i+1,j+1}}{\bar{P}_{i,j}}$ is decreasing in i,j.

Theorem 4. *Let $\bar{P}_{t_1,t}(k,l)$ and $\bar{Q}_{t_1,t}(k,l)$ be the survival functions corresponding to probability mass functions $p_{t_1,t}(k,l)$ and $q_{t_1,t}(k,l)$, respectively.*

(a) If $\beta_{k,l}$ is TP_2 in k and l and $\frac{\beta_{k+1,l+1}}{\beta_{k,l}}$ is decreasing in k and l then $\bar{P}_{t_1,t}(k,l)$ is $BIFR$.

(b) If $\bar{S}_{k,l}$ is TP_2 in k and l and $\frac{\bar{S}_{k+1,l+1}}{\bar{S}_{k,l}}$ is decreasing in k and l then $\bar{Q}_{t_1,t}(k,l)$ is $BIFR$.

The following example present an application of Theorem 4.

Example 1. Figure **??** presents a network consists of 5 nodes and 10 links. Assume that links are subject to failures. The states of the network are defined as $K = 2$ if all nodes are connected, $K = 1$ if nodes are divided into two disconnected sets, and $K = 0$ if nodes are divided into at least three disconnected sets.

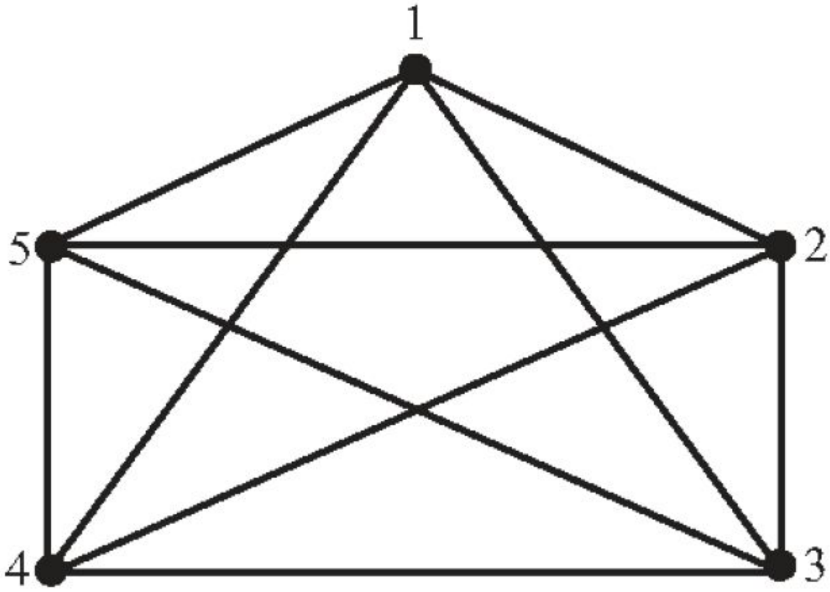


Figure 1: Network with 5 nodes and 10 links

The signature matrix (S) of this network is given in Gertsbakh and Shpungin (2012). It can be seen that $\beta_{k,4} = 0.0241$, $\beta_{k,5} = 0.1183$, $\beta_{k,6} = 0.4049$, $\beta_{k,7} = 0.9166$, $k = 0, ..., 3$ and $\beta_{4,5} = 0.0942$, $\beta_{4,6} = 0.3808$, $\beta_{4,7} = 0.8972$, $\beta_{5,6} = 0.2866$, $\beta_{5,7} = 0.8221$, $\beta_{6,7} = 0.5951$. It can be shown that $\beta_{k,l}$ is TP_2 in k and l and $\frac{\beta_{k+1,l+1}}{\beta_{k,l}}$ is decresing in k and l.

References

[1] Asadi, M. and Berred, A. (2012), On the number of failed components in a coherent operating system. *Statistics & Probability Letters*, **82**(12), 2156-2163.

[2] Eryilmaz, S. (2010), Number of working components in consecutive k-out-of-n system while it is working. *Comm. Statist. Simulation Comput.*, **39**, 683-692.

[3] Gertsbakh, I. and Shpungin, Y. (2011). *Network reliability and resilience.* Springer Science & Business Media.

[4] Gertsbakh, I. and Shpungin, Y. (2012), Stochastic models of network survivability. *QTQM*, **9**, 45-58.

[5] Harris, R. (1970), A multivariate definition for increasing hazard rate distribution functions. *The Annals of Mathematical Statistics*, 713-717.

Bayesian Sensitivity Analysis for Competing Risks Data With Missing Cause of Failure

Azizi, F. [1] , Eftekhari Mahabadi, S. [2] and Mosayebi, E. [3]

Department of Mathematics, University of Tehran

Abstract

Competing risks data are often summarized using a failure time and an indicator of cause of failure that may not be observed for some subjects. In such case, standard analysis through complete case may lead to biased inferences when the missing mechanism is not ignorable. In this paper, we propose a Bayesian Index of local Sensitivity to Non-ignorability (ISNI) for modeling competing risks data in the presence of hybrid censoring when the competing risks have Weibull distribution with the same shape parameter, but different scale parameters. The results of applying the above index on a set of real data show that the model could have potential sensitivity to non-ignorability for scale parameters but not for the common shape parameter.

Keywords: Competing risks, Missing data, Sensitivity analysis, Type-I Hybrid censoring, Weibull distribution.

1 Introduction

Competing risks arise when a subject is exposed to many causes of failure in a survival analysis. To analyze multiple causes of failure in the framework of competing risks models, it is often assumed that the data consists of a failure time and an indicator, denoting the cause of failure. The competing risks models also have been studied by several authors using parametric (such as [1]) and nonparametric setups (such as [4]).

However, in applied studies the cause of failure may not be observed for some subjects where some researchers have applied multiple imputation procedures which needs the ignorability assumption (see [3], [2]). Since ignorability is a critical assumption, in

[1]fariba.azizi@khayam.ut.ac.ir
[2]S.Eftekhari@khayam.ut.ac.ir
[3]elham_mosayebi2@ut.ac.ir

this paper, we extend the Bayesian index of local sensitivity to non-ignorability (ISNI) method proposed by Zhang and Heitjan [5] to competing risks with missing cause of failure. We derive formula for the Bayesian ISNI when the competing risks have Weibull distribution with the same shape parameter, but different scale parameters with Type-I hybrid censoring scheme.

The model assuming non-ignorable missing cause of failure will be studied In Section 2. We will give the ISNI calculation In Section 3. In Section 4 we will illustrate the method with a set of real data. Finally some concluding remarks will be given in Section 5.

2 Model Description

Consider a multiple causes of failure experiment with M possible causes on a sample of n items. Let $T_{ij} \sim Weibull(\alpha, \lambda_j)$ denote the latent failure time for the i-th individual and $T_i = min\{T_{i1}, ..., T_{iM}\}$ be the actual observable life time for the i-th individual. We consider Type-I hybrid censoring scheme that terminates the experiment either if the terminal time T is reached or the R-th failure happens. In other words, the experiment terminates at $T_* = min\{T_{R:n}, T\}$ and we will assume that d indicates the total number of failures before T_*.

With the possibility of missing causes of failure, we assume that the missingness indicator M_i has a Bernoulli distribution with the success probability $\pi_i = P(M_i = 1|t_i, \delta_i)$, where $logit\ \pi_i = \gamma_{00} + \gamma_{01} t_i + \gamma_1 \delta_i$. Actually this missing model introduces a non-ignorable missing mechanism while $\gamma_1 \neq 0$. Let $\Theta = (\alpha, \lambda_1, \lambda_2)$, $\Gamma = (\gamma_{00}, \gamma_{01}, \gamma_1)$, then the full likelihood function with the nonignorable missing mechanism would be (for $M = 2$):

$$
\begin{aligned}
&L_{full}(\Theta, \Gamma|t_i, M_i, (1-M_i)\delta_i) \propto \\
&\prod_{i=1}^{d} \Bigg\{ \left[\left(\alpha\lambda_1 t_{i:n}^{\alpha-1} e^{-(\lambda_1+\lambda_2)t_{i:n}^{\alpha}}\right)^{\delta_i} \left(\alpha\lambda_2 t_{i:n}^{\alpha-1} e^{-(\lambda_1+\lambda_2)t_{i:n}^{\alpha}}\right)^{1-\delta_i} (1-\pi_i) \right]^{1-M_i} \\
&\times \left[\sum_{\delta^*=0,1} \left(\alpha\lambda_1 t_{i:n}^{\alpha-1} e^{-(\lambda_1+\lambda_2)t_{i:n}^{\alpha}}\right)^{\delta^*} \left(\alpha\lambda_2 t_{i:n}^{\alpha-1} e^{-(\lambda_1+\lambda_2)t_{i:n}^{\alpha}}\right)^{1-\delta^*} \pi_i \right]^{M_i} \Bigg\} \\
&\times e^{-(n-d)(\lambda_1+\lambda_2)T_*^{\alpha}}
\end{aligned}
$$

where,

$$
\delta_i = \begin{cases} 1 & \text{the first cause leads to failure} \\ 0 & \text{the second cause leads to failure} \end{cases}
$$

3 Bayesian ISNI

In this section, we use the Bayesian ISNI as the derivative of posterior mean of parameters η with respect to a nonignorability parameter, γ_1, evaluated locally at the ignorable model

Table 1: Parameters estimation, standard deviation, ISNI and SET indexes for different missing rates.

Missing Rate	Parameter	Estimate	SD	ISNI	SET
0.04	α	0.9617	0.1889	0.0368	5.1332
	λ_1	0.0004	0.0007	0.0567	0.0123
	λ_2	0.0007	0.0001	-0.0584	0.0017
0.08	α	0.9661	0.1805	-0.0143	12.6224
	λ_1	0.0004	0.0006	0.0845	0.0071
	λ_2	0.0006	0.0010	-0.0402	0.0249
0.12	α	0.9315	0.1744	-0.0126	13.8413
	λ_1	0.0005	0.0011	0.0210	0.0524
	λ_2	0.0008	0.0014	-0.0610	0.0230
0.20	α	0.8995	0.1708	-0.0450	3.79556
	λ_1	0.0007	0.0010	0.1429	0.0070
	λ_2	0.0008	0.0013	0.0146	0.0890
0.32	α	0.9721	0.1622	0.0368	4.4076
	λ_1	0.0005	0.0007	0.0117	0.0598
	λ_2	0.0004	0.0010	-0.0535	0.0187

$(\gamma_1 = 0)$:

$$\begin{aligned} ISNI(\widetilde{\eta}(\gamma_1)) &= \frac{\partial E(\eta|\gamma_1, Data)}{\partial \gamma_1}|_{\gamma_1=0} \\ &= COV_I(\eta, \frac{\partial \ell_{full}(\eta; \gamma_1, Data)}{\partial \gamma_1}|_{\gamma_1=0}) \end{aligned} \tag{1}$$

where $\widetilde{\eta}(\gamma_1)$ denotes the posterior mean of η when the non-ignorability parameter is fixed at $\gamma_1 = 0$. ℓ_{full} and $COV_I(.)$ denote the log-likelihood and the posterior covariance under the ignorable model respectively. Smaller absolute value of ISNI implies smaller local sensitivity. For the competing risks model, we have:

$$\begin{aligned} \frac{\partial \ell_{full}(\eta; \gamma_1, t, M, \delta^{obs})}{\partial \gamma_1}|_{\gamma_1=0} = &-\sum_{i=1}^{d} \delta_i (1 - M_i)(\pi_i|_{\gamma_1=0}) \\ &+ \sum_{i=1}^{d} M_i \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{1}{1 + exp(\gamma_{00} + \gamma_{01} t_i)} \end{aligned} \tag{2}$$

4 An illustrative example

In this section, we use a real complete hybrid data set which has been taken from [1]. To study the behavior of the Bayesian sensitivity index of previous section, we have created some artificially missing values with different rates of $0.04, 0.08, 0.12, 0.2, 0.32$. Also we have used low-informative priors for the parameters ($\gamma_{00}, \gamma_{01} \sim N(0, 100)$ and $\alpha, \lambda_1, \lambda_2 \sim \Gamma(0.01, 0.01)$). The results of Bayesian MAR estimation and its corresponding sensitivity are presented in Table 1 where $SET = |SD/ISNI|$.

According to this table, the SET index is considerably large for the common shape parameter α, while it has a small value for the two scale parameters λ_1, λ_2. Hence, we could conclude that using this competing risks model framework, the scale parameters could be highly sensitive to the non-ignorability of the missingness if it exists.

5 Conclusion

Sensitivity analysis is necessary to assess robustness of the model to non-ignorability of missingness. In this paper we have presented a Bayesian index to study this sensitivity for competing risks with hybrid censoring.

References

[1] Bhattacharyaa, S. et al. (2014). Analysis of hybrid censored competing risks data, *Statistics*, 2014,**48**, 1138-1154.

[2] Christian NJ. (2011). Hierarchical likelihood inference on clustered competing risk data. PhD Thesis, Department of Biostatistics, University of Pittsburgh, USA.

[3] Minjung, Lee et al. (2014). Frailty modeling for clustered competing risks data with missing cause of failure. *Stat Methods Med Res*, 0962280214545639, first published on August 14.

[4] Peterson, Jr., A.P. (1977). Expressing the Kaplan-Meier estimator as a function of empirical survival functions. *J. Amer. Statist. Assoc.*, **72**, 854-858.

[5] Zhang, J., Heitjan, D. F. (2007). Impact of nonignorable coarsening on Bayesian inference. *Biostatistics*, **8**,722-43.

Goodness-of-Fit Test Based on Kullback-Leibler Information for Progressively First-Failure Censored Data

Bitaraf, M. [1] , Rezaei, M. [2] and Yousefzadeh, F. [3]

Department of Statistics, University of Birjand

Abstract

In this article, We constructed a goodness-of-fit test statistic based on Kullback-Leibler information for exponential distribution by using maximum likelihood estimate of the model parameter. A Monte Carlo simulation is performed to evaluate the power of the proposed test for several alternatives under different sample sizes and progressive first-failure censoring schemes.

Keywords: Entropy, Goodness-of-fit test, Kullback-Leibler information, Monte Carlo simulation, Progressively frist-failure censored data.

1 Introduction

Censoring is very important in determining the distribution of life-time products and where as units test are often censored based on cost and time. Although progressively Type- II shortens the test duration, but it is still too long for products having a high reliability that made Johnson [1] proposed a new censoring scheme known as the first-failure. Wu & Kus [6] combined the concepts of fist-failure and progressively censoring to introduce a new concept called progressively first-failure censoring scheme.

[1]m.bitaraf@birjand.ac.ir
[2]mjrezaei@birjand.ac.ir
[3]fyousefzadeh@birjand.ac.ir

1.1 Progressively First-Failure Censored Data

Suppose n independent groups with k items in each group are placed on a life-testing experiment that their life-times are identically distributed with probability density (p.d.f), $f(x;\theta)$, and cumulative distribution function (c.d.f), $F(x;\theta)$. Where θ is the unkown the vector of parameters and $m(n)$ is fixed prior to the exprement. When the first failure (X_1) occurs, R_1 groups and the group with observed failure are randomly withdrawn from the experiment. When second failure (X_2) observed, R_2 groups and the group with observed failure are randomly withdrawn from the experiment, and so on. Finally, when the failure m is observed, the remaining l_m groups and the group with observed failure are all withdrawn from the experiment. This censoring is called a progressive first-failure censoring scheme. The joint p.d.f of all progressively first-failure censored order statistics $(X_{1:m:n:k}, X_{2:m:n:k}, \ldots, X_{m:m:n:k})$ with progressive censoring scheme proposed by Wu & Ku? [?] that is given by

$$f_{X_{1:m:n:k},\ldots,X_{m:m:n:k}}(x_1,\ldots,x_m) = ck^m \prod_{i=1}^{m} f(x_i;\theta)\left(1-F(x_i;\theta)\right)^{k(R_i+1)-1}$$
$$,0 < x_1, \ldots, < x_m$$

where $c = n(n-R_1-1), \ldots, (n-\sum_{i=1}^{m-1} R_i - m + 1)$.

1.2 Nonparametric Entropy Estimate of Progressively First-Failure Censored Data

Balakrishnan et al. [1] has been simplified the joint entropy of progressively Type-II censored order statistics in terms of an integral involving the hazard function $h(x)$. Since the joint p.d.f Progressively first-failure censored is similar to the joint p.d.f progressively Type-II censored, the nonparametric estimate of the joint entropy $H_{1\cdots:m:n:k}$ is given by

$$H_{1\cdots:m:n:k} = -\log c + nkH(w,n,m,k)$$

where

$$\begin{aligned} H(w,n,m) &= \frac{1}{nk}\sum_{i=1}^{m} log\left(\frac{x_{i+w:m:n:k}-x_{i-w:m:n:k}}{E(U_{i+w:m:n:k})-E(U_{i-w:m:n:k})}\right) \\ &+\frac{m}{nk}-\frac{1}{nk}\sum_{i=1}^{m}\sum_{j=1}^{i}\frac{D_i}{\gamma_j^2} \end{aligned}$$

where $D_i = \prod_{j=1}^{i}$, $\gamma_i = m-i+1+\sum_{j=i}^{m} R_i$ for $1 \le i \le m$.

1.3 Kullback-Leibler Information

For a null density function $f^0(x_i;\theta)$, the KL information from progressively first-failure censored data can be estimated by

$$\begin{aligned} T &= -H(w,n,m,k) - \frac{1}{nk}\sum_{i=1}^{m}\log f^0(x_i;\widehat{\theta}) \\ &-\frac{1}{nk}\sum_{i=1}^{m}\left(k\left(R_i+1\right)-1\right)\log\left(1-F^0(x_i;\widehat{\theta})\right) \end{aligned}$$

where $\widehat{\theta}$ is a MLE estimator of θ.

1.4 Goodness-of-Fit Test for Exponential

Suppose we are interested in a goodness-of-fit test for

$$H_0 : f^0 = \left(\frac{1}{\theta}\right) exp\left(-\frac{x}{\theta}\right) \quad VS. \quad H_A : f^0 \neq \left(\frac{1}{\theta}\right) exp\left(-\frac{x}{\theta}\right)$$

where θ is unknown. If we replace the maximum likelihood estimate in place of the unknown parameter θ, then the KL information for progressively first-failure censored data can be estimated by

$$T = -H(w, n, m, k) + \frac{m}{nk}\left[\log\left(\frac{1}{m}\sum_{i=1}^{m} k\,(R_i + 1)X_{i:m:n:k}\right) + 1\right].$$

If $T(w, n, m, k)$ is close to 0, H_0 will be acceptable, and therefore large values of $T(w, n, m, k)$ will lead to the rejection of H_0.

Table 1: VALUE OF THE WINDOWS SIZE m WHICH MINIMUM CRITICAL VALUES OFα FOR 0.1

nk	k	m	w
20	(2,2)	(5,7)	(3,4)
30	(2,2,3)	(5,7,10)	(3,4,6)
40	(2,2,2,4,4)	(5,10,15,5,7)	(3,6,8,3,4)
50	(2,2,2,2,5,5)	(5,10,15,20,5,7)	(3,6,8,11,3,4)

2 Implementation of Test

Because the sampling distribution of $T\,(w, n, m, k)$ is intractable, we determine the percentage points using 10,000 Monte Carlo simulations from an exponential distribution. In determining the window size w which depends on n, m, k and α, we consider the optimal window size to be one which gives minimum critical points. However, we understood from the simulated percentage points that the optimal window size w varies much according to m rather than n, k and does not vary much according to α, if $\alpha \leq 0.1$. In view of these observations, our recommended values of w for different m are presented in Table 1.

3 Main results

Since the suggested test statistic is related to the hazard function of the distribution, we consider the following alternatives according to the type of hazard function as

(a) **Monotone increasing hazard including Gamma and Gexp (shape parameter 2) and Chi-square (degree of freedom 3),**

(b) **Monotone decreasing hazard including Gamma and Gexp (shape parameter 0.5) and Chi-square (degree of freedom 1), and**

(c) **Non-monotone hazard including Beta and Log-logistic (shape parameter 0.5) and Burr (shape1 and shape2 1).**

To estimated the power of proposed test statistic, We used 10,000 Monte Carlo simulations for $nk = 20(10)40$, each with own different $k's$, and some m under null hypothese. However, we understood following results when the alternative is either monotone decreasing hazard or monotone increasing hazard functions:

(a) **Censoring scheme** $R = (m, \ldots, 0)$ **and** $R = (0, m, \ldots, 0)$ **show better power than other censoring schemes when the alternative is a monotone increasing hazard function.**

(b) **It is observed that for fixed n and k, as m increases the power is improved but when k increases the power is decreased .**

(c) **for** $nk = (20, 30, 40)$**, the best power is shown at** $k = 2$ **and** $m = (7, 10, 15, 20)$ **respectively.**

References

[1] Johnson, L.G. (1964), Theory and Technique of Variation Research. *Elsevier*, Amsterdam .

[2] Wu, S. J. and Ku, C. (2009), On estimation based on progressive first-failure-censored sampling, *Computational Statistics and Data Analysis*, **53**, 3659-3670.

[3] Balakrishnan, N. and Rad, A. and Arghami, N. R. (2007),Testing Exponentiality Based on Kullback-Leibler Information With Progressively Type-II Censored Data, *IEEE Transactions on Reliability*, **56**, 301-307.

A Representation of the Residual Lifetime of a Repairable System

Chahkandi, M. [1] and Ahmadi, J. [2]

[1] Department of Statistics, University of Birjand
[2] Department of Statistics, Ferdowsi University of Mashhad

Abstract

In this paper, the residual lifetime of a repairable system is studied when the failure status of the system is known. A mixture representation of the reliability function of the conditional residual lifetime of a repairable system in terms of the reliability function of residual records is provided. Some stochastic properties of the conditional probabilities and the residual lifetimes also are given.

Keywords: Aging properties, Minimal repair, Residual lifetime, Stochastic ordering.

1 Introduction

For a repairable system, carrying out minimal repairs is a natural approach, because it can keep the system working at a minimal cost. That is, minimal repair restores the system to its functioning condition just prior to failure with the failure rate of the system remaining undisturbed. Many authors have been followed this model, see e.g., for example [1], [2], [3], [4] and [5]. The present paper explores some applications of the residual life of record values in analysis of a repairable system. For this purpose we consider a repairable system with minimal repairs, whose number of repairs is a positive random variable with a given probability vector. We obtain some mixture representations for residual lifetime of a repairable system and compare two systems. For briefness, we just mention the following orders for comparison of arithmetic random variables.

[1] ma.chahkandi@yahoo.com
[2] ahmadi-j@um.ac.ir

Definition 1. *If X and Y are discrete random variables taking on values in $\mathbb{Z}^+$, with distributions $\mathbf{p} = (p_1, p_2, \ldots)$ and $\mathbf{q} = (q_1, q_2, \ldots)$, where $p_i = Pr(X = i)$ and $q_i = Pr(Y = i)$, $i \in \mathbb{Z}^+$. Then*

1. *$X \leq_{st} Y$ if and only if $\sum_{j=i}^{\infty} p_j \leq \sum_{j=i}^{\infty} q_j$, for all $i \in \mathbb{Z}^+$;*
2. *$X \leq_{hr} Y$ if $\sum_{j=i}^{\infty} p_j / \sum_{j=i}^{\infty} q_j$ is decreasing in i, for all $i \in \mathbb{Z}^+$;*
3. *$X \leq_{lr} Y$ if p_i/q_i is decreasing in i, for all $i \in \mathbb{Z}^+$ when $p_i, q_i > 0, \forall i \in \mathbb{Z}^+$.*

We refer the reader to [6] and [7] for more details on stochastic orderings and their applications.

2 Model description

Consider a repairable coherent system under the condition that, at time t, some information about the status of the system lifetime is available. Suppose the system can be repaired $N-1$ times and the Nth failure is fatal to the system with probability vector $\mathbf{p}$, where

$$\mathbf{p} = (0, \ldots, 0, p_{\ell_1}, p_{\ell_1+1}, \ldots, p_{\ell_2-1}, p_{\ell_2}), \ \ell_1 = 1, 2, \ldots, \ell_2, \quad \ell_2 = 1, 2, \ldots, n. \tag{1}$$

Chahkandi et al. [8] investigated some reliability properties of this model. Sometimes, an operator may know that, at time $t > 0$, the system is still operating, i.e. $T > t$, which is related to the residual lifetime of the system. Thus, we are interested in the probability that the system can be repaired $(i-1)$ times by assuming that $T > t$. Let us denote $\Pr(T = T(i)|T > t)$ by $b_i(t)$, then we have

$$b_i(t) = \frac{p_i \bar{F}_{T(i)}(t)}{\sum_{j=\ell_1}^{\ell_2} p_j \bar{F}_{T(j)}(t)} \quad i = \ell_1, \ldots, \ell_2. \tag{2}$$

3 Mixture representations

Navarro et al. [9] considered the residual lifetime of a coherent system and obtained a mixture representation for the system's residual lifetimes in terms of the order statistics of its components. Here, we consider the situation that one may have some partial information about the system lifetime and interested in finding the dynamic probability of system failure. We derive a mixture representation for the survival function of used but working repairable system, i.e. for distribution function of the system lifetime T given that its lifetime is greater than t, $(T - t|T > t)$. In the next result, we obtain a mixture representation for the residual lifetime of a repairable system in terms of the record values of its original distribution.

Theorem 1. *If T is the lifetime of a repairable system that can be repaired $(N-1)$ times and the Nth failure is fatal to the system with probability vector $\mathbf{p}$. Then, for all $x \geq 0$ and $t > 0$, such that $\bar{F}_T(t) > 0$, we have*

$$\Pr(T - t > x|T > t) = \sum_{i=\ell_1}^{\ell_2} b_i(t) \Pr(T(i) - t > x|T(i) > t), \tag{3}$$

where the coefficient $b_i(t)$ *is given in* (2) *such that* $\sum_{i=\ell_1}^{\ell_2} b_i(t) = 1$.

Consider two repairable systems with different probability vectors for their repairable numbers. For example two manufactures may guaranty their products with different number of repairs:

$$\mathbf{p} = (0, \dots, 0, p_{\ell_1}, p_{\ell_1+1}, \dots, p_{\ell_2-1}, p_{\ell_2})$$

and

$$\mathbf{q} = (0, \dots, 0, q_{\ell_1}, q_{\ell_1+1}, \dots, q_{\ell_2-1}, q_{\ell_2}),$$

for $\ell_1 = 1, 2, \dots, \ell_2$, $\ell_2 = 1, 2, \dots, n$, respectively. Suppose two systems are repairable $(N-1)$ and $(M-1)$ times, and the Nth and Mth failures are fatal to the systems with probability vectors $\mathbf{p}$ and $\mathbf{q}$, respectively. Take

$$\mathbf{b}(t) = (0, \dots, 0, b_{\ell_1}(t), \dots, b_{\ell_2}(t)), \tag{4}$$

where $b_i(t)$ is given as in (2). In this case, if $\mathbf{p} \le_{lr} \mathbf{q}$, then $\mathbf{b}_{\mathbf{p}}(t) \le_{st} \mathbf{b}_{\mathbf{q}}(t)$, where $\mathbf{b}_{\mathbf{p}}(t)$ is the dynamic probability vectors corresponding to $\mathbf{p}$.

Theorem 1 shows that the reliability function of the residual lifetime of a repairable system can be expressed in terms of a weighted summation of record values' residual lifetimes. Here, we consider a situation in which the system is alive after a known number of repairs and investigate its residual. For this purpose, we study the residual lifetime of a repairable system when the system is working, and at least $k-1$ repairs are done on the system at time t; namely the conditional random variable $[T - t | T > t, T(k) \le t]$, $k = \ell_1, \dots, \ell_2 - 1$. The next theorem presents a mixture representation for the conditional residual lifetime of $[T - t | T > t, T(k) \le t]$, $k = \ell_1, \dots, \ell_2 - 1$.

Theorem 2. *Consider a repairable system with probability vector* $\mathbf{p}$*, as in* (1)*, for the random maximum number of minimal repairs that can be performed. If* $\Pr(T > t, T(k) \le t) > 0$*, then*

$$\Pr(T - t > x | T > t,\, T(k) \le t) = \sum_{i=k+1}^{\ell_2} b_i(t,k) \Pr(T(i) - t > x | T(i) > t, T(k) \le t), \tag{5}$$

where $\mathbf{b}(t,k) = (0, \dots, 0, b_{k+1}(t,k), \dots, b_{\ell_2}(t,k))$*, with*

$$b_i(t,k) = \frac{p_i \Pr(T(k) \le t < T(i))}{\sum_{j=k+1}^{\ell_2} p_j \Pr(T(k) \le t < T(j))}, \tag{6}$$

such that $\sum_{j=k+1}^{\ell_2} b_j(t,k) = 1$.

References

[1] Park, K. S. (1979). Optimal number of minimal repairs before replacement. *IEEE Trans. Reliab.*, **28**, 137–140.

[2] Nakagawa, T. and Kowada, M. (1983). Analysis of a system with minimal repair and its application to replacement policy. *Eur. J. Oper. Res.*, **12**, 176–182.

[3] Kijima, M. (1989). Some results for repairable systems with general repair. *J. Appl. Prob.*, **26**, 89–102.

[4] Aven, T. and Jensen, U. (2000). A general minimal repair model. *J. Appl. Prob.*, **37**, 187–197.

[5] Finkelstein, M. (2008). *Failure Rate Modelling for Reliability and Risk*. Springer: London.

[6] Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*, Wiley: Chichester.

[7] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer: New York.

[8] Chahkandi, M., Ahmadi, J. and Baratpour, S. (2014). Some results for repairable systems with minimal repairs. *Appl. Stoch. Models Bus. Ind.*, **30**, 218–226.

[9] Navarro, J., Balakrishnan, N. and Samaniego, F. J. (2008). Mixture representations of residual lifetimes of used systems, *J. Appl. Probab.*, **45**, 1097–1112.

A Switch Model in Redundant Systems

Chahkandi, M. [1] and Ahmadi, J. [2]

[1] Department of Statistics, University of Birjand
[2] Department of Statistics, Ferdowsi University of Mashhad

Abstract

Redundancy is a technique that has been widely applied to improve the system reliability and its availability. In this paper, a new switching model is proposed to increase the reliability of a unit (system) with a cold standby backup. It is assumed that the switch over to the standby unit is not failure-free, contrary to what we have in standby redundancy. The optimal time to switch between the key unit and its cold standby backup is find such that the mean lifetime of the system to be maximized. Finally, an example is presented to compare the mean lifetime of the proposed switching model and a system with parallel redundancy.

Keywords: Parallel system, Redundancy, Survival function, Switching.

1 Introduction

Redundancy is a common method to increase system reliability. There are various methods, techniques, and terminologies for implementing the redundancy. Standby redundancy is one of the main methods. In general, there are three types of standby, i.e. cold, hot and warm standby. In cold standby, the secondary unit is powered off, thus preserving the reliability of the unit. In hot standby, an inactive unit undergoes the same operational environment as when it is in active state. Warm standby is an intermediate case. In this case an inactive unit undergoes operational environment that is milder than the environment of the same component in active state. The performance of the standby system was studied by some of researchers such as [1], [2], [3], [4] and [5]. For the simplicity of the standby redundancy models, we assume that the switch over to the standby unit is perfect, i.e. instantaneous and failure-free. But there are some real situations that we

[1]ma.chahkandi@yahoo.com
[2]ahmadi-j@um.ac.ir

haven't any time to switch the failed unit to its backup. Because after the unit failure, the system would be failed. Here, we focus on this case, i. e. the case that the switch over to the standby unit is not failure-free. In these situations, we can use the dual modular redundancy (DMR) or parallel redundancy, where the key component (system) and its backup begins to operate together, to increase the system reliability. But, parallel redundancy increases the cost and complexity of the system. Because, the lifetime of the key unit and its backup decrease simultaneously. In this paper we present a new model for a system with a cold standby component such that it is not required to work the key component and its backup continuously. In the next section, we present the new model that allows to switch between the key unit and its backup before the units failures. The optimal switching time for increasing (decreasing) failure rate distributions is obtained in Section 3. A parametric example is given to compare the mean lifetime of our new model and a system with parallel redundancy.

2 A new switching model

In this section we present a new switching model for a unit (system) with a cold standby backup. Consider a system consisting of two units A (the key unit) and B (the cold standby unit) connected in parallel branches, and a switcher (S) as shown in Figure 1.

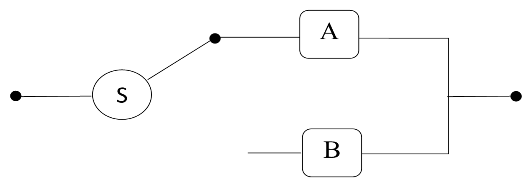


Figure 1: Standby system with two units.

At the beginning, the units are new with a good reliability. Thus, it may not be required to operate both of them. We consider a switcher key before the units to allows us to switch between the ones in specified times. First, the key components begins to operate and its back up is in an 'off' state. After a specified time t_1, a switch turns on the 'standby' backup (while the key component goes on 'off' state), and the system continues to operate. After specified time t_2 a switch turns on the key components and its backup together. This arrangement implies that the system may be failed before the first switch, when the key unit is working, between the first and second switch, when the backup unit is operating, or after the second switch, when both of the units are operating. For simplicity, suppose that $t_1 = t_2 = c$, and the units' lifetimes are the same, with distribution function F. After some manipulations, the survival function of the system can be expressed as

$$\bar{F}_{T_1}(t) = \begin{cases} \bar{F}(t) & t \leq c \\ \bar{F}(c)\,\bar{F}(t-c) & c < t \leq 2c \\ 2\bar{F}(c)\bar{F}(t-c) - \bar{F}^2(t-c) & t > 2c. \end{cases} \tag{1}$$

The failure rate function of T_1 also is given as

$$r_{T_1}(t) = \begin{cases} r(t) & t < c \\ r(t-c) & c < t < 2c \\ 2r(t-c)\left[\dfrac{\bar{F}(c) - \bar{F}(t-c)}{2\bar{F}(c) - \bar{F}(t-c)}\right] & t > 2c, \end{cases} \tag{2}$$

Table 1. The values of mean lifetimes

(λ, β)	c_0 (the optimal switching time)	$E(T_1)$	$E(T_2)$
(1, 0.5)	0.000	3.500	3.500
(1, 1)	0.000	1.500	1.500
(1, 1.5)	0.164	1.286	1.236
(1, 2)	0.347	1.303	1.145
(1, 3)	0.507	1.359	1.077
(1, 4)	0.586	1.472	1.037
(1, 5)	0.637	1.530	1.028
(1, 7)	0.703	1.616	1.023

where $r(t)$ is the units' failure rate function. Utilizing (1), the expectation lifetime of the system can be found as

$$E(T_1) = \int_0^c \bar{F}(t)dt + \bar{F}(c)[E(X) + \int_c^\infty \bar{F}(t)dt] - \int_c^\infty \bar{F}^2(t)dt = g(c). \tag{3}$$

We are interested in finding a value of c that maximize the mean lifetime of the system.

3 Main results

In this section, we find the optimal time of switching to maximize the mean lifetime of the system. In the next results the optimal time of switch is found in DFR and IFR distributions.

Theorem 1. *Let F be a DFR distribution, then the function $g(c)$ in (3) would be maximized at $c = 0$.*

Theorem 2. *Let F be an IFR distribution and $f(0) < \frac{1}{2\mu}$, where f is the density function of F and μ is its mean, then the function $g(c)$ in (3) takes its maximum value at a point on its domain (on the interval $[0, \infty)$) not on the boundary points.*

Now, we compare the mean lifetime of the system in switch model, $E(T_1)$, and the mean life of a parallel system with two units, $E(T_2)$. $E(T_1)$ is obtained in equation (3) and $E(T_2)$ can be obtained as

$$E(T_2) = 2\mu - \int_0^\infty \bar{F}^2(t)dt,$$

where X_1, X_2 are the units' lifetime with distribution F and mean μ. The failure rate function of T_2 also is given by

$$r_{T_2}(t) = 2r(t)a(t), \tag{4}$$

where $a(t) = \frac{F(t)}{1+F(t)}$. Note that $0 \leq a(t) \leq 0.5$, and is an increasing function of t. By comparing the equations (3) and (4), it would be found that $E(T_2) = g(0)$. In the next example we find the optimal time for our switching model when the units' distribution lifetimes are Weibull. Consider the following distribution for the components' lifetimes of the system given in Figure 1

$$\bar{F}(t; \lambda, \beta) = e^{-(\lambda t)^\beta}.$$

Table 1 confirms the results of Theorems 1 and 2.

The failure rate functions of T_1 and T_2 are also plotted in Figure 2 for $\lambda = 1$, $\beta = 3$ and $c_0 = 0.507$. Note that the failure rate of the new switching model is less than the failure rate of a parallel system after c_0.

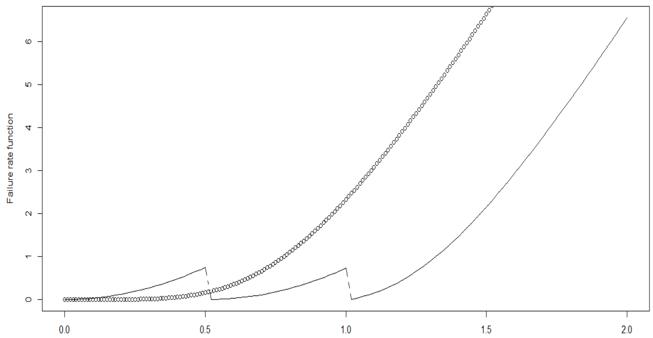


Figure 2: The failure rate functions of T_1 (thick line) and T_2 (dotted line) for the Weibull distribution.

References

[1] Vanderperre E. J. (2004), Reliability analysis of a renewable multiple cold standby system, *Operations Research*, **32**, 288–292.

[2] Cha, J. H., Mi, J. and Yun, W. Y. (2008), Modelling a general standby system and evaluation of its performance, *Applied Stochastic Models in Business and Industry*, **24**, 159–169.

[3] Li, X., Zhang, Z. and Wu, Y. (2009), Some new results involving general standby systems, *Applied Stochastic Models in Business and Industry*, **25**, 632–642.

[4] Van Gemund, A. J. C. and Reijns, G. L. (2012), Reliability analysis of k-out-of-n systems with single cold standby using Pearson distributions, *IEEE Transactions on Reliability*, **61**, 526–532.

[5] Eriylmaz, S. (2013), Reliability of a k-out-of-n System Equipped With a Single Warm Standby Component, *IEEE Transactions on Reliability*, **62**, 499-503.

On Additive-Multiplicative Hazards Model

Esna-Ashari, M. [1] and Asadi, M. [2]

Department of Statistics, University of Isfahan

Abstract

In survival analysis and reliability theory, a fundamental problem is the study of lifetime properties of a live organism or system. In this regard, there have been considered and studied several models based on different concepts of aging such as hazard rate and mean residual life. In this paper, we consider an additive-multiplicative hazard model (AMHM) and study some of reliability and aging properties of the proposed model. We then specify the bivariate models whose conditionals satisfy AMHM. Several properties of the proposed bivariate model are investigated.

Keywords: Conditionally specified distributions, Bivariate Pareto distribution, Additive hazard, Multiplicative hazard.

1 Introduction

In order to study the lifetime properties of a live organism or system, different approaches have been considered in the literature. In the the context of reliability and survival analysis some of approaches are based on aging concepts such as hazard rate, reversed hazard rate, mean residual life etc. Among the well known models, one can refer to proportional hazards model, proportional mean residual lives model, proportional odds ratio etc, see, for example, Cox (1972), Navarro et al. (2015)). Assume that X denotes the lifetime of a live organism or a device. In the study of aging and stochastic of the system in addition to the main variable (X), to be more realistic one has to consider other observable or unobservable random variable (covariate) which effects the aging characteristics of X such as hazard rate, reversed hazard rate, mean residual life, odds ratio etc. In many applications, the effect of some covariates to the lifetime characteristics are additive while others are multiplicative. There are situation where the effect of covariates on the lifetime

[1] m.esnaashari.stat@gmail.com
[2] m.asadi@sci.ui.ac.ir

characteristic are both the additive and multiplicative (see, for example, Cox (1972), Unnikrishnan and Sankaran (2012)). In additive-multiplicative hazard model (AMHM), one considers a baseline hazard rate, $r(x)$, corresponding to a non-negative random variable and a random variable, Z, representing the covariate with an additive-multiplicative effect on $r(x)$. In other words, in AMHM it is assumed that there are non-negative functions $a(.)$ and $b(.)$ such that the conditional hazard rate $r(x)$, given that $Z = z$, is expressed as:

$$r(x \mid z) = b(z)r(x) + a(z), \tag{1}$$

The different choices for distribution function of Z and flexibility of choosing functions $a(z)$ and $b(z)$ provide situations that one can have a very flexible model to describe variate phenomena in reliability and survival analysis. In particular the model has proportional hazards model as a special case when $a(z) \equiv 0$ and it reduces to the additive hazard model when $b(z) \equiv 1$, which is recently studied by Unnikrishnan and Sankaran (2012). This paper is an investigation on different aging and stochastic properties of the model in (1).

2 Additive−multiplicative hazards model

In this section, we study several properties of AMHM in (1). Note that, if X^* is a random variable satisfying to (1), the survival function of X^* is represented as

$$S^*(x \mid z) = S^{b(z)}(x)\exp(-xa(z)), \quad x, z > 0. \tag{2}$$

The marginal distribution of X^* is given as follows

$$S^*(x) = \int_0^\infty S^{b(z)}(x)\exp(-xa(z))\ g(z)dz. \tag{3}$$

From (6) the joint density of (X^*, Z) is

$$f^*(x, z) = (b(z)r(x) + a(z))S(x)g(z), \quad x, z > 0. \tag{4}$$

It can be easily seen that the following expression is equivalent to (3)

$$r^*(x) = r(x)\frac{\int_0^\infty b(z)\ S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz}{\int_0^\infty S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz} + \frac{\int_0^\infty a(z)\ S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz}{\int_0^\infty S^{b(z)}(x)\ e^{-xa(z)}\ g(z)dz}. \tag{5}$$

Proposition 1. *Suppose that $S(x)$ is a baseline survival function and X^* has the CDF (3) for some functions $a(z)$ and $b(z)$. If both $a(z)$ and $b(z)$ are increasing or both are decreasing, then*

$$r^*(x) \le r(x)E(b(Z)) + E(a(Z)).$$

In the following we give an example.

Example 1. Assume that Y has generalized gamma distribution with density function

$$g(z) = (\Gamma(\alpha))^{-1}c\lambda^{c\alpha}z^{c\alpha-1}\exp(-(\lambda z)^c), \quad z > 0.$$

Now assume that $a(z) = b(z) = z^c,\ c > 0$. Then, we have

$$S^*(x) = \left(\frac{\lambda^c}{x - \ln S(x) + \lambda^c}\right)^\alpha, \quad x, \lambda, c > 0,$$

or, equivalently, $r^*(x) = \frac{\alpha(r(x)+1)}{x - \ln S(x) + \lambda^c}$.

Theorem 1. *Let $a(z)$ and $b(z)$ be increasing (decreasing) functions and also suppose that $S(x)$ is IFR distribution. Then, the joint density of X^* and Z is RR_2 (TP_2).*

Theorem 2. *Suppose that the functions $a(z)$ and $b(z)$ are both increasing or both decreasing.*

(a) If X^* (X) is IFRA (DFRA) , then so is X (X^*);

(b) If X^* (X) is NBU (NWU), then so is X (X^*);

Theorem 3. *In two AMHM's, suppose that $Z_1 \stackrel{d}{=} Z_2$, where $\stackrel{d}{=}$ stands for equality in distribution. Then*

(i) $X_1 \leq_{st} X_2$ if and only if $X_1^* \leq_{st} X_2^*$.

(ii) If $X_1 \geq_{cx} X_2$,then $X_1^* \geq_{cx} X_2^*$.

Theorem 4. *In two AMHM's, suppose that $X_1 \stackrel{d}{=} X_2$. If $Z_1 \leq_{st} Z_2$ and the functions $a(z)$ and $b(z)$ are both decreasing (increasing), then $X_1^* \leq_{st} X_2^*$ ($\geq_{st}$).*

Constructing bivariate distributions based on conditional distributions is a subject that has been explored by many researchers (see, e.g., Arnold et al. (1993)). In the sequel, we study the bivariate models whose conditional distributions satisfy in AMHM. That is, we are interested in specifying the joint distribution for (X, Y) such that the following conditions are met.

$$P(X > x \mid Y > y) = S_1^{b_1(y)}(x) \exp(-xa_1(y)), \quad x, y > 0, \tag{6}$$

and

$$P(Y > y \mid X > x) = S_2^{b_2(x)}(y) \exp(-ya_2(x)), \quad x, y > 0. \tag{7}$$

Theorem 5. *Suppose that $S_1(x)$ and $S_2(y)$ are baseline reliability functions of X and Y, respectively, and let (X, Y) has common support $(0,\infty) \times (0,\infty)$. Then, the bivariate reliability function with conditionals satisfying (6) and (7) is given by.*

$$S(x, y) = S_1^{\lambda_1 y+\lambda_2}(x) S_2^{\lambda_3 x+\lambda_4}(y) \exp\Big(-(\lambda_5 \ln S_1(x) \ln S_2(y) + \lambda_6 x + \lambda_7 y + \phi\lambda_6\lambda_7 xy) \Big), \tag{8}$$

for $x, y > 0$, where λ_i, ϕ is nonnegative constants.

In the follwing, we study some reliability and aging properties of the model in (8) in special case when $b_1(y) = b_2(x) = 1$. In other words we are absorbed in specifying the joint distribution for (X, Y) such that the following conditions be satisfied.

$$P(X > x \mid Y > y) = S_1(x) \exp(-xa_1(y)), \quad P(Y > y \mid X > x) = S_2(y) \exp(-ya_2(x)). \tag{9}$$

In this case the the joint reliability function of (X, Y) has the from

$$S(x, y) = S_1(x)S_2(y) \exp(-(\lambda_1 x + \lambda_2 y + \phi\lambda_1\lambda_2 xy)), \tag{10}$$

where λ_1, λ_2 and ϕ are nonnegative constants. The choice of $b_1(y) = b_2(x) = 1$, enable us to have more insight to the properties of the joint distribution of (X, Y). First note

that, from definitions of IFRA and NBU, one can easily conclude that when S_1 and S_2 are univariate IFRA, then (X, Y) is bivariate IFRA and when S_1 and S_2 are both univariate NBU property, then (X, Y) is bivariate NBU property.
From (10), the marginal distributions of X and Y are given respectively by

$$S_X(x) = S(x, 0) = S_1(x)\exp(-\lambda_1 x), \quad S_Y(y) = S(0, y) = S_2(y)\exp(-\lambda_2 y)$$

for $x \geq 0$, $y \geq 0$. Hence, the marginal distributions belong to additive hazards model. From these, under the assumption that the derivatives exist, the marginal hazards rates of X and Y are given by

$$r_X(x) = r_1(x) + \lambda_1, \quad r_Y(y) = r_2(y) + \lambda_2. \tag{11}$$

Hence, $r_X(x)(r_Y(y))$ is increasing (decreasing) iff $r_1(x)(r_2(y))$ is increasing (decreasing). The conditional survival functions are also given by

$$P(X > x \mid Y > y) = S_1(x)\exp(-(\lambda_1 + \phi\lambda_1\lambda_2 y)x)$$

and

$$P(Y > y \mid X > x) = S_2(y)\exp(-(\lambda_2 + \phi\lambda_1\lambda_2 x)y)$$

for $x, y \geq 0$. That is , the conditional survival functions belong to additive hazard model, given in (9) , with both $a_1(y)$ and $a_2(x)$increasing functions and

$$a_1(y) = \lambda_1 + \phi\lambda_1\lambda_2 y, \qquad a_2(x) = \lambda_2 + \phi\lambda_1\lambda_2 x, \tag{12}$$

From (10), the bivariate PDF can be expressed as

$$f(x, y) = \Big((r_1(x) + a_1(y))(r_2(y) + a_2(x)) - \phi\lambda_1\lambda_2\Big)S(x, y), \tag{13}$$

where $r_1(x)$ and $r_2(y)$ are the hazards rate of the baseline distributions S_1 and S_2, respectively.

Theorem 6. *The joint survival function defined in (10) is RR_2 on $(0, \infty) \times (0, \infty)$.*

Theorem 7. *Suppose that the baseline distributions S_1 and S_2 are IFR. Then, the joint PDF of (X, Y) obtained in expression (13) is RR_2 on $(0, \infty) \times (0, \infty)$.*

References

[1] Arnold, B.C., Castillo, E., Sarabia, J.M., 1993. Multivariate distributions with generalized Pareto conditionals. Statistics Probability Letters 17, 361-368.

[2] Cox, D.R., 1972. Regression models and life tables (with discussion). Journal of the Royal Statistical Society Series 34, 187-220.

[3] Navarro, J., Esna-Ashari, M., Asadi, M., Sarabia, J.M., 2015. Bivariate distributions with conditionals satisfying the proportional generalized odds rate model. Metrika DOI 10.1007/s00184-014-0523-7.

[4] Unnikrishnan Nair, N., Sankaran, P.G., 2012. Some results on an additive hazards model. Metrika 75, 389-402.

Estimation with Non-Homogeneous Sequential K-out-of-N System Lifetimes

Hashempour, M. [1] and Doostparast, M. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

In this paper, sequential k-out-of-n systems with coming non- homogeneous exponential component lifetimes are considered. Point estimates of parameters as well as equal-tail and approximate confidence intervals and Fisher Information are derived on the basis of observed multiply system lifetimes.

Keywords: Bayesian approach, Estimation, Maximum likelihood, Sequential order statistics.

1 Introduction

Kamps [7] introduced the concept of the *sequential order statistics* (SOSs), as an extension of the (usual) order statistics (OSs). SOSs may be used for modelling lifetimes of sequential r-out-of-n systems. Specifically, in (usually) the k-out-of-n system failing a component does not change the lifetimes of the surviving components. Motivated by Cramer and Kamps [3, 4], in practice, the failure of a component may results in a higher load on the remaining components and hence causes the distribution of the surviving components change. In these cases, the system lifetimes may be modelled by SOSs. The mentioned system is called *sequential r-out-of-n system* and the system lifetime is then r-th component failure time, denoted by $X^{\star}_{(r)}$. In the literature, $(X^{\star}_{(1)}, \cdots, X^{\star}_{(n)})$ is called SOSs; See, Kamps [7]. The problem of estimating parameters on the basis of SOS has been considered in the literature. For example, Cramer and Kamps [3] considered the problem of estimating the parameters on the basis of s independent SOSs samples under a

[1]ma.hashempour@stu.um.ac.ir
[2]doustparast@um.ac.ir

proportional hazard rates (PHR) model, defined by $\bar{F}_j(t) = \bar{F}_0^{\alpha_j}(t)$ for $j = 1, \cdots, r$, where the underlying CDF $F_0(t)$ is the exponential distribution, i.e.

$$F_0(x;\sigma) = 1 - \exp\left\{-\left(\frac{x}{\sigma}\right)\right\}, \quad x > 0, \quad \sigma > 0. \tag{1}$$

In this case, the hazard rate function of the CDF F_j, for $t > 0$ and $j = 1, \cdots, n$, is $h_j(t) = \alpha_j h_0(t)$. See also, Schenk *et al.* [9], Esmailian and Doostparast [5], Beutner and Kamps [1] and references therein.

2 Main results

We assume that $s \geq 2$ independent SOS samples of equal size r from s non-homogeneous populations are available. The data may be represented by $[[x_{ij}]]_{i=1,\cdots,s,j=1,\cdots,r}$ where the i-th row of the matrix $\mathbf{x}$ in (2) denotes the SOS sample coming from the i-th population. The LF of the available data is

$$L(\underline{F_j^{[i]}};\mathbf{x}) = B^s \prod_{i=1}^{s}\left(\prod_{j=1}^{r-1}\left[f_j^{[i]}(x_{ij})\left(\frac{\bar{F}_j^{[i]}(x_{ij})}{\bar{F}_{j+1}^{[i]}(x_{ij})}\right)^{n-j}\right] f_r^{[i]}(x_{ir})\bar{F}_r^{[i]}(x_{ir})^{n-r}\right), \tag{2}$$

where $B = \Gamma(n+1)/\Gamma(n-r+1)$, and for $i = 1, \cdots, s$, $j = 1, \cdots, r$. By substituting Equation (1) into Equation(2), under the earlier mentioned PHR model, the LF of the available data reduces to

$$L(\underline{\sigma};\mathbf{x}) = A^s\left(\prod_{j=1}^{r}\alpha_j\right)^s\left(\prod_{i=1}^{s}\frac{1}{\sigma_i}\right)^r \exp\Big\{-\sum_{i=1}^{s}\sum_{j=1}^{r}\Big(\frac{x_{ij}m_j}{\sigma_i}\Big)\Big\}. \tag{3}$$

where $m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1}$, for $j = 1, \cdots, r$, with convention $\alpha_{r+1} \equiv 0$. For sake of brevity, we assumed that the proportional parameter vector $\boldsymbol{\alpha}$ are the same among the s sequential r-out-of-n systems. First suppose that the vector parameter $\boldsymbol{\alpha}$ in Equation (3) is known. The solutions of the likelihood equations yields the ML estimate of σ_i $(i = 1, \cdots, s)$ as

$$\hat{\sigma}_i = \frac{\sum_{j=1}^{r} x_{ij}m_j}{r} = \frac{\sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij}}{r}, \tag{4}$$

where $D_{ij} = x_{ij} - x_{i,j-1}$, for $j = 1, \cdots, r$. Cramer and Kamps [4] showed that under the PHR with the one-parameter exponential baseline CDF,

$$T_i = \sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} \sim \Gamma(r,\sigma_i), \quad i = 1, \cdots, s, \tag{5}$$

where $\Gamma(a,b)$ calls for the gamma distribution. Thus, for $i = 1, \cdots, s$, $\hat{\sigma}_i \sim \Gamma(r, \sigma_i/r)$, and then $E(\hat{\sigma}_i) = \sigma_i$ and $Var(\hat{\sigma}_i) = \sigma_i^2/r$. From Equation (5), $2r\,(\hat{\sigma}_i/\sigma_i) \sim \chi_{2r}$, where χ_ν stands for the chi-square distribution with ν degrees of freedom. So, an equal-tail confidence interval at level $100\gamma\%$ for σ_i $(i = 1, \cdots, s)$ is

$$\left(\frac{2r\hat{\sigma}_i}{\chi_{2r,(1+\gamma)/2}}, \frac{2r\hat{\sigma}_i}{\chi_{2r,(1-\gamma)/2}}\right), \tag{6}$$

where $\chi_{\nu,p}$ calls for the p-th percentile of the χ_ν-distribution. The observed FI, denoted by $[i(\hat{\sigma}_1, \cdots, \hat{\sigma}_s)]$, on the basis of available SOSs data is equal to minus of the Hessian matrix (HM) evaluated at the MLEs of the parameters, i.e. $i(\hat{\sigma}_1, \cdots, \hat{\sigma}_s) = [[(-\partial^2 \log(L)/\partial\sigma_i\partial\sigma_j)_{1\leq i,j\leq s}]]|_{\sigma_1=\hat{\sigma}_1,\cdots,\sigma_s=\hat{\sigma}_s}$. It is well known that the MLEs have asymptotically normal distribution with mean σ and the variance $[i(\hat{\sigma}_1, \cdots, \hat{\sigma}_s)]^{-1}$. Therefore, an approximate equi-tailed confidence interval for σ_i is

$$\left(\hat{\sigma}_i - z_{\gamma/2}\sqrt{\frac{\hat{\sigma}_i^2}{r}}\ ,\ \hat{\sigma}_i + z_{\gamma/2}\sqrt{\frac{\hat{\sigma}_i^2}{r}}\right), \tag{7}$$

where z_γ stands for the γ-percentile of the standard normal distribution. When the vector parameter $\boldsymbol{\alpha}$ in Equation (3) is unknown, see, e.g., Cramer and Kamps [4] and Hashempour and Doostparast [6].

We here consider the problem of estimating unknown parameters via a strict Bayesian approach. To do this, we assume that $\boldsymbol{\alpha}$ is known and suggest the conjugate prior distributions for the scale parameters $\sigma_i, i = 1, \cdots, s$, i.e.

$$\sigma_i \sim IG(a_i, b_i),\ i = 1, \cdots, s, \tag{8}$$

From Equation (8) and the LF (3), the joint posterior density function of $\sigma_1, \ldots, \sigma_s$ is

$$\pi(\sigma_1, \ldots, \sigma_s \mid \underline{\mathbf{x}}) \propto \prod_{i=1}^{s}\left(\prod_{j=1}^{r}\alpha_j\frac{b_i^{a_i}\sigma_i^{-(a_i+r)-1}}{\Gamma(a_i)}\exp\left\{-\left(\frac{\sum_{j=1}^{r}x_{ij}m_j + b_i}{\sigma_i}\right)\right\}\right). \tag{9}$$

which implies $\sigma_i \mid \underline{\mathbf{x}} \sim IG\left(a_i + r, \sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} + b_i\right),\ i = 1, \cdots, s$. As we expected given $\underline{\mathbf{x}}$, the parameter σ_i are independent. Under the squared error loss (SEL) function, the Bayes estimate of the parameter $\sigma_i (i = 1, \cdots, s)$ is

$$\hat{\sigma}_{i,B} \ = \ \frac{\sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} + b_i}{a_i + r - 1} = \frac{r\hat{\sigma}_i + b_i}{a_i + r - 1}, \tag{10}$$

where $\hat{\sigma}_i$ is the ML estimate of σ_i given by Equation (4). For $i = 1, \cdots, s$, the Bayes estimates (10) is bias, admissible and may be written as a weighted mean of the mean of the prior (8) and the ML estimate (4). The risk function of the Bayes estimates (10) is

$$R(\hat{\sigma}_{i,B}, \sigma_i) = \frac{\sigma_i^2\left(r + (1-a_i)^2\right) + 2b_i\left(1-a_i\right)\sigma_i + b_i^2}{(a_i + r - 1)^2}, \tag{11}$$

which its minimum, as a function of σ_i, occurs at point $b_i(a_i-1)/[(1-a_i)^2+r]$. Notice for $r = n$ and $\alpha_1 = \cdots = \alpha_n = 1$, $\hat{\sigma}_{i,n} = \sum_{j=1}^{n} x_{ij}/n$ and $\hat{\sigma}_{i,B} = (\sum_{j=1}^{n} x_{ij} + b_i)/(a_i + n - 1)$, which are, respectively, the well-known ML and the Bayes estimates of the exponential parameters on the basis of a random sample of size n; See, e.g., Lawless [8].

References

[1] Beutner, E. and Kamps, U. (2009), Order restricted statistical inference for scale parameters based on sequential order statistics, *Journal of Statistical Planning and Inference,* **139**, 2963-2969.

[2] Burkschat, M. , kamps, U. and Kateri, M. (2010), Sequential order statistics with an order statistics Prior, *Journal of Multivariate Analysis,* **101**, 1826-1836.

[3] Cramer, E. and Kamps, U. (1996), Sequential order statistics and k-out-of-n systems with sequentially adjusted failure rates, *Annals of the Institute of Statistical Mathematics,* **48**(3), 532-542.

[4] Cramer, E. and Kamps, U. (2001), Estimation with Sequential order statistics from exponential distributions, *Annals of the Institute of Statistical Mathematics,* **53**(2), 307-324.

[5] Esmailian, M. and Doostparast, M. (2014), Estimation based on sequential order statistics with random removals, *Probability and Mathematical Statistics* **43**(1), 81-95.

[6] Hashempour, M. and Doostparast,M. (2015), Inference based on multiply sequential order statistics from heterogeneous exponential populations, *Communications in Statistics-Theory and Methods.* to appear.

[7] Kamps, U. (1995), *A concept of generalized order statistics. Journal of Statistical Planning and Inference,* **48**, 1-23.

[8] Lawless, J. F. (2003), *Statistical models and methods for lifetime Data*, John Wiley & Sons, Inc., Hoboken, New Jersey.

[9] Schenk, N. , Burkschat, M. , Cramer, E. and Kamps, U. (2011), *Bayesian estimation and prediction with multiply type-II censored samples of sequential order statistics from one-and two-parameter exponential distributions. Journal of Statistical Planning and Inference,* **141**, 1575-1587.

Kaplan-Meier Estimator for Associated Random Variables Under Left Truncation and Right Censoring

Jabbari, H. [1]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

It is assumed that in long term studies the lifetimes are positively (negatively) associated random variables. Under some regular conditions, the strong convergence rates of Kaplan-Meier estimator of marginal distribution function F and cumulative hazard function Λ are obtained. In order to demonstrate the empirical performance of the results, simulation studies are done.

Keywords: Censored data, Kaplan-Meier estimator, Negative association, Positive association, Strong consistency, Truncation.

1 Introduction

Let $\{X_n, n \geq 1\}$ be sequence of the lifetime variables which may not be mutually independent, but have a common continuous marginal distribution function (df) F. Let $\{T_n, n \geq 1\}$ be a sequence of iid rv's with continuous df G. Suppose that the rv's X_i be censored on the right by the rv's Y_i, so that one observe only $Z_i = X_i \wedge Y_i$ and $\delta_i = I(X_i \leq Y_i)$ where $\wedge$ denotes minimum and $I(.)$ is the indicator function. In this random censorship model, the censoring times $Y_i,\ i = 1, ..., n$ are assumed to be iid rv's with df H and be independent of the X_i's and T_i's. The problem at hand is that of drawing nonparametric inference about F, based on the right censored and left truncated observations $(Z_i, T_i, \delta_i),\ i = 1, ..., n$. In the left truncated model, (Z_i, T_i) is observed only when $Z_i \geq T_i$. Let $\gamma \equiv P(T_1 \leq Z_1) > 0$. Assume, without loss of generality, that X_i, T_i and Y_i are nonnegative rv's, $i = 1, ..., n$. For any df L denotes the left and right endpoints of its support by $a_L = \inf\{x;\ L(x) > 0\}$ and $b_L = \sup\{x;\ L(x) < 1\}$, respectively. Then under the current model, we assume that $a_G \leq a_W$ and $b_G \leq b_W$, where W be the df of Z. Let $\Lambda(x)$ denotes the cumulative hazard function of F, $C(x) = P(T_1 \leq x \leq Z_1 | T_1 \leq Z_1) = \gamma^{-1} P(T_1 \leq x \leq Y_1) \times (1 - F(x))$

[1] Jabbarinh@um.ac.ir

and $W_1(x) = P(Z_1 \leq x, \delta_1 = 1|T_1 \leq Z_1)$. Let $C_n(x)$ and $W_{1n}(x)$ be the empirical estimators of $C(x)$ and $F_1(x)$, respectively, i.e. $C_n(x) = n^{-1}\sum_{i=1}^n I(T_i \leq x \leq Z_i)$ and $W_{1n}(x) = n^{-1}\sum_{i=1}^n I(Z_i \leq x,\ \delta_i = 1)$, where $F_1(x) = P(Z_1 \leq x,\ \delta_1 = 1)$. Then, the PL estimator of F and $\Lambda(x)$ are $\hat{F}_n(x) = \begin{cases} 1 - \prod_{Z_i<x}(1 - \frac{1}{nC_n(Z_i)})^{\delta_i} & ; x < Z_{(n)} \\ 1 & ; x \geq Z_{(n)} \end{cases}$ and $\hat{\Lambda}_n(x) = \sum_{i=1}^n \frac{I(Z_i \leq x,\ \delta_i=1)}{nC_n(Z_i)}$, respectively.

For independent failure time observations, the PL estimator has been studied extensively by many investigators. However, there are preciously few results available for the dependent case. Our focus in present paper is to study asymptotic properties of PL estimator for the right censored and left truncated data under PA (NA) failure times. So in Section 2, we introduce preliminary results and discuss strong uniform consistency and rates of convergence for the estimators $\hat{F}_n$ and $\hat{\Lambda}_n(x)$. Finally in Section 3, we use a simulation study to show the convergence rates. We now introduce general assumptions to be used throughout the article.

(A1). $\{X_n,\ n \geq 1\}$ is stationary sequence of PA (NA) rv's having bounded density function and finite second moment.

(A2). The censoring time variables $\{Y_n,\ n \geq 1\}$ and truncated time variables $\{T_n,\ n \geq 1\}$ are iid rv's with bounded density and are independent of $\{X_n,\ n \geq 1\}$.

(A3). $\sum_{j=2}^{\infty} j^{-2}\sum_{i=1}^{j-1}|Cov(X_i, X_j)|^{1/3} < \infty$.

(A4). $\sum_{j=n+1}^{\infty}|Cov(X_1, X_j)|^{1/3} = O(n^{-(r-2)/2})$, for some $r > 2$.

For the dfs F, G and H (the possibly infinite) times τ_F, τ_G and τ_H by $\tau_F = \inf\{y;\ F(y) = 1\}$, $\tau_G = \inf\{y;\ G(y) = 1\}$, $\tau_H = \inf\{y;\ H(y) = 1\}$, $a_F = \sup\{y;\ F(y) = 0\}$, $a_G = \sup\{y;\ G(y) = 0\}$ and $a_H = \sup\{y;\ H(y) = 0\}$. Then for the marginal df W of the Z_i's, it holds $\tau_W = \tau_F \wedge \tau_H$ and $a_W = a_F \wedge a_H$.

2 Strong uniform consistency with rates

In this section, we introduce preliminary and main results. Let $\{X_n,\ n \geq 1\}$ be a stationary sequence of rv's. Then:

i) If rv's are PA under (A3), $\frac{1}{n}\sum_{i=1}^n (X_i - EX_i) \to 0$ *a.s.*

ii) If rv's are NA with finite first moment, the convergence of (i) holds. Suppose that (A1) and (A2) hold. Then

i) If the rv's $\{X_i,\ i \geq 1\}$ are PA and (A3) is fulfilled, it holds

$$\sup_{a_w \leq x \leq \tau_W} |C_n(x) - G(x)[1 - W(x)]| \longrightarrow 0 \ \ a.s., \tag{1}$$

$$\sup_{a_w \leq x \leq \tau_W} |W_{1n}(x) - F_1(x)| \longrightarrow 0 \ \ a.s. \tag{2}$$

ii) If the rv's $\{X_n,\ n \geq 1\}$ are NA, (1) and (2) hold true.

Theorem 1. *Under (A1) and (A2) and for any $a_W < \tau < \tau_W$,*
i) If the rv's $\{X_i,\ i \geq 1\}$ are PA and (A3) is satisfied, it holds

$$\sup_{a_W \leq x \leq \tau_W} |\hat{\Lambda}_n(x) - \Lambda(x)| \longrightarrow 0. \qquad a.s. \tag{3}$$

ii) If the rv's $\{X_i,\ i \geq 1\}$ are NA, then (3) holds.

Theorem 2. *Under (A1) and (A2) and the additional assumptions either in part (i) or part (ii) of Theorem 1, it holds*

$$\sup_{a_W \le x \le \tau_W} |\hat{F}_n(x) - F(x)| \longrightarrow 0 \ \ a.s., \qquad \sup_{Z_{1:n} \le x \le Z_{n:n}} |\hat{F}_n(x) - F(x)| \longrightarrow 0 \ \ a.s.,$$

where $Z_{n:n} = \max_{i \le n} Z_i$ *and* $Z_{1:n} = \min_{i \le n} Z_i$.

Theorem 3. *Suppose that (A1), (A2) and (A4) hold. Then, for any* $a_W \le a < \tau \le \tau_W$,

$$\sup_{a \le x \le \tau} |\hat{\Lambda}_n(x) - \Lambda(x)| = O(n^{-\theta}) \ \ a.s., \tag{4}$$

where $0 < \theta < (r-2)/(2r+2+\delta)$, *for any* $\delta > 0$ *and* r *in (A4).*

Theorem 4. *Under the assumptions of Theorem 3, it follows*

$$\sup_{a \le x \le \tau} |\hat{F}_n(x) - F(x)| = O(n^{-\theta}). \ \ a.s. \tag{5}$$

3 Simulation study

In this section, we intend to compare our results with simulation of such generated NA (PA) data of size n=10(1)1000 to check the goodness convergence of the estimators. For generating NA data as introduced by Cai and Roussas (1998), we could use n-variate normal distribution with $\mu' = (12, 12, ..., 12)$ and

$$\Sigma = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho & -\rho^2 \dots & -\rho^{n-1} \\ -\rho & 1 & -\rho \dots & -\rho^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho^{n-1} & -\rho^{n-2} & -\rho & 1 \end{pmatrix}. \tag{6}$$

We have a vector with NA property when $\rho > 0$. We set $\rho = 0.2$ and the censored and truncation samples are generated from $N(13, 1)$ and $N(11, 1)$, respectively. So, we calculate $\hat{F}_n(x)$, $\hat{\Lambda}_n(x)$, $d_{F_n} = \sup_{a_W \le x \le \tau_W} |\hat{F}_n(x) - F(x)|$ and $d_{\Lambda_n} = \sup_{a_W \le x \le \tau_W} |\hat{\Lambda}_n(x) - \Lambda(x)|$ for some n. Figure 1 shows the results for this two functions against n and the green line is the convergence rates (4) and (5) using θ is equal to 0.27 and 0.12, respectively. In

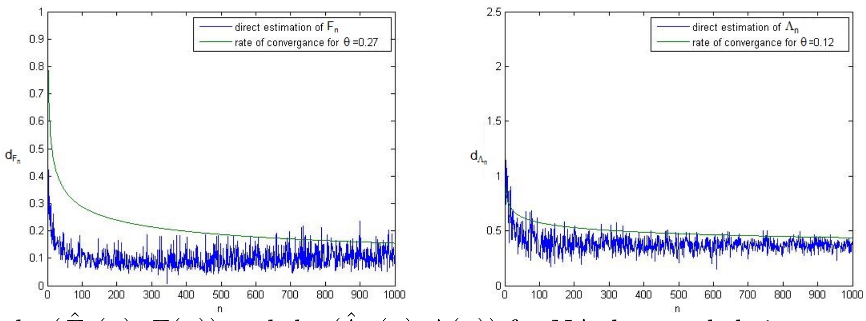


Figure 1: $d_{F_n}(\hat{F}_n(x), F(x))$ and $d_{\Lambda_n}(\hat{\Lambda}_n(x), \Lambda(x))$ for NA data and their convergence rates (green line).

both graphs of Figure 1, we can see that the convergence rates are reasonable in NA case i.e.:

(a) in the left graph the convergence rate could get sharper and this graph shows that the convergence behavior of $\hat{F}_n(x)$ is good.
(b) in the right graph however the convergence rate isn't reasonable as well as left graph, but it is good enough to present. Since $d_{\Lambda(x)} \in [0, +\infty)$, the differences more than one could be reasonable.
For generating PA sample, we follow the same way as in NA case with

$$\Sigma = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & \rho & \rho^2 \dots & \rho^{n-1} \\ \rho & 1 & \rho \dots & \rho^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho & 1 \end{pmatrix}. \tag{7}$$

We have a vector with PA property when $\rho > 0$. We set $\rho = 0.2$ and generate the censored and truncation samples as NA case. Figure 2 shows the trend of d_{F_n} and d_{Λ_n} with respect to n and the green line is the convergence rates (4) and (5) using θ is equal to 0.27 and 0.12, respectively. Figure 2 shows the same results as in NA case.

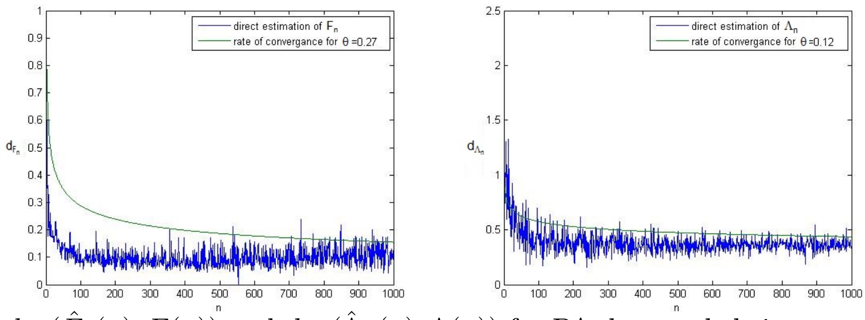


Figure 2: $d_{F_n}(\hat{F}_n(x), F(x))$ and $d_{\Lambda_n}(\hat{\Lambda}_n(x), \Lambda(x))$ for PA data and their convergence rates (green line).

References

[1] Cai, Z.W. and Roussas, G.G., (1998). Kaplan-Meier estimator under association, *J. Multivariate Anal.*, 67, 318-348.

Reliability Estimation in Burr X Distribution Based on Fuzzy Lifetime Data

Jafari, A.A. [1] and Pak, A. [2]

[1] Department of Statistics, Yazd University, Yazd, Iran
[2] Department of Computer Sciences, Shahrekord University

Abstract

In this paper we consider estimation of the reliability characteristics of Burr type X distribution based on fuzzy lifetime data. The Bayes estimates of the parameter and reliability function of the Burr type X model are obtained using a Markov Chain Monte Carlo method. Simulation studies are conducted to demonstrate the efficiency of the proposed method.

Keywords: Fuzzy lifetime data, Reliability estimation, Bayesian estimation, Markov Chain Monte Carlo method.

1 Introduction

Burr type X distribution has increasing importance in several areas of applications such as lifetime tests, health, agriculture, biology, and other sciences. The probability density function (pdf) and reliability function of Burr type X distribution is given by

$$f(t;\theta) = 2\theta t e^{-t^2}(1-e^{-t^2})^{\theta}, \quad t>0,\ \theta>0 \tag{1}$$

and

$$R(t;\theta) = 1-(1-e^{-t^2})^{\theta} \tag{2}$$

, respectively. From now on Burr type X model with the shape parameter θ will be denoted by $Burr(\theta)$.

In various fields of science such as biology, engineering and medicine, it is not possible to obtain the measurements of a statistical experiment exactly, but is possible to classify

[1] aajafari@yazd.ac.ir
[2] abbas.pak1982@gmail.com

them into imprecise quantities. For example, "The lifetime of a bearing is around 8.17×10^6 revolutions" and "The lifetime of some shaft be between 1,500 and 2,000 h, but near to 2,000 h" etc. are imprecise quantities relating to lifetime. The lack of precision of lifetime data can be described using fuzzy sets. The classical statistical estimation methods are not appropriate to deal with such imprecise cases. Therefore, we need suitable statistical methodology to handle these data as well. In this paper, we study Bayesian estimation of the reliability characteristics of Burr type X distribution when the lifetime observations are reported in the form of fuzzy numbers. In Section 2, we obtain the Bayes estimates of the parameter θ and reliability function $R(t)$ using Markov Chain Monte Carlo technique. Then, simulation study is presented in Section 3 in order to assess the accuracy of the proposed method. For a review about the main definitions of fuzzy sets and some of the formula used in this paper, see Pak et al. [2] and the references therein.

2 Bayesian estimation

Suppose that n identical units are placed on a life test with the corresponding lifetimes $X_1, ..., X_n$. It is assumed that these variables are independent and identically distributed as $Burr(\theta)$. Let $\mathbf{X} = (X_1, ..., X_n)$ denotes the vector of lifetimes. If a realization $\mathbf{x}$ of $\mathbf{X}$ was known exactly, we could obtain the complete-data likelihood function as

$$L(\theta; \mathbf{x}) = (2\theta)^n \prod_{i=1}^{n} x_i e^{-x_i^2}(1 - e^{-x_i^2})^{\theta}.$$

Now consider the problem where $\mathbf{x}$ is not observed precisely and only partial information about $\mathbf{x}$ is available in the form of a fuzzy subset $\tilde{\mathbf{x}} = (\tilde{x}_1, ..., \tilde{x}_n)$ with the Borel measurable membership function $\mu_{\tilde{\mathbf{x}}}(\mathbf{x}) = (\mu_{\tilde{x}_1}(x_1), ..., \mu_{\tilde{x}_n}(x_n))$. The observed data likelihood function can then be obtained, using Zadeh's definition of the probability of a fuzzy event (see [2]), as

$$\ell(\theta; \tilde{\mathbf{x}}) = \theta^n \prod_{i=1}^{n} \int 2xe^{-x^2}(1 - e^{-x^2})^{\theta} \mu_{\tilde{x}_i}(x)dx. \tag{3}$$

For computing the Bayes estimate of the unknown parameter θ and reliability function $R(t)$, we assume that θ has $Gamma(a, b)$ density with pdf given by

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{(-\theta b)}, \qquad \theta > 0, \tag{4}$$

where $a > 0$ and $b > 0$. Based on this prior, the posterior density function of θ given the data can be written as

$$\pi(\theta \mid \tilde{\mathbf{x}}) \propto \theta^{n+a-1} e^{-\theta b} \prod_{i=1}^{n} \int xe^{-x^2}(1 - e^{-x^2})^{\theta} \mu_{\tilde{x}_i}(x)dx. \tag{5}$$

Then, under a squared error loss function, the Bayes estimate of any function of θ, say $h(\theta)$, becomes

$$E(h(\theta) \mid \tilde{\mathbf{x}}) = \frac{\int_0^{\infty} h(\theta)\theta^{n+a-1} e^{-\theta b} \prod_{i=1}^{n} \int xe^{-x^2}(1 - e^{-x^2})^{\theta} \mu_{\tilde{x}_i}(x)dxd\theta}{\int_0^{\infty} \theta^{n+a-1} e^{-\theta b} \prod_{i=1}^{n} \int xe^{-x^2}(1 - e^{-x^2})^{\theta} \mu_{\tilde{x}_i}(x)dxd\theta}. \tag{6}$$

The ratio of integrals in (2.4) does not seem to take a closed form. Therefore, in the following, we adopt Markov Chain Monte Carlo (MCMC) method for approximating (2.4).

Noting that the density function $\pi(\theta \mid \tilde{\mathbf{x}})$ is not known, but by experimentation, we observed that it appears similar to normal distribution. So to generate random samples from $\pi(\theta \mid \tilde{\mathbf{x}})$, we can use MetropolisHastings algorithm with normal proposal distribution as follows.

1) Start with an initial guess $\theta^{(0)}$ and set $j = 1$.

2) Generate $\theta^{(j)}$ from $\pi(\theta \mid \tilde{\mathbf{x}})$ with the proposal distribution $q(\theta) \equiv I(\theta > 0)N(\theta^{(0)}, 1)$, where $I(.)$ is the indicator function, as follows:

(a). Let $\gamma = \theta^{(j-1)}$.

(b). Generate ω from the proposal distribution q.

(c). Let $p(\gamma, \omega) = \min\left\{1, \frac{\pi(\omega|\tilde{\mathbf{x}})q(\gamma)}{\pi(\gamma|\tilde{\mathbf{x}})q(\omega)}\right\}$.

(d). Accept ω with probability $p(\gamma, \omega)$ or accept γ with probability $1 - p(\gamma, \omega)$.

3) Compute $R^{(j)}(t)$ from (1.2).

4) Set $j = j + 1$.

5) Repeat Steps 2-4, M times and obtain $\theta^{(j)}$ and $R^{(j)}(t)$ for $j = 1, ..., M$.

Now the Bayes estimates of the parameter θ and reliability function $R(t)$ with respect to the squared error loss function become

$$\hat{\theta}_{BM} = \hat{E}(\theta \mid \tilde{\mathbf{x}}) = \frac{1}{M}\sum_{j=1}^{M}\theta^{(j)}.$$

and

$$\hat{R}_{BM}(t) = \hat{E}(R(t) \mid \tilde{\mathbf{x}}) = \frac{1}{M}\sum_{j=1}^{M}R^{(j)}(t).$$

3 Simulation study

In this section, a Monte Carlo simulation study is presented in order to investigate the performance of the proposed method. First, for fixed $\theta = 1$ and different choices of n, we have generated fuzzy samples from Burr X distribution using the method proposed by Pak et al. [2]. Then, the Bayes estimates of the parameter θ and reliability function $R(t)$, at $t = 1$, for the fuzzy sample were computed using the MCMC technique. As a conjugate prior for θ, we take the $Gamma(a, b)$ density with $a = b = 0.0001$. The average values (AV) and mean squared errors (MSE) of the estimates over 1000 replications are presented in Table 1. In viewing the table, we find that the performance of the Bayes estimates are quite satisfactory in terms of AVs and MSEs. It can be further observed that, as the sample size increases, the MSEs of the estimates decrease as expected.

References

[1] A., Pak, G.H., Parham, and M., Saraj, Inference for the Weibull Distribution Based on Fuzzy Data. Revista Colombiana de Estadistica, 36(2), (2013), 339-358.

[2] L. A., Zadeh, Probability measures of uzzy events, Journal of Mathematical Analysis and Applications. 10, (1968), 421-427.

Table 1: AVs and MSEs of the Bayes estimates of θ and $R(t)$ for different sample sizes.

n	$\hat{\theta}_{BM}$		$\hat{R}_{BM}(t)$	
	AV	MSE	AV	MSE
15	1.1673	0.0631	0.4276	0.0093
20	1.1509	0.0580	0.4103	0.0081
30	1.1221	0.0519	0.4062	0.0076
40	1.0960	0.0428	0.3984	0.0070
50	1.0631	0.0297	0.3913	0.0058
70	1.0409	0.0156	0.3825	0.0043
100	1.0285	0.0117	0.3794	0.0026

A Non-Parametric Test Against Renewal Increasing Mean Residual Life Distributions

Jamshidian, A.R. [1]

Department of Mathematics and Computer, Sheikh Bahaee University

Abstract

In this paper we introduce a new aging class of life distributions when a device is operating in a realistic environment. We study the behavior of such life distributions through the mean residual life notion, when a device is experiencing number of shocks. Due to these shocks the lifetime of such device has become shortened or prolonged. These tempered events are governed by a homogenous Poisson process. A moment inequality which characterizes this new aging class, namely renewal increasing mean residual life, is derived. We propose a new U-statistic test procedure to address the problem of testing exponentiality against such class of life distributions. It is shown that the proposed test enjoys a superior power for some commonly used alternative.

Keywords: Poisson Shock model, Increasing mean residual life, Exponential distribution, Moment inequalities, U-statistics.

1 Introduction

The mean residual lifetime, MRL, is the remaining lifetime of a component alive at time t. If X denotes a nonnegative random variable with a continuous life distribution function F and finite mean $\mu = \int_0^\infty \bar{F}(x)dx$, then the MRL function at time t is defined as

$$m(t) \ = E\left(X - t | \, X > t\right) = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}\left(u\right) du. \tag{1}$$

The properties of mean residual life (MRL) of a component (subjected to no shocks) have been widely used for deriving maintenance and replacement policies.

The goal of this paper is to study the age replacement models through the remaining life time of a device in a more real life environment in which unit fail by physical deterioration

[1] jamshidian@shbu.ac.ir

suffered from some damage. In the latter case, units fail when the total damage due to shocks has exceeded a critical level. Such damage models may apply to the actual units that are working in industry, service, information, computers etc. In the past decade, various properties of failure distributions when shocks occur in a Poisson process were extensively investigated. We refer our reader to Li and Xu (2008), Ahmad and Mugdadi (2010), and Izadkhah and Kayid (2013), Sepehrifar et al. (2014), among others. We study the cumulative damage model for an operating unit through the increasing mean residual life function, which follows by the related definitions. We use the innovative features of this function to introduce the related moment inequality function and to derive a new U-statistic testing exponentiality against renewal increasing mean residual life under shock (RIMRl$_{shock}$) alternatives. Finally, the power simulation ane some numerical results are presented.

2 Basic definitions and properties

Consider a unit which is subjected to successive shocks and each shock causes some damage to this unit in some amount. Let random variables $\{T_j$: j=1, 2,...$\}$ denote the sequence of time intervals between successive shocks, and random variables $\{W_i$: i=1, 2,...$\}$ denote the amount of life-damage produced by the i-th shock, where $W_0 \equiv 0$. It is assumed that the sequence of $\{W_i\}$ is nonnegative, independent and identically distributed. We also assume that W_i is independent of T_j. Let random variable $N(t)$ denote the total number of shocks up to time t. Then, the total cumulative life-damage up to time t is defined as $Z\left(t\right)$where $Z\left(t\right) = \sum_{i=0}^{N(t)} W_i$.

It is assumed that the unit fails when the total damage exceeds a pre-specified level $x(>0)$. Usually, the failure level x is statistically estimated and is already known. Let X be the life variable of a device with survival function $\bar{F}\left(t\right) = P\{X \geq t\}$ which is subjected to *N(t)* shocks with$P\left\{N\left(t\right) = j\right\} = F^{(j)}\left(t\right) - F^{(j+1)}\left(t\right),\ j = 0,\ 1,\ 2,\ .$ Let the random variable W_i be the amount of hidden lifetime absorbed by the i^{th} shock, with common distribution $G\left(x\right) = P\{W_j \leq x\}$. Then the distribution of$Z\left(t\right)$ is $Q\left(x\right) = P\{Z\left(t\right) \leq x\} = \sum_{j=0}^{\infty} G^{(j)}\left(x\right)\left[F^{(j)}\left(t\right) - F^{(j+1)}\left(t\right)\right]$. Glynn and Witt (1993), studied the distribution of$Q\left(x\right)$.

Let $X^* = X - Z\left(t\right)$ be the lifetime of an item (with lifetime X, and survival function $\bar{F}\left(t\right)$) in a service with total life-damaged $Z\left(t\right)$. Set $m^*\left(t\right) = E\left[X_t^*\right] = E\left[X^* - t| \geq t\right]$, which is the mean residual lifetime of such item in the age replacement model subjected to N(t) shock, given that the item is in operating situation hours after installation or the total life-damaged is not exceeding the threshold level x, whichever comes last. We assume that X and Z(t) are independent. First, we present definitions and basic properties that will be used in the sequel.

Definition 1. The mean residual life of a device under shock model (MRL$_{shock}$) at time t, is defined as

$$m^*\left(t\right) \ = \frac{\int_t^{\infty} \bar{v}\left(z\right) dz}{\bar{v}(t)}, \tag{2}$$

where $\bar{v}\left(z\right) = \int_0^x \bar{F}\left(z + w\right) dQ\left(w\right).$

Definition 2. The distribution function F is said to be a renewal increasing mean residual life under shock models (RIMRL$_{shock}$) if $m^*(t)$ is an increasing function in $t \geq 0$.

Corollary 1. The distribution F belongs to RIMRL$_{shoch}$ if

$$(\bar{v}(t))^2 \leq \int_0^x f(t+w)\, dQ(w) \int_t^\infty \bar{v}(z)\, dz\ .$$

Corollary 2. The distribution function F belongs to RIMRL$_{shock}$ if

$$E_{f^*}\left[Min(X_1^*, X_2^*)\right] \leq \frac{1}{2}\left(E_{f^*}\left(X_1^*\right)\right)$$

where $X_i^* = X_i - W_i$.

3 Testing exponentiality against RIMRL$_{shock}$ alternatives

Consider the problem of testing H$_0$: F is exponential with mean,$\mu < \infty$ versus H$_1$: F is RIMRL$_{shock}$ and not exponential. We consider corollary 2 for RIMRL$_{shock}$ as the measure of departure from the null hypothesis H$_0$:

$$\delta = \frac{1}{\mu^*}\left\{E_{f^*}[Min(X_1^*, X_2^*) - \frac{1}{2}X_1^*]\right\}$$

where $\mu^* = E(X_i^*)$.
This measure may be estimated by the following statistics:

$$\hat{\delta} = \frac{1}{\bar{X^*}} \times \frac{2}{n(n-1)}\left\{\sum_{\substack{i=1\\ i<j}}^{n}\sum_{j=1}^{n}\left\{Min(X_i^*, X_j^*) - \frac{1}{2}X_i^*\right\}\right\} \qquad (3)$$

where $\bar{X^*} = \frac{1}{n}\sum_{i=1}^{n} X_i^*$ is the sample mean based on a random sample from distribution F. Note that $\hat{\delta}$ is derived based on the standard U-statistic theory.Let

$$\phi(X_1^*, X_2^*) = \frac{1}{\mu^*}\left[Min\left(X_1^*, X_2^*\right) - \frac{X_1^*}{2}\right]$$

$$\phi(X_2^*, X_1^*) = \frac{1}{\mu^*}\left[Min\left(X_2^*, X_1^*\right) - \frac{X_2^*}{2}\right]$$

and define the symmetric kernel

$$\psi\left(X_1^*, X_2^*\right) = \frac{1}{2!}\sum \phi(X_{i1}^*, X_{i2}^*)$$

where the sum is an overall arrangement of X_1^* and X_2^*. It can be shown that $\hat{\delta}$ in equation (3.1) is equivaltne to U-statistic given by

$$U = \frac{1}{\binom{n}{2}}\sum_{i<j}\phi(X_i^*, X_j^*).$$

The following theorem gives the large sample properties of $\hat{\delta}$ or U.
Theolem 1. As $n \to \infty$, $\sqrt{n}(\hat{\delta} - \delta)$is asymptotically normal whit mean 0 and variance

$$Var\left(\frac{1}{2}\left\{\frac{2}{\mu^*}\{X_1^* F\left(X_2^*\right) + \mu^* \bar{F}\left(X_2^*\right)\} - \frac{X_1^*}{2\mu^*} - \frac{1}{2}\right\}\right).$$

Under the null hypothesis H_0 ,$X_i^* \sim Exp\,(1)$, the variance is calculated as $\sigma_0^2 = \frac{7}{48}$.

To carry out this test, we calculate $\sqrt{n}\hat{\delta}\sigma_0^{-1}$ and reject H_0 if this value is larger than $\mathrm{Z}_{1-\alpha}$.

References

[1] Ahmad, I. A. and Mugdadi, A. R. (2010), Higher order equilibriumlife distributions, *IEEE Trans. Rel.*, **59**, 66-73.

[2] Glynn, P.W., Whitt, W. (1993), Limit theorems for cumulative processes, *Stoch Process Appl*, **47**,299-314.

[3] Izadkhah, S. and Kayid, M. (2013), Reliability analysis of the harmonic mean inactivity time order, *IEEE Trans. Rel.*, **62**, 329-337.

[4] Li, X., and Xu, M. (2008), Reversed hazard rate order of equilibrium distributions and a related aging notion, *Statist. Papers*, **49**, 749-767

[5] Sepehrifar, M. B., Khorshidian, K., Jamshidian, A. R. (2013),On renewal increasing mean residual life distributions: An age replacement model with hypothesis testiag application, *Stntistics & Probability Letters Available online*, **49**, 749-767

Residual Lifetime of Coherent System with Dependent Identically Distributed Components

Kelkinnama, M. [1]

Department of Mathematical Sciences, Isfahan University of Technology

Abstract

In this paper, we study the residual lifetime of coherent system with possibly dependent identically distributed component lifetimes. These results are based on the representation of system reliability function as a distorted function of common reliability function of components.

Keywords: Coherent systems, Residual lifetime, Survival copula, Distorted function, Stochastic orders.

1 Introduction

Consider a coherent system consisting of n possibly dependent components with lifetimes $X_1, ..., X_n$. Suppose that these random variables are identical with common distribution function F and reliability function $\bar{F}$. The dependence among components is represented by the joint reliability function of $(X_1, ..., X_n)$,

$$\bar{F}(x_1, ..., x_n) = Pr(X_1 > x_1, ..., X_n > x_n).$$

Using the Sklar's copula representation, we have

$$\bar{F}(x_1, ..., x_n) = K(\bar{F}(x_1), ..., \bar{F}(x_n)),$$

where, $K(u_1, ..., u_n)$ is reliability copula and $0 < u_i < 1$. In fact, K is an useful tool for modeling dependence between the components. Denote the lifetime of coherent system with $T = \phi(X_1, ..., X_n)$ where ϕ is the structure function of system. Navarro et al. [1] provided an useful representation for system reliability function as a distorted function of

[1] m.kelkinnama@cc.iut.ac.ir

the $\bar{F}$. Let T be the lifetime of a coherent system with identically distributed component lifetimes having the common reliability function $\bar{F}$ and the joint reliability copula K. Then

$$\bar{F}_T(t) = Pr(T > t) = h(\bar{F}(t)),$$

where, h is a function that depends only on copula K and system structure (minimal path sets of system). The function h is a distorted function which is an increasing continuous function from $[0, 1]$ to $[0, 1]$, $h(0) = 0$ and $h(1) = 1$. The function h is called domination function. If $K(u_1, ..., u_n)$ is exchangeable, i.e. it is permutation invariant, then

$$h(u) = \sum_{i=1}^{n} a_i K(\underbrace{u, ..., u}_{i-times}, \underbrace{1, ..., 1}_{(n-i)-times}),$$

where, $\mathbf{a} = (a_1, ..., a_n)$ is the minimal signature of the system. . In particular, in the i.i.d. case, K is the product copula, hence $h_I(u) = \sum_{i=1}^{n} a_i u^i$ As an example, consider the system $T = \max(X_1, \min(X_2, X_3))$. The minimal path sets are $P_1 = \{1\}$ and $P_2 = \{2, 3\}$.

$$\begin{aligned} Pr(T > t) =& Pr(X_{P_1} > t) + Pr(X_{P_2} > t) - Pr(X_{P_1 \cup P_2} > t) \\ =& \bar{F}(t, 0, 0) + \bar{F}(0, t, t) - \bar{F}(t, t, t) \\ =& K(\bar{F}(t), 1, 1) + K(1, \bar{F}(t), \bar{F}(t)) - K(\bar{F}(t), \bar{F}(t), \bar{F}(t)) = h(\bar{F}(t)) \end{aligned}$$

where, $h(u) = K(u, 1, 1) + K(1, u, u) - K(u, u, u)$. If K is exchangeable then $a = (1, 1, -1)$ is the minimal signature of the system.

In this paper, we study the aging properties and stochastic comparisons of residual lifetimes of coherent systems with dependent identically distributed (DID) component lifetimes. The results derived in this paper can also be applied to coherent systems with exchangeable or i.i.d. components.

2 Main results

For a fixed $t > 0$, the residual lifetime of the coherent system at time t, is denoted by $T^t = [T - t|T > t]$. The reliability function of T^t is

$$\bar{F}_{T^t}(x) = Pr(T^t > x) = \frac{\bar{F}_T(t + x)}{\bar{F}_T(t)} = \frac{h(\bar{F}(t + x))}{h(\bar{F}(t))}.$$

The hazard rate function of T^t can be written as

$$r_{T^t}(x) = r(t + x)\alpha(\bar{F}(t + x)),$$

where, $\alpha(u) = \frac{uh'(u)}{h(u)}$ and r is the hazard rate function of $\bar{F}$.

Theorem 1. *If $\bar{F}$ is IFR and $\alpha(u)$ is a decreasing function of $u \in (0, 1)$, then for the conditional random variable T^t, we have*

(i) $T^t \geq_{hr} T^{t'}$, *for* $t \leq t'$.

(ii) T^t *is IFR for all* $t > 0$.

Note that part (i) implies that $T^t \geq_{st} T^{t'}$, i.e. T is IFR.
For the reversed hazard rate function of T^t we have

$$\tilde{r}_{T^t}(x) = \tilde{r}(t+x)\beta(\bar{F}(t+x))\frac{1-h(\bar{F}(t+x))}{h(\bar{F}(t)) - h(\bar{F}(t+x))},$$

where, $\beta(u) = \frac{(1-u)h'(u)}{1-h(u)}$, and $\tilde{r}$ is the reversed hazard rate of $\bar{F}$.

Theorem 2. *Suppose that $\bar{F}$ is $DRHR$, and $\beta(u)$ is increasing in $u \in (0,1)$.*

(i) If $1-h(u)$ is log-concave in u, then $T^t \geq_{rh} T^{t'}$ for $t \leq t'$.

(ii) T^t is $DRHR$ for all $t > 0$.

The Glaser's function (eta function) of T^t can be written as

$$\begin{aligned} \eta_{T^t}(x) = -\frac{f'_{T^t}(x)}{f_{T^t}(x)} &= \eta(t+x) + r(t+x)\gamma(\bar{F}(t+x)) \\ &= \eta(t+x) + \tilde{r}(t+x)\bar{\gamma}(\bar{F}(t+x)), \end{aligned}$$

where, $\gamma(u) = \frac{uh''(u)}{h'(u)}, \bar{\gamma}(u) = \frac{(1-u)h''(u)}{h'(u)}$, and η is the eta function of $\bar{F}$.

Theorem 3. *If the common density function, f, is log-concave and there exist $a \in [0,1]$ such that $\gamma(u)$ is non-negative and decreasing in $u \in (0,a)$ and $\bar{\gamma}(u)$ is non-positive and decreasing in $u \in (a,1)$ then*

(i) $T^t \geq_{lr} T^{t'}$, for $t \leq t'$.

(ii) f_{T^t} is log-concave for all $t > 0$.

Navarro et al. [2] showed that, if $\bar{F}$ is $NBU(NWU)$ and $h(u)h(v) \geq (\leq) h(uv)$ for all $0 \leq u, v \leq 1$, then T is $NBU(NWU)$, it means that $T \geq_{st} (\leq_{st}) T^t$ for all $t > 0$. Now, in the next theorem, we give sufficient conditions for some other stochastic orders between T and T^t.

Theorem 4. *(i) If $\bar{F}$ is $IFR(DFR)$ and $\alpha(u) \geq (\leq) 1$, then $T \geq_{hr} (\leq_{hr}) T^t$ for all $t > 0$.*

(ii) If $\bar{F}$ is $DRHR$ and $\beta(u) \leq 1$, then $T \geq_{rh} T^t$ for all $t > 0$.

(iii) If f is log-concave (log-convex) and $\gamma(u) \geq (\leq) 0$ then $T \geq_{lr} (\leq_{lr}) T^t$ for all $t > 0$.

The following theorems provide conditions under which the residual lifetimes of two coherent systems with DID components can be compared.

Theorem 5. *Let $T_1 = \phi_1(X_1, ..., X_n)$ and $T_2 = \phi_2(Y_1, ..., Y_m)$ be the lifetimes of two coherent systems with DID components having common reliability function $\bar{F}$. Let h_1 and h_2 be their respective domination functions. Then, we have the following properties for all $t > 0$.*

(i) $T_1^t \leq_{st} (\geq_{st}) T_2^t$ for all $\bar{F}$ if and only if $\frac{h_2(u)}{h_1(u)}$ is decreasing (increasing) in $u \in (0,1)$.

(ii) $T_1^t \leq_{hr} (\geq_{hr}) T_2^t$ for all $\bar{F}$ if and only if $\frac{h_2(u)}{h_1(u)}$ is decreasing (increasing) in $u \in (0,1)$.

(iii) $T_1^t \leq_{rhr} (\geq_{rhr}) T_2^t$ *for all* $\bar{F}$ *if and only if* $\frac{h_2(q)-h_2(u)}{h_1(q)-h_1(u)}$ *is decreasing (increasing) in* $u \in (0, q)$.

(iv) $T_1^t \leq_{lr} (\geq_{lr}) T_2^t$ *for all* $\bar{F}$ *if and only if* $\frac{h_2'(u)}{h_1'(u)}$ *is decreasing (increasing) in* $u \in (0, 1)$.

Theorem 6. *Let* $T_1 = \phi(X_1, ..., X_n)$ *and* $T_2 = \phi(Y_1, ..., Y_n)$ *be the lifetimes of two coherent systems with the same structure and with DID component lifetimes having the same copula and common absolutely continuous reliability functions* $\bar{F}$ *and* $\bar{G}$*, respectively. Let* h *be the domination function and assume that it is twice differentiable. Then, we have the following properties for all* $t > 0$.

(i) *If* $X_1 \leq_{st} Y_1$ *and* $h(u)$ *is log-concave in* u*, then* $T_1^t \leq_{st} T_2^t$.

(ii) *If* $X_1 \leq_{hr} Y_1$ *and* $\frac{uh'(u)}{h(u)}$ *is decreasing in* u*, then* $T_1^t \leq_{hr} T_2^t$.

(iii) *If* $X_1 \leq_{rhr} Y_1$, $\frac{(1-u)h'(u)}{1-h(u)}$ *is increasing in* u*, and* $\frac{1-h(u_1)}{h(q_1)-h(u_1)} \leq \frac{1-h(u_2)}{h(q_2)-h(u_2)}$ *for* $u_1 \leq u_2$, $u_1 \in (0, q_1), u_2 \in (0, q_2), q_1 \leq q_2$*, then* $T_1^t \leq_{rhr} T_2^t$.

(iv) *If* $X_1 \leq_{lr} Y_1$ *and* $\frac{uh''(u)}{h'(u)}$ *is non-negative and decreasing in* u*, then* $T_1^t \leq_{lr} T_2^t$.

References

[1] Navarro, J., Aguila, Y., Sordo, M.A. and Suarez-Llorens, A. (2012), Stochastic ordering properties for systems with dependent identically distributed components. *Appl. Stoch. Models Bus. Ind.*, **29**, 264–278

[2] Navarro, J., Aguila, Y., Sordo, M.A. and Suarez-Llorens, A. (2013), Preservation of reliability classes under the formation of coherent systems. *Appl. Stoch. Models Bus. Ind.*, **30**, 444–454.

Stochastic Comparicons of Generalized Residual Entropy of Order Statistics

Khammar, A. H. [1] and Baratpour, S. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

In modeling of biological and engineering systems often requires use of concepts of information theory, and in particular of entropy. The concept of residual entropy is applicable to a system which has survived for some units of time. In this paper, we propose a generalized residual entropy based on order statistics and obtain some results on the stochastic comparisons of it.

Keywords: Generalized residual entropy, Hazard rate function, Order statistics, Stochastic comparisons.

1 Introduction

Throughout this paper, X and Y will denote two random variables and the distribution function, density function and hazard rate function of X be denoted by $F(t)$, $f(t)$ and $\lambda_F(t)$ and those of Y be denoted by $G(t)$, $g(t)$ and $\lambda_G(t)$, respectively. We will be particularly interested in X_t, the remaining lifetime of a unit of age $t \geq 0$. That is, $X_t \overset{d}{=} X - t|X > t$ where $\overset{d}{=}$ stands for equality in distribution. For each $t \geq 0$, the probability distribution of X_t is absolutely continuous with distribution function $F_t(x) = P(X - t \leq x|X > t) = \frac{[F(x+t)-F(t)]}{\bar{F}(t)}, x > 0$, survival function $\bar{F}_t(x) = 1 - F_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}, x > 0$, and probability density function $f_t(x) = \frac{f(x+t)}{\bar{F}(t)}, x > 0$.
As is well known, an early definition of a measure of the entropy has been introduced by Shannon (1948). Further, Nanda and Paul (2006) introduced a measure of residual

[1] amirhamzeh.kh@gmail.com
[2] baratpour@um.ac.ir

entropy over (t,∞) based on the Tsallis entropy that is a generalisation of order β of the Shannon entropy (Tsallis, 1988), given by

$$H^{\beta}(X;t)=\frac{1}{\beta-1}\left[1-\frac{\int_t^{+\infty}f^{\beta}(x)dx}{\bar{F}^{\beta}(t)}\right],\ \beta\neq 1,\ \beta>0. \tag{1}$$

Obviously $H^{\beta}(X;0)$ results in Tsallis entropy and $\beta\longrightarrow 1$ gives Shannon entropy. In this paper, we extend this generalized residual entropy based on order statistics and we derive some stochastic comparisons based on the generalized residual entropy and order statistics version of it.

2 Generalized residual entropy of order statistics

Suppose that $X_1,X_2,\dots,X_n$ are independent and identically distributed observations from cdf $F(t)$ and pdf $f(t)$. The order statistics of the sample is defined by the arrangement of $X_1,X_2,\dots,X_n$ from the smallest to the largest, denoted as $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$. Generalized residual entropy associated with the i^{th} order statistics $X_{i:n}$ is given by

$$H^{\beta}(X_{i:n};t)=\frac{1}{\beta-1}\big[1-\frac{\int_t^{\infty}f_{i:n}^{\beta}(x)dx}{\overline{F}_{i:n}^{\beta}(t)}\big],\quad \beta\neq 1,\beta>0, \tag{2}$$

where $f_{i:n}(x)$ and $\overline{F}_{i:n}(x)$ are the density function and survival function of $X_{i:n}$, respectively (see Davide and Nagaraja, 2003).

Now, using probability integral transformation $U=F(X)$, where U is standard uniform distribution (2) can be expressed as

$$H^{\beta}(X_{i:n};t)=\frac{1}{\beta-1}\left[1-\frac{\overline{B}_{F(t)}(\beta(i-1)+1,\beta(n-i)+1)E[f^{\beta-1}(F^{-1}(Y_i))}{\overline{B}_{F(t)}^{\beta}(i,n-i+1)}\right],$$

where $Y_i\sim\overline{B}_{F(t)}(\beta(i-1)+1,\beta(n-i)+1)$.

3 Stochastic comparisons

Notation: a. The definitions of stochastic comparisons used in this section is available in Shaked and Shanthikumar (1994).
b. The proof theorems stated in brief.

Definition 1. *A random variable X is said to be smaller than Y in Tsallis entropy ordering (denoted by $X\overset{GRE}{\leqslant}Y$) if $H^{\beta}(X;t)\leqslant H^{\beta}(Y;t)$ for all $t>0$.*

It is well known that $X\overset{LR}{\leqslant}Y\Rightarrow X\overset{HR}{\leqslant}Y\Rightarrow X\overset{ST}{\leqslant}Y$ and $X\overset{DIDP}{\leqslant}Y\Rightarrow X\overset{ST}{\leqslant}Y$ and $X\overset{LR}{\leqslant}Y\Rightarrow X\overset{ST}{\leqslant}Y$ (Bickel and Lehmann, 1976; and Shaked and Shanthikumar, 1994). We first prove the following preliminary results for generalized residual entropy.

Theorem 1. *Let X and Y be two random variables, then $X\overset{DISP}{\leq}Y$ implies $X\overset{GRE}{\leq}Y$.*

Proof. From (1), we have

$$H^{\beta}(X;t)=\frac{1}{\beta-1}\left[1-\frac{B(1,\beta)}{\overline{F}^{\beta}(t)}E(\lambda_F^{\beta-1}(F^{-1}(W)))\right],$$

where $W\sim B(1,\beta)$. we also note that $X\overset{DISP}{\leq}Y$ if and only if $\lambda_G(G^{-1}(u))\leq\lambda_F(F^{-1}(u))$ for all $u\in(0,1)$ (see Shaked and Shanthikumar, 1994). First, we assume that $\beta>1$, on the other hand from Remark 3, $X\overset{DISP}{\leq}Y$ implies that $X\overset{ST}{\leq}Y$. Hence, we find

$$\begin{aligned}H^{\beta}(X;t)-H^{\beta}(Y;t)&\leq\frac{B(1,\beta)}{\beta-1}\left[\frac{1}{\overline{G}^{\beta}(t)}-\frac{1}{\overline{F}^{\beta}(t)}\right]\cdot E(\lambda_F^{\beta-1}(F^{-1}(W)))\\&\leq 0.\end{aligned}$$

Thus, we obtain $X\overset{GRE}{\leq}Y$. For $0<\beta<1$ the proof is similar. □

Theorem 2. *Let X and Y be two random variables, at least one of them is DFR. Then $X\overset{HR}{\leq}Y$ implies $X\overset{GRE}{\leq}Y$.*

Proof. First, we assume that $0<\beta<1$ and X is DFR. Since $X\overset{HR}{\leq}Y$ implies that $X_t\overset{ST}{\leq}Y_t$ (see Shaked and Shanthikumar, 1994) and from (1), we have

$$\begin{aligned}H^{\beta}(X;t)&=\frac{1}{\beta-1}\left[1-E_{f_{X_t,\beta}}(\lambda_F^{\beta-1}(t+X_t))\right]\\&\leq\frac{1}{\beta-1}\left[1-E_{g_{Y_t,\beta}}(\lambda_F^{\beta-1}(t+Y_t))\right]\\&\leq\frac{1}{\beta-1}\left[1-E_{g_{Y_t,\beta}}(\lambda_G^{\beta-1}(t+Y_t))\right]=H^{\beta}(Y;t),\end{aligned}$$

where $f_{X_t,\beta}=\frac{-d\bar{F}_t^{\beta}(x)}{dx}$. For $\beta>1$ the proof is similar. □

Now, by the fact that, $X\overset{DISP}{\leqslant}Y$ implies that $X_{i:n}\overset{DISP}{\leqslant}Y_{i:n}$ (Shaked and Shanthikumar, 1994) and by Theorem 1, we have the following result. Let X and Y be two random variables. Then $X\overset{DISP}{\leqslant}Y$ implies $X_{i:n}\overset{GRE}{\leqslant}Y_{i:n}$.

Theorem 3. *Suppose X has a DFR distribution. Then $X_{i:n}\overset{GRE}{\leqslant}X_{j:n},\ i<j$.*

Proof. Using the result of Chan et al. (1991), we have $X_{i:n}\overset{LR}{\leqslant}X_{j:n}$. By Remark 3, this implies that $X_{i:n}\overset{HR}{\leqslant}X_{j:n}$. Since X has a DFR distribution , $X_{i:n}$ has a DFR distribution, (see Takahasi, 1988). So, by using Theorem 2 , we can conclude that $X_{i:n}\overset{GRE}{\leqslant}X_{j:n}$. □

Theorem 4. *Let $X_1,X_2,\ldots,X_{n+1}$ be iid random variables with distribution function $F(t)$. Suppose X has a DFR distribution. Then, $X_{1:n+1}\overset{GRE}{\leqslant}X_{1:n}$ and $X_{n:n}\overset{GRE}{\leqslant}X_{n+1:n+1}$.*

Proof. We use the fact that $X_{j:m}\overset{LR}{\leqslant}X_{i:n}$ whenever $j\leqslant i$ and $m-j\geqslant n-i$ (Shaked and Shanthikumar, 2007), and the method used in the proof of Theorem 3. □

References

[1] Bickel, P. J., Lehmann, E. L. (1976), Descriptive Statistics for nonparametric models. III. Dispersion. *Ann. Statist.* **4**, 1139-1158.

[2] Chan, W., Proschan, F., Sethuraman, J. (1991), Convex ordering among functions with applications to reliability and mathematical statistics. *IMS Lecture Notes* **16**, 121-134.

[3] David, H. A., Nagaraja, H. N. (2003), *Order Statistics.* New York, Wiley.

[4] Nanda, A. K, Paul, P. (2006), Some results on generalized residual entropy. *Information Sciences* **176**, 27-47.

[5] Shaked, M., Shanthikumar, J. G. (1994), *Stochastic Orders and Their Applications.* New York, Academic Press.

[6] Shaked, M., Shanthikumar, J. G. (2007), *Stochastic Orders.* Springer Verlag.

[7] Shannon, C. E. (1948), A mathematical theory of communication. *Bell Syst. Tech. J.* **27**, 379-432.

[8] Takahasi, K. (1988), A note on hazard rates of order statistics. Comm. Statist. Theory Methods. **17**(12), 4133-4136.

[9] Tsallis, C. (1988), Possible generalization of Boltzmann-Gibbs Statistics. *Statistical Physics* **52**, 479-487.

On the Effect of Dependent Components on the Mean Time To Failure (MTTF) of the System

Khanjari-Sadegh, M. [1]

Department of Statistics, University of Birjand

Abstract

In many practical applications in system reliability, the assumption that the component lifetimes are independent is not valid and realistic.
In this talk we Consider the effect of dependency between system components on MTTF of the system. For example if we increase (decrease) the degree of dependency between system components wether the MTTF of the system has the same behavior or not? We see that the answer of this question depends on the structure of the system (it also may depend on the structure of dependency between system components).

Keywords: Quadrant Dependency, Mean Time To Failure, Diagonally Dependency, System Reliability, Stochastic Ordering.

1 Introduction

In this paper we are trying to find some answers for the following questions.

1. In all systems, if the degree of dependency between component lifetimes of the system changed, (for example increased from d_1 to $d_2 > d_1$ in any sense), will the system reliability or at least the MTTF of the system always be increased (or decreased)?

2. If the answer of Q.1 depends on the structure of the system, what is its answer for a system with specific structure?

3. Even in a system with specific structure, is it possible the answer of Q.2 may depend on the dependence structure between system components?

[1] mkhanjarii@birjand.ac.ir

In order to compare the systems containing dependent components with that of independent components, we assume that when the degree of dependency between component lifetimes changed their marginal distributions will not changed. We will see the answer of Q.1 is no. It seems that the Q.2 and Q.3 have no specific answer in general till now. But at least for series and parallel systems with positive (or negative) dependent components (or even under some weaker conditions) the answer of Q.2 is yes and Q.3 is no. For other structures it remains as an open problem.

Motivated by the recent work of Lai and Lin (2014) who defined the concept of more diagonal dependent here we define a weaker dependence concept than PQD(NQD) and call it positive(negative) diagonal dependent PDD(NDD) to use it for stochastic comparisons of series and parallel systems.

References

[1] Cui, L. and Li, H.(2007), Analytical Method for Reliability and MTTF Assessment of Coherent Systems with Dependent Components. Reliability Engineering and System Safety, 92, 300-307.

[2] Kotz, S., Lai, C.D. and Xie, M.(2003). On the Effect of Redundancy for Systems with Dependent Components. II Transactions, 35, 1103-1110.

[3] Lai, C.D. and Lin, G.D.(2014). Mean Time to Failure of Systems with Dependent Components. Applied Mathematics and Computation, 246, 103-111.

[4] Lai, C.D. and Xie, M.(2006). Stochastic Ageing and Dependence for Reliability. Springer.

[5] Navarro, J. and Lai, C.D.(2007). Ordering Properties of Systems with Two Dependent Components. Communications in Statistics-Threory and Methods, 36, 645-655.

On the Dynamic Proportional Odds Model

Kharazmi, O. [1]

Department of Statistics, Vli-e-Asr University of Rafsanjan

Abstract

The proportional odds model plays an important role in analyzing survival data.This note develops the definition of dynamic proportional odds (DPO) model and its properties including some results on stochastic comparisons. One application of DPO model is considered as Marshall and Olkin family of distribution in dynamic situation.

Keywords: Proportional odd model , Survival analysis, Marshall and Olkin distribution.

1 Introduction

The proportional odds model which was introduced by Bennett (1983) is appropriate to analyze data in survival analysis. In survival studies, heterogeneity in the population of lifetimes is usually represented by covariates. The main objective in such studies is to understand and exploit the relationship between lifetime and covariates. Parametric and semi parametric regression models are used to analyze such lifetime data. Commonly used semi parametric regression model is Coxs (1972) proportional hazards model. In practical situations, it is not uncommon for the hazard func- ions obtained for two groups to converge with time. In the situations where the data exhibit non-proportional hazards, proportional odds model can be employed. For more details, one could refer to Kirmani and Gupta (2001) and Wang et al. (2003). The proportional odds (PO) frailty model is defined by Marshall and Olkin (1997). Also, see Marshall and Olkin (2007). Its extensions andmodications have been studied by various authors including Gupta and Peng (2009). Since different distributions of frailty give rise to different population-level distribution for analyzing survival data, it is appropriate to investigate how the comparative effect of two frailties translates into the comparative effect on the resulting survival distribution.

[1]O.kharazmi@vru.ac.ir

Frequently, in reliability and survival analysis the problem of interest is the lifetime beyond an age t. For example, when a system is working at time t, one is interested in obtaining the reliability of the system beyond$t$. In such case, the random variable of interest, for computing the reliability of the system, is the residual random variable $X_t = X - t|X > t$ with survival function

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)} & \text{if } x\epsilon S_t, \\ 1 & \text{o.w} \end{cases}$$

where $\bar{F}$ denotes the survival function of X and $S_t = \{x; x > t\}$

Here we briefly recall the definition of proportional odds and proportional odds frailty model .See Bennett (1983) , Kirmani and Gupta (2001),Marshall and Olkin (1997) and the referencess therein for more details about these models. Let X_0 and X be two nonnegative random variables ,with cumulative distribution functions $F_0(x)$ and $F(x)$ and survival functions $\bar{F}_0(x)$ and $F\bar{(x)}$, respectively.The odds functions of X_0 and X are given by

$$\phi_0(x) = \frac{\bar{F}_0(x)}{F_0(x)} \tag{1}$$

and

$$\phi(x) = \frac{\bar{F}(x)}{F(x)} \tag{2}$$

for $x \geq 0$,respectively.We say that X_0 and X satisfy the proporttional odds model with positive proportional constant α if

$$\phi(x) = \alpha\phi_0(x), x \geq 0, \tag{3}$$

where $\phi(x)$ and $\phi_0(x)$ are the population odds function and the baseline , respectively.

Recently, Marshall and Olkin introduced a family of distributions by adding a new parameter to a survival function. Suppose that the $F(. \mid \gamma)$ is defined in terms of the underlying distribution F by the formula

$$\frac{\bar{F}(x \mid \gamma)}{F(x \mid \gamma)} = \gamma\frac{\bar{F}(x)}{F(x)} \tag{4}$$

The family $\{F(. \mid \gamma), \gamma > 0\}$ is said proportional odds frality model.

2 Dynamic proportional odds model

In this section we will give the defenition and some results about DPO and PO frailty models.

Definition 1. *Let X_0 and X be two nonnegative random variables with cumulative distribution functions $F_0(x)$ and $F(x)$ and survival functions $\bar{F}_0(x)$ and $F\bar{(x)}$, respectively. The odds functions of X_0 and residual odds function X are given by*

$$\phi_0(x) = \frac{\bar{F}_0(x)}{F_0(x)} \tag{5}$$

and

$$\phi_t(x) = \frac{\bar{F}_t(x)}{F_t(x)} \tag{6}$$

for $x \geq 0$,respectively.We say that X_0 and X satisfy the dynamic proportional odds (DPO)model with positive continuous proportional function $\alpha(x,t)$ if

$$\phi_t(x) = \alpha(x,t)\phi_0(x) \tag{7}$$

where $\phi_t(x)$ and $\phi_0(x)$ are the dynamic population odds function and the baseline one,respectively.

Example 1. *By appling residul life time distributions in PO model then DPO has the following structure.*

$$\phi_t(x) = \alpha \frac{F_0(x)}{F_0(x) - F_0(t)} \phi_0(x) \tag{8}$$

It is easily seen that PO model is a special case of DPO model as $t \to 0$.

Now we develop some properties on stochastic comparisons of the dynamic proportional odds model.

Theorem 1. *Suppose (2.3)holds,*

1. *if $0 < \alpha(x,t) \leq 1$, then $X_t \leq_{lr} X_0$*
2. *if $\alpha(x,t) \geq 1$,then $X_0 \leq_{lr} X_t$*

where Lr denotes likelihood ratio order , X_0 and X_t are the baseline variable and the residual of population variable respectively.

Let X_0 and X be two nonnegative random variables that satisfy the dynamic proportional odds (DPO)model

$$\phi_t(x) = \alpha(x,t)\phi_0(x) \tag{9}$$

and Y_0 and Y be two nonnegative random variables that satisfy the dynamic proportional odds (DPO)model

$$\psi_t(x) = \beta(x,t)\psi_0(x) \tag{10}$$

Theorem 2. *Suppose (2.4) and (2.5) are satisfied,*
if $\alpha \leq \beta$, $X_0 \leq_{st} Y_0$ and $G_0(x) - G_0(t) \leq F_0(x) - F_0(t)$ then

$$X \leq_{st} Y \tag{11}$$

Definition 2. *Suppose that the residul life time $F_t(. \mid \gamma)$ is defined in terms of the underlying residual distribution F_t by the formula*

$$\frac{\bar{F}_t(x \mid \gamma)}{F_t(x \mid \gamma)} = \alpha \frac{\bar{F}_t(x)}{F_t(x)}. \tag{12}$$

then the family $\{F_t(. \mid \gamma), \gamma > 0\}$ is said dynamic proportional odds frality model.

Example 2. *Suppose underlying distribution has exponnetial distribution then by (2.8) we have*

$$\bar{F}_t(x|\lambda) = \frac{\alpha e^{-\lambda x}}{e^{-\lambda t} - \bar{\alpha} e^{-\lambda x}} \tag{13}$$

Theorem 3. *If F has a density f and hazard rate r. then for $\gamma > 0$, the hazard rate distribution $F_t(x \mid \gamma)$ is given by*

$$r_t(x|\gamma) = \frac{r(x)\bar{F}(t)}{\bar{F}(t) - \bar{\gamma}\bar{F}(x)} \tag{14}$$

References

[1] Bennett, S. (1983),Analysis of survival data by the proportional odds model, *Statistics in medicine*, **2**(2), 273-277.

[2] Cox, DR. (1972), Regression models and life tables , *J R Stat Soc Ser B*, **34**, 187-220.

[3] Gupta, R. C. and Peng, C. (2009), Estimating reliability in proportional odds ratio models, *Computational Statistics and Data Analysis*, **53**(3), 1495-1510.

[4] Kirmani, S. N. U. A. and Gupta, R. C. (2001), On the proportional odds model in survival analysis, *Annals of the Institute of Statistical Mathematics*, **53**(2), 203-216.

[5] Marshall, A. W. and Olkin, I. (2007), *Life distributions*, Springer, New York.

On Properties of Log-Odds Function

Khorashadizadeh, M. [1] and Mohtashami Borzadaran, G.R. [2]

[1] Department of Statistics, University of Birjand
[2] Department of Statistics, Ferdowsi University of Mashhad

Abstract

In this paper, first we introduce the log-odds (LO) and log-odds ratio (LOR) functions and their relations with reliability concepts such as hazard and reversed hazard rate. Then, we proposed a new measure of skewness based on LO function in discrete and continuous lifetime distributions and compare it with Pearson's moment coefficient of skewness and also Groeneveld-Meeden measure of skewness via some examples. Also some results due to bivariate log-odds are discussed.

Keywords: Log-odds rate, Hazard rate, Reversed hazard rate, Second hazard rate, Second reversed rate of failure.

1 Introduction

Zimmer et al. [6] and Wang et al. [4, 5] introduced a new model for continuous time to failure based on the log-odds rate (LOR) which is comparable to the model based on the failure rate. Also Khorashadizadeh et al. [2] defined the discrete log-odds rate and have obtained some characterization results for discrete lifetime distributions.

Suppose that X be a non-negative continuous random variable with probability density function (pdf) $f_X(x)$, cumulative density function (cdf) $F_X(x) = P(X \leq x)$ and reliability function $R_X(x) = P(X > x)$, then the LOR function is defined by $\mathrm{LOR}_X(x) = \frac{\partial}{\partial x}\mathrm{LO}_X(x)$, where $\mathrm{LO}_X(x) = \ln \frac{F_X(x)}{R_X(x)}$ is the log-odds function. Hence,

$$\mathrm{LOR}_X(x) = \frac{f_X(x)}{F_X(x)R_X(x)} = \frac{h_X(x)}{F_X(x)} = \frac{r_X(x)}{R_X(x)} = h_X(x) + r_X(x),$$

[1] m.khorashadizadeh@birjand.ac.ir
[2] grmohtashami@um.ac.ir

where $h_X(x) = \frac{f_X(x)}{R_X(x)}$ is the hazard rate and $r_X(x) = \frac{f_X(x)}{F_X(x)}$ is the reversed hazard rate. The log-odds rate function characterizes the distribution uniquely [4].

Let T be a non-negative discrete random variable with probability mass function (pmf) $p_T(t)$, cdf $F_T(t) = P(T \leq t)$ and reliability function $R_T(t) = 1 - F(t) = P(T > t)$. Then the LO function is defined by $\mathrm{LO}_T(t) = \ln \frac{F_T(t)}{R_T(t)}$. Khorashadizadeh et al. [2] have shown that,

$$\mathrm{LOR}^*_T(t) = \mathrm{LO}_T(t) - \mathrm{LO}_T(t-1) = r^*_T(t) + h^*_T(t),$$

where $r^*_T(t) = \ln \frac{F_T(t)}{F_T(t-1)}$ is the second reversed rate of failure and $h^*_T(t) = -\ln \frac{R_T(t)}{R_T(t-1)}$ is the second rate of failure. The $\mathrm{LOR}^*_T(t)$ function also characterizes the distribution function uniquely [2].

By changing the variables, $Y = \ln X$, $(X = e^Y)$, in continuous case the log-odds rate in terms of $\ln x$, we have $\mathrm{LOR}_Y(y) = \frac{h_Y(y)}{F_Y(y)} = e^y \frac{h_X(e^y)}{F_X(e^y)}$ for $\quad y \geq 0$. Wang et al. [4, 5] proved the following relation, under mild condition, which is usually satisfied in reliability practice,

$$\mathrm{ILOR} \quad \text{in} \quad x \Rightarrow \mathrm{IFR} \Rightarrow \mathrm{ILOR} \quad \text{in} \quad \ln x.$$

The class of log-odds rate in terms of $\ln x$ is more interesting than log-odds rate in terms of x, because the class of LOR in terms of $\ln t$ is weaker than the class of IFR.

Also, for discrete case in terms of $K = \ln T, (T = e^K)$ it has been shown that [2], $\mathrm{LOR}^*_K(k) = \sum_{i=1}^{t} (r^*_T(i) + h^*_T(i)) - \ln \frac{F_T(te^{-1})}{R_T(te^{-1})} + a$, where $a = \ln \frac{p_T(0)}{1-p_T(0)}$.

In general for continuous lifetime distribution we have:

- F has constant LOR in x ($\ln x$) if and only if F has a logistic (log logistic) distribution.
- If F has a Burr XII distribution with parameters α and β, then, for $\beta = 1$, it reduces to log logistic distribution and has constant LOR in $\ln x$, and for $\beta > 1(\beta < 1)$, it is ILOR (DLOR) in $\ln x$.

Also, in discrete lifetime distribution we have,

- F is ILOR in terms of $t\,(k = \ln t)$ if and only if the LO function is convex with respect to (w.r.t), $t\,(k = \ln t)$. Also, for dual class DLOR it is true for concave function.
- If T has a discrete standard logistic distribution, then $LO_T(t) = t + 1$ and by simple transformation the discrete truncated logistics distribution has constant LOR in t.
- If T has a discrete Burr XII distribution with parameters α and θ, then in terms of $\ln t$, F is ILOR for $\theta < e^{-1}$, constant for $\theta = e^{-1}$, and DLOR for $\theta > e^{-1}$.

2 Measure of skewness based on LO

If we define $SM = \int LO(x)dx$, in continuous case and $SM = \sum LO(t)$ in discrete distributions, then these measures may be measure of skewness.

Theorem 1. *Let X be a continuous random variable with cdf, $F(x)$ and log-odds function, $LO(x) = \ln \frac{F(x)}{1-F(x)}$, then if SM be finite such that, $SM = \int_{-\infty}^{\infty} LO(x)dx$, we have, $F(x)$ is symmetric (positive or negative skewed) if and only if $SM = (\geq \text{ or } \leq)0$.*

Proof:
Suppose X has a symmetric distribution, then we have, $F(x) = \overline{F}(2M - x)$, where M is its median (or mean) and therefore $LO(x) = -LO(2M - x)$. Thus, $SM = \int_{-\infty}^{M} LO(x)dx + \int_{M}^{+\infty} LO(x)dx$, so, using the transformation $x = 2M - t$, we have $SM = 0$. Also, when F is positive (negative) skewed, $F(x) > (<)\overline{F}(2M - x)$ and therefore $LO(x) > (<) - LO(2M - x)$, so the "only if" part is proved. The "if" can be proved on contrary.□

Similar results of Theorem 1 can be proved for discrete distribution, using $SM = \sum_{-\infty}^{\infty} LO(t)$. It should be noted that, since SM is just related to cdf, it estimating is more easier than other measures of skewness like Pearson's moment coefficient of skewness [3], $\gamma_1 = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$ and also Groeneveld-Meeden measure of skewness [1] $\gamma_2 = \frac{(\mu - M)}{E|X-M|}$, where M is median.

3 Bivariate case

Let $LO_1(x)$ and $LO_2(y)$ denote the marginal log odds functions of $F_1(x)$ and $F_2(y)$ respectively. The bivariate log odds function can be defined as,

$$LO(x, y) = \ln\left(\frac{F(x, y)}{1 - F(x, y)}\right).$$

We obtained the following properties for $LO(x, y)$:

- The joint distribution can be determined uniquely by,

$$F(x, y) = \frac{1}{1 + e^{-LO(x,y)}}.$$

- If X and Y be two independent random variables, then we have,

$$LO(x, y) = LO_1(x) + LO_2(y) - \ln\left(1 + e^{LO_1(x)} + e^{LO_2(y)}\right).$$

In similar way of Theorem 1, we can proved the following theorem.

Theorem 2. *The bivariate distribution of the random variable* (X, Y)*, is radial symmetric if and only if,*

$$BSM = \int_R \int_R LO^*(x, y)dxdy = 0,$$

where $LO^*(x, y) = \ln \frac{F(x,y)}{\overline{F}(x,y)}$.

Future of the Work

Studying the estimation of the skewness based on data and also a similar definition of skewness and symmetric in bivariate cases are the future of the work.

References

[1] Groeneveld, R.A. and Meeden, G. (1984), Measuring Skewness and Kurtosis, *Journal of the Royal Statistical Society*, Series D (The Statistician), **33**, 391-399.

[2] Khorashadizadeh, M., Rezaei Roknabadi, A.H. and Mohtashami Borzadaran, G.R. (2013), Characterization of life distributions using Log-odds rate in discrete aging, *Communications in Statistics - Theory and Methods*, **42**, 1, 76-87.

[3] Pearson, K. (1895), Contributions to the Mathematical Theory of Evolution, II: Skew Variation in Homogeneous Material, *Transactions of the Royal Philosophical Society*, Series A, **186**, 343-414.

[4] Wang, Y., Hossain, A. M., and Zimmer, W. J. (2003), Monotone log-odds rate distributions in reliability analysis, *Comm. in Statis. Theo. and Meth.,* 32, 11, 2227-2244.

[5] Wang, Y., Hossain, A. M., and Zimmer, W. J. (2008), Useful properties of the three-parameter Burr XII distribution, *Applied Statistics Research Progress, Editor: Mohammad Ahsanullah,* 11-20.

[6] Zimmer, W. J., Wang, Y. and Pathak, P. K. (1998), Log-odds rate and monotone log-odds rate distributions, *Journal of Quality Technology,* **30**, 4, 376-385.

Some Properties of Multivariate Skew-Normal Distribution, with Application to Strength-Stress model

Mehrali, Y. [1]

Department of Statistics, University of Isfahan

Abstract

In recent years, a large number of research works are appeared in the literature dealing with the properties and applications of the skew distributions. Skew distributions are shown to be flexible models for describing different kind of data. In the present study, we consider multivariate skew-normal distribution, and obtain some of its properties. These properties help us to explore the stress-strength model based on the multivariate skew-normal distribution.

Keywords: Linear combination, Multivariate skew-normal distribution, Skew-normal distribution, Stress-strength model.

1 Introduction

Let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density and cumulative distribution functions, respectively. Then, a random variable Z_λ is said to have a *standard skew-normal distribution* with parameter $\lambda \in \mathbb{R}$, denoted by $Z_\lambda \sim SN(\lambda)$, if its probability density function (pdf) is given by (Azzalini 1985, 1986)

$$\phi_{SN}(z;\lambda) = 2\phi(z)\Phi(\lambda z), \qquad z \in \mathbb{R}. \tag{1}$$

Azzalini and Dalla Valle (1996) presented the multivariate skew-normal distribution with the following pdf

$$\phi_{SN_n}(\mathbf{z};\mathbf{\Omega},\boldsymbol{\alpha}) = 2\phi_n(\mathbf{z};\mathbf{\Omega})\Phi(\boldsymbol{\alpha}^T\mathbf{z}), \qquad \mathbf{z} \in \mathbb{R}^n, \tag{2}$$

where $\mathbf{\Omega}$ is $n \times n$ dimensional dispersion matrix, $\boldsymbol{\alpha} \in \mathbb{R}^n$ is vector of shape parameter and $\phi_n(\cdot;\mathbf{\Omega})$ denotes the pdf of the multivariate normal distribution with covariance matrix

[1] yasermehrali@gmail.com

$\mathbf{\Omega}$. We denote by $\mathbf{Z} \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$ and in special case that $\mathbf{\Omega} = \mathbf{I}_n$(Identity matrix), we denote by $\mathbf{Z} \sim SN_n(\boldsymbol{\alpha})$.

Azzalini and Dalla Valle (1996) presented representation of $\mathbf{Z} \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$ as fallow. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)^T$ and

$$\begin{pmatrix} Y_0 \\ \mathbf{Y} \end{pmatrix} \sim N_{n+1}\left(\mathbf{0}, \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{\Gamma} \end{pmatrix}\right), \tag{3}$$

where $\mathbf{\Gamma} = [\gamma_{i,j}]$ is $n \times n$ dimensional correlation matrix. Now if define $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ as

$$Z_i = \delta_i |Y_0| + \sqrt{1-\delta_i^2} Y_i, \tag{4}$$

where $\delta_i = \lambda_i / \sqrt{1+\lambda_i^2}$, $i = 1, \ldots, n$, then $Z_i \sim SN(\lambda_i)$ and $\mathbf{Z} \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$, where $\mathbf{\Omega} = \mathbf{\Delta}\left(\mathbf{\Gamma} + \boldsymbol{\lambda}\boldsymbol{\lambda}^T\right)\mathbf{\Delta}$, $\boldsymbol{\alpha}^T = \frac{\boldsymbol{\lambda}^T \mathbf{\Gamma}^{-1} \mathbf{\Delta}^{-1}}{\sqrt{1+\boldsymbol{\lambda}^T \mathbf{\Gamma}^{-1} \boldsymbol{\lambda}}}$, $\boldsymbol{\lambda}= (\lambda_1, \ldots, \lambda_n)^T$ and $\mathbf{\Delta} = diag\left\{\sqrt{1-\delta_1^2}, \ldots, \sqrt{1-\delta_n^2}\right\}$. In matrix form, we can represent

$$\mathbf{Z} = \boldsymbol{\delta} |Y_0| + \mathbf{\Delta} \mathbf{Y}. \tag{5}$$

In case $n = 2$, Gupta and Brown (2001) evaluated $P(Z_1 < Z_2)$ as follow

$$P(Z_1 < Z_2) = \frac{1}{\pi} \tan^{-1}\left(\frac{\delta_2 - \delta_1}{\sqrt{2 - \delta_1^2 - \delta_2^2}}\right) + \frac{1}{2}. \tag{6}$$

where $\delta_i = \lambda_i / \sqrt{1+\lambda_i^2}$, $i = 1, 2$. Let $X_i \stackrel{d}{=} \mu_i + \sigma_i Z_i$, $i = 1, 2$, where Z_1 and Z_2 represented as in (4). Mehrali and Asadi (2010) evaluated $P(X_1 < X_2)$ as follow

$$P(X_1 < X_2) = \Phi_{SN}\left(k/\sqrt{1+\delta^2}; \delta\right), \tag{7}$$

where $\Phi_{SN}(\cdot; \delta)$ is the cdf of $SN(\delta)$, $k = \frac{1}{\sigma}\frac{\mu_2 - \mu_1}{\sigma_1}$ and $\delta = \frac{a_1\delta_1 + a_2\delta_2}{\sigma}$, where δ_i, $i = 1, 2$ are as in (6), $\sigma^2 = a_1^2\left(1-\delta_1^2\right) + a_2^2\left(1-\delta_2^2\right)$, $a_1 = 1$ and $a_2 = -\frac{\sigma_2}{\sigma_1}$. Here we are interested in evaluation of the following model of which presented by Kotz et al. (2003) as

$$P(X_1 < X_2 < \cdots < X_n) \tag{8}$$

where $X_i \stackrel{d}{=} \mu_i + \sigma_i Z_i$, $i = 1, \ldots, n$, where $\mathbf{Z} = (Z_1, \ldots, Z_n)^T \sim SN_n(\mathbf{\Omega}, \boldsymbol{\alpha})$ with representation 5. For this purpose we study some properties of multivariate skew-normal distribution which help us to explore the stress-strength model based on the multivariate skew-normal distribution.

2 Some properties of multivariate skew-normal distribution

In this section, we present some properties of multivariate skew-normal distribution. These results help us to evaluate the stress-strength model based on the multivariate skew-normal distribution.

Let $Z_\lambda \sim SN(\lambda)$ independent of $\mathbf{W} \sim N_n(\mathbf{0}, \mathbf{\Sigma})$, where $N_n(\mathbf{0}, \mathbf{\Sigma})$ denotes the multivariate normal distribution.

a) If we define $\mathbf{Y} = \mathbf{H}^T\mathbf{W} + \mathbf{k}Z_\lambda$, then $\mathbf{Y} \sim SN_n(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, where $\mathbf{H}$ is $n \times n$ symmetric matrix, $\mathbf{k} \in \mathbb{R}^n$, $\boldsymbol{\Omega} = \mathbf{H}^T\Sigma\mathbf{H} + \mathbf{k}\mathbf{k}^T$, $\boldsymbol{\alpha}^T = \frac{\delta \mathbf{k}^T \boldsymbol{\Omega}^{-1}}{\sqrt{1-\delta^2 \mathbf{k}^T \boldsymbol{\Omega}^{-1}\mathbf{k}}}$ and $\delta = \lambda/\sqrt{1+\lambda^2}$.

Let $Z_\lambda \sim SN(\lambda)$. Then

$$E\left[\Phi_n\left(\mathbf{k}Z_\lambda + \mathbf{u}; \boldsymbol{\Sigma}\right)\right] = \Phi_{SN_n}\left(\mathbf{u}; \boldsymbol{\Omega},\boldsymbol{\alpha}\right),$$

where $\Phi_n(\cdot; \boldsymbol{\Sigma})$ is the cdf of $N_n(\mathbf{0}, \boldsymbol{\Sigma})$, $\Phi_{SN_n}(\cdot; \boldsymbol{\Omega},\boldsymbol{\alpha})$ is cdf of $SN_n(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, $\boldsymbol{\Omega}$ is same as lemma 2 part (c) and $\boldsymbol{\alpha}^T = -\frac{\delta \mathbf{k}^T \boldsymbol{\Omega}^{-1}}{\sqrt{1-\delta^2 \mathbf{k}^T \boldsymbol{\Omega}^{-1}\mathbf{k}}}$.

Let define

$$\Psi_n(\mathbf{k}, \mathbf{u}, \boldsymbol{\Sigma}) = \int_0^\infty \Phi_n(\mathbf{k}z + \mathbf{u}; \boldsymbol{\Sigma})\, \phi(z)\, dz.$$

Then

$$\Psi_n(\mathbf{k}, \mathbf{u}, \boldsymbol{\Sigma}) = \frac{1}{2}\Phi_{SN_n}(\mathbf{u}; \boldsymbol{\Omega},\boldsymbol{\alpha}),$$

where $\boldsymbol{\Omega}$ is same as lemma 2 part (c) and $\boldsymbol{\alpha}^T = -\frac{\mathbf{k}^T \boldsymbol{\Omega}^{-1}}{\sqrt{1-\mathbf{k}^T \boldsymbol{\Omega}^{-1}\mathbf{k}}}$.

Let $\mathbf{Z} \sim SN_n(\boldsymbol{\Omega}, \boldsymbol{\alpha})$ with representation 5 and $\mathbf{D}$ be an $(n-1) \times n$ matrix, then

$$P(\mathbf{D}\mathbf{Z} < \mathbf{u}) = \Phi_{SN_{n-1}}(\mathbf{u}; \boldsymbol{\Omega}^*,\boldsymbol{\alpha}^*),$$

where where $\boldsymbol{\Omega}^* = \mathbf{D}^T\boldsymbol{\Omega}\mathbf{D}$ and $\boldsymbol{\alpha}^{*T} = \frac{\boldsymbol{\delta}^T \boldsymbol{\Omega} \mathbf{D}^{T^{-1}}}{\sqrt{1-\boldsymbol{\delta}^T \boldsymbol{\Omega}^{-1} \boldsymbol{\delta}}}$, where $\boldsymbol{\delta}$ is same as 4.

3 Stress-strength models in multivariate skew-normal distribution

Theorem 1. *Let $X_i \stackrel{d}{=} \mu_i + \sigma_i Z_i$, $i = 1, \ldots, n$, with representation as in (4). Then*

$$P(X_1 < X_2 < \cdots < X_n) = \Phi_{SN_{n-1}}(\mathbf{u}; \boldsymbol{\Omega}^*,\boldsymbol{\alpha}^*),$$

where $\mathbf{u} = (u_1, \ldots, u_{n-1})^T$, $u_i = \mu_{i+1} - \mu_i$, $\boldsymbol{\Omega}^*$ and $\boldsymbol{\alpha}^*$ are same as lemma 2 part (a) with

$$\mathbf{D} = \begin{bmatrix} a_1 & b_1 & & & \mathbf{0} \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & a_{n-1} & b_{n-1} \end{bmatrix}.$$

and $a_i = \sigma_i$ and $b_i = -\sigma_{i+1}$.

In special cases, we can find main results of Mehrali and Asadi (2010) and Gupta and Brown (2001) as in 6 and 7.

References

[1] Azzalini, A. (1985) A class of distributions which includes the normal ones, *Scandinavian Journal of Statistics,* **12(2)**, 171-178.

[2] Azzalini, A. (1986) Further results on a class of distributions which includes the normal ones, *Statistica*, **46,** 199-208.

[3] Azzalini, A. and Chiogna, M. (2004) Some results on the stress–strength model for skew-normal variates, *METRON - International Journal of Statistics*, **62(3)**, 315-326.

[4] Azzalini, A., and Dalla Valle, A. (1996). The multivariate skew-normal distribution. *Biometrika*, **83(4)**, 715-726.

[5] Chiogna, M., (1998) Some results on the scalar skew-normal distribution, *J. Ital. Statist. Soc.,* **7(1)**, 1-13.

[6] Gupta, R.C. and Brown, M. (2001) Reliability studies of skew normal distribution and its application to a strength-stress model, *Commun Stat Theory Methods,* **30(11)**, 2427-2445.

[7] Kotz, S., Lumelskii, Y., and Pensky, M. (2003) *The stress-strength model and its generalizations*, World Scientific Publishing, ISBN 981-238-057-4.

[8] Mehrali, Y. and Asadi, M. (2012). On the stochastic and reliability properties of skew distributions. *The 11th Iranian Statistical Conference*, Iran University of Science and Technology, Tehran, Iran.

[9] Roberts, H. V. (1988) *Data Analysis for Managers with Minitab*, Scientific Press: Redwood City, CA.

Nonparametric and Parametric Estimation of Survival Function

Mireh, S. [1] and Khodadadi, A. [2]

Department of Statistics, University of Shahid Beheshti

Abstract

This paper considers a general degradation path model and failure time data with traumatic failure mode. It provides a review of the nonparametric estimator of survival function, studied by Bagdonavicius, and considers the parametric estimation of survival function of failure times with a hazard rate in the degradation space. In addition, we discuss the comparison of both parametric and nonparametric methods according to simulated and real data.

Keywords: Degradation models, Failure times, Hazard rate, Nonparametric and parametric estimation, Survival function.

1 Introduction

Analyzing survival data is historically based on $(T_1, \ldots, T_n)$ each measuring an individual time to event. It is difficult to assess reliability with traditional life tests that record only time to the failure. In some cases, degradation is measured directly by passage of time. Thus, it is necessary to define a level of degradation at which a failure is said to have occurred. We define soft and hard failures in terms of a specified level of degradation and traumatic failures.

Usually, one attempts to conditionally define the hazard rate such as Bagdonavicus[3] that define $\lambda(t|A) = \lambda_0(t) \times \lambda(g(t, A))$ where g is a given non-decreasing function.

Statistical analysis of linear degradation and multiple failure modes using nonparametric method are discussed by Bagdonavicuset al.[1]. They have presented reliability characteristics using a semiparametric method[2]. In this work, we estimate the survival

[1]s_mireh@sbu.ac.ir
[2]khodadadi@sbu.ac.ir

function from the degradation and failure time data using parametric and nonparametric methods.

This paper is organized as follows. Section 2 defines the joint models for degradation and failure time. Section 3 is devoted to a review on the estimation of survival function. Section 4 deals with the performance of the two methods through a set of real data. In section 5, a simulation study is performed for the comparison of both methods usingin various sample sizes.

2 Joint models for degradation and failure time

Assume that the degradation of an item is given by stochastic process $Z(t)$. We denote the true degradation path of particular unit by $g(t)$, but the observed degradation processes is a degradation path plus error: $Z(t) = g(t, A) + e$, where A is a vector of unknown parameters. Bagdonavicus[1] has studied a linear degradation model with multiple failure modes.

Suppose the life time T^0 is the first time of crossing a ultimate threshold z_0 for $Z(t)$. If we denote h for the inverse function of g and h' for its partial derivative then: $T^0 = h(z_0, A)$. Let T^1 be traumatic failure time. Thus the moment of the observed failure is: $T = \min(T^0, T^1)$.

Suppose the random variable T^1 has the intensity $\lambda^{(1)}(z)$ and the cumulative intensity $\Lambda^{(1)}(z)$, depending on the degradation level. The conditional survival function of T^1 given A is:

$$S^{(1)}(t|A) = \exp\Big\{-\int_0^t \lambda^{(1)}(g(y,a))dy\Big\} = \exp\Big\{-\int_0^{g(t,a)} h'(z,a)d\Lambda^{(1)}(z)\Big\}.$$

We can obtain the survival function of the random variable T:

$$S(t) = \int_{g(t,a)<z_0} \exp\Big\{-\int_0^{g(t,a)} h'(z,a)d\Lambda^{(1)}(z)\Big\}d\pi(a) \tag{1}$$

where π is the distribution function of A.

3 Estimation of the reliability functions

Suppose the data are collected from n unit: $(T_1, Z_1, \delta_1), \ldots, (T_n, Z_n, \delta_n)$ where T_i is the failure time, Z_i is the degradation level and δ_i is the indicator of the failure modes. In the parametric method, we set a distribution on A and the parameters are estimated using MLE. However in the nonparametric method, the estimators are given by the following: The estimation of the distribution function and the cumulative hazard function:

$$\hat{\pi}(a) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i \le a\}} \quad , \quad \hat{\Lambda}(z) = \sum_{Z_i \le z, \delta_i = 1} \frac{1}{\sum_{j, Z_j \le Z_i} h'(Z_j, A_i)}$$

4 Estimation by real data

The real data are the wear and failure time data of 79 bus tires. The critical tire wear value is $z_0 = 15mm$. Set $g(T, A) = T/A$. We have used the Exponentiated Weibull family

Table 1: Maximum Likelihood Estimators: Exponentiated Weibull distribution and intensity functions

Parameter	Estimation
β	1.4836
σ	1.4772
θ	130.439
α_1	0.0425
ν_1	6.4332

as the parametric family of π and let $\lambda^{(1)}(z) = (\alpha_1 z)^{\nu_1}$. The MLE of the parameters are summarized in Table 1.

We obtain a parametric estimation of survival function by substituting parameter estimates in (1). In addition, we calculate the nonparametric estimation. Figure 1 gives graphs of empirical cdf of A and the estimators of $S(t)$.

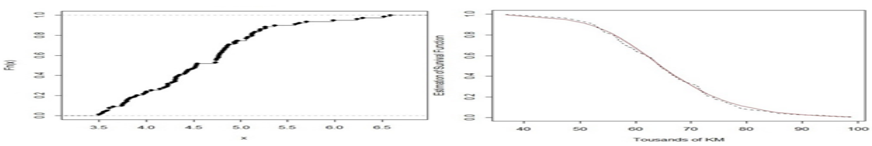


Figure 1: (Left) Empirical cdf of A; (Right) parametric(solid line) and nonparametric(dotted line) estimators of $S(t)$

5 Simulation study

Example 1. *In this example, we compare the parametric and nonparametric estimations by using small, moderate, and large sample sizes. We have generated vector A from the Weibull distribution with parameters (5, 2) and set* $z_0 = 10$.

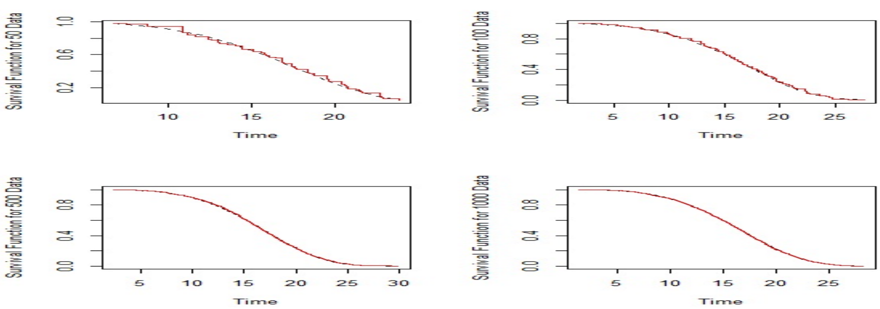


Figure 2: Parametric(dotted line) and nonparametric(solid line) estimators of $S(t)$ in different sample sizes

Example 2. *We consider simulations of n=100 degradation curves* $Z(t, \theta_1, \theta_2) = e^{\theta_1}(1 + t)^{\theta_2}, t \in [0, 12]$ *with a hazard rate in the degradation space of Weibull-type(*$\alpha = 5, \beta = 2.5$*) and* $A = (\theta_1, \theta_2)$ *is a Gaussian vector with mean (-2,2) and* $Var\theta_1 = Var\theta_2 = 0.1^2$.

Figure 3 shows the distribution function of θ_2 and the nonparametric estimation of the cumulative hazard rate.

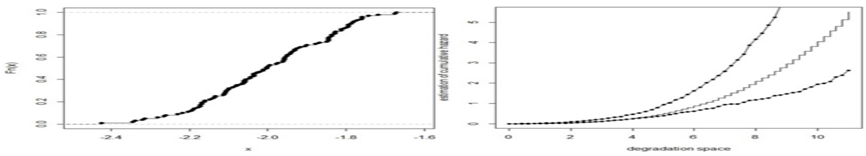


Figure 3: (Left) Empirical cdf of θ_1; (Right) Nonparametric hazard rate and 95% confidence band

References

[1] Bagdonavicus, V., Bikelis, A. and Kasakevicius, V. (2004), Statistical Analysis of Linear Degradation and Failure Time Data with Multiple Failure Modes. *Lifetime Data Analysis*, **10**, 65-81.

[2] Bagdonavicus, V., Haghighi, F. and Nikulin, M. (2005), Statistical Analysis of Degradation and Failure Time Data using the General Path Model. *Communication in Statistics-Theory and Methods*, **34**, 1771-1793.

[3] Bagdonavicius, V. and Nikulin, M. (2004), *Semiparametric Analysis of Degradation and Failure Times Data with Covariates*, Birkauser.

Determining the Warranty Period Using Pitman Measure of Closeness

Mirfarah, E. [1] and Ahmadi, J. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

In this paper, we study the determination of the warranty period in view of a warranty policy where the manufacture accept to minimally repaired the failure product. To do this, the problem of predicting the time of minimal repair based on a progressive Type-II censored sample is considered. We utilize the property of Pitman measure of closeness and propose a method to find the closest predictor. Since, over-predication may be more important in a warranty problem, asymmetry loss is also considered in the probability of closeness.

Keywords: Pitman measure of closeness, Prediction, Warranty period, Progressively Type-II censored order statistics, Minimal repair.

1 Introduction

A warranty is a contractual agreement in which the manufacturer accept to rectify all failures occurring up to a given amount of time (warranty period) from the date of purchase. Manufacturers offer many types of warranties to promote their products such as repair, replacement or cash refund. Offering warranty leads to additional costs to the manufacturer, so choosing the best policy reduces the servicing costs of manufacturer. A detailed discussion of various issues related to warranties can be found in [5].

In this paper, we consider a policy where warranty is not renewed on product failure but it is minimally repaired. This means that, on repair, the failure rate of the item remains the same as just prior to failure. Such policies are suitable for complex and expensive products where repair typically involves a small part of the product. We are interested to predict the time of ith minimal repair to determine the perfect warranty

[1]elham_mirfarah@yahoo.com
[2]ahmadi-j@um.ac.ir

period. Moreover, the minimal repair times have the same joint distribution as record (R) values (see [2]), so to simplify the notation, in the rest of this paper, let R_i be the ith R, which has the same distribution as the ith minimal repair times.

Now, consider a life testing experiment involving n experimental units. Suppose m complete failure are to be observed, such that when the ℓth failure is observed, a_i items are randomly removed from the test. The vector $(a_1, \cdots, a_m)$ is fixed prior to the experiment. Let $X_{\ell:m:n}$ denotes the ℓth progressively Type-II censored (PTC) order statistic (OS) of the observed sample. We want to use this information to predict the minimal repair times. Statistical prediction play an important role in determining the warranty length. Many researches consider the prediction of a subset of ordered data based on an independent observed sample of ordered data and different methods are considered in the literatures, for more details see [3]. Here, the concept of Pitman's measure of closeness (PMC) is used to proposed a method for prediction.

The concept of PMC was introduced by [6] and faced a considerable attention in ordered data topics after [1]. For more review about the PMC, see the monograph by [4]. More formally, the PMC in prediction context is defined as follows.

Definition 1. *If T_1 and T_2 are two predictors of a random variable Y, then T_1 is a Pitman closer predictor than T_2, under loss function $L(\cdot,\cdot)$, if* $\Pr[L(T_1, Y) < L(T_2, Y)] \geq \frac{1}{2}$. *Moreover, let $\Lambda = \{T_1, T_2, ..., T_n\}$ be a non-empty class of predictors of Y. Then, T_i is the Pitman-closest predictor if, for every $T_j \in \Lambda$ such that $i \neq j$, we have* $\Pr[L(T_i, Y) < L(T_j, Y)] \geq \frac{1}{2}$.

Depending on the situation of problem, one can use different loss functions in the probability of PMC. Absolute loss function, i.e., $L(T, Y) = |T - Y|$, is the most common loss in PMC concept. However, in many warranty problem, under-prediction is more important than over-prediction or vice versa. So, apart from absolute loss function, in this paper, we consider the following loss function

$$L_1(T, Y) = \begin{cases} 0, & T < Y; \\ T - Y, & T > Y. \end{cases}$$

In the rest of this paper, we formulate the warranty issue as a prediction problem and study the PMC of OSs from current PTC sample to R values from a future sequence. Considering two loss functions in the probability of PMC, results have been compared.

2 Main result

Let $X_{\ell:m:n}$ denote the ℓth OS from a PTC sample with an absolutely continuous cumulative distribution function $F(\cdot)$ and probability density function $f(\cdot)$ and R_i be the ith R with the same parent distribution as $X_{\ell:m:n}$. Since PMC has the transitivity property in a class of ordered data, we consider the PMC of two adjacent OSs, i.e.,

$$\mathrm{PMC}(X_{\ell:m:n}, X_{\ell+1:m:n}|R_i) = \Pr(|X_{\ell:m:n} - R_i| < |X_{\ell+1:m:n} - R_i|), \tag{1}$$

The exact expression for (1) is given as follows

$$\begin{aligned} \mathrm{PMC} \quad & (X_{\ell:m:n}, X_{\ell+1:m:n}|R_i) = \Pr(X_{\ell:m:n} + X_{\ell+1:m:n} > 2R_i) \\ = \quad & \Pr(X_{\ell:m:n} > R_i) + \Pr(X_{\ell:m:n} < R_i, X_{\ell:m:n} + X_{\ell+1:m:n} > 2R_i) \\ = \quad & \sum_{t=1}^{\ell} c_{\ell-1}^{\mathcal{R}} a_t^{\mathcal{R}}(\ell) \left\{ \frac{1}{\gamma_t^{\mathcal{R}}} \left(\frac{1}{\gamma_t^{\mathcal{R}} + 1} \right)^{i+1} + \frac{1}{\gamma_{\ell+1}^{\mathcal{R}}} B(t, i) \right\}, \end{aligned}$$

where

$$B(t,i) = \int_0^1 \int_y^1 u^{\gamma_t^{\mathcal{R}} - \gamma_{\ell+1}^{\mathcal{R}} - 1} [\bar{F}(2F^{-1}(1-y) - F^{-1}(1-u))]^{\gamma_{\ell+1}^{\mathcal{R}}} \frac{\{-\log y\}^i}{i!} du dy.$$

PMC depends on the parent distribution of OSs. In the next section, we will find the result for exponential distribution.
Now, let us consider the problem of prediction using PMC with $L_1(\cdot,\cdot)$. Given a PTC sample, the PC probability to a R from a future independent sequence, under loss function $L_1(\cdot,\cdot)$, is given by

$$\Pr(L_1(X_{\ell:m:n}, R_i) < L_1(X_{\ell+1:m:n}, R_i)) = \sum_{t=1}^{\ell+1} c_\ell^{\mathcal{R}} a_t^{\mathcal{R}}(\ell+1) \left\{ \frac{1}{\gamma_t^{\mathcal{R}}} \left(\frac{1}{\gamma_t^{\mathcal{R}}+1} \right)^{i+1} \right\}.$$

It is important to note that, in the case of $L_1(\cdot,\cdot)$, PMC is non-parametric. In the next section, we will compare this results.

3 Example

We present our result in the previous section for the standard exponential in the case of absolute loss function. Then, the probability of closeness is compared with the results of non-parametric PMC.

Let the parent distribution be standard exponential, then $B(t,i)$ in the case of exponential is given as below

$$B(t,i) = \begin{cases} \frac{1}{\gamma_t^{\mathcal{R}} - 2\gamma_{\ell+1}^{\mathcal{R}}} \left\{ \left(\frac{1}{1+2\gamma_{\ell+1}^{\mathcal{R}}} \right)^{i+1} - \left(\frac{1}{1+\gamma_t^{\mathcal{R}}} \right)^{i+1} \right\}, & \gamma_t^{\mathcal{R}} \neq 2\gamma_{\ell+1}^{\mathcal{R}}, \\ (i+1) \left(\frac{1}{1+2\gamma_{\ell+1}^{\mathcal{R}}} \right)^{i+2}, & \gamma_t^{\mathcal{R}} = 2\gamma_{\ell+1}^{\mathcal{R}}. \end{cases}$$

Table 1 present the PMC of PTC OSs with censoring scheme $R = (20, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ to the first 6 Rs of future sequence. Table 2 is the non-parametric PMC when the loss function is $L_1(\cdot,\cdot)$.

Table 1: PMC for standard exponential.

ℓ	i = 0	1	2	3	4	5
1	0.038	0.002	0.000	0.000	0.000	0.000
2	0.135	0.015	0.002	0.000	0.000	0.000
3	0.233	0.040	0.006	0.001	0.000	0.000
4	0.331	0.077	0.015	0.003	0.000	0.000
5	0.430	0.129	0.032	0.007	0.001	0.000
6	0.530	0.199	0.061	0.016	0.004	0.001
7	0.631	0.292	0.110	0.036	0.011	0.003
8	0.739	0.421	0.199	0.082	0.031	0.011
9	0.871	0.639	0.405	0.228	0.118	0.056

Table 2: Non-parametric PMC for $L_1(\cdot,\cdot)$.

ℓ	i = 0	1	2	3	4	5
1	0.129	0.014	0.001	0.000	0.000	0.000
2	0.226	0.037	0.005	0.001	0.000	0.000
3	0.323	0.073	0.014	0.002	0.000	0.000
4	0.419	0.123	0.029	0.006	0.001	0.000
5	0.516	0.188	0.056	0.014	0.003	0.001
6	0.613	0.273	0.099	0.031	0.009	0.002
7	0.710	0.382	0.170	0.066	0.023	0.008
8	0.806	0.524	0.288	0.140	0.062	0.026
9	0.903	0.713	0.501	0.320	0.191	0.109

To find the Pitman closest OS for the specific R value, find the first ℓ which PMC is greater than 0.5. For example $X_{6:10:20}$ is the Pitman closest predictor for the first R when the loss function is absolute error. From Table 1 and 2, it can be seen that by ignoring the under-predict error, smaller OSs get closer to R value comparing with the time that we use absolute loss function.

References

[1] Balakrishnan, N., Davies, K. F., and Keating, J. P. (2009). Pitman closeness of order statistics to population quantiles. *Communications in Statistics-Simulation and Computation*, **38**, 802–820.

[2] Balakrishnan, N., Kamps, U., and Kateri, M. (2009). Minimal repair under a step-stress test. *Statistics and Probability Letters*, **79**, 1548–1558.

[3] Geisser, S. (1993). *Predictive Inference: An Introduction.* Chapman and Hall, New York.

[4] Keating, J. P., Mason, R. L. and Sen, P. K. (1993). *Pitman's Measure of Closeness: A Comparison of Statistical Estimators.* Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

[5] Murthy, D. N. P. and Blischke, W. R. (1992). Product warranty management-III: a review of mathematical models, *European Journal of Operational Research*, **63**, 1–34.

[6] Pitman, E. J. G. (1937). The "closest" estimate of statistical parameters. *Proceedings of the Cambridge Philosophical Society*, **33**, 212–222.

Bayesian Inference for the Rayleigh Distribution Based on Record Ranked Set Samples

MirMostafaee, S.M.T.K. [1] and Aminzadeh, M. [2]

Department of Statistics, University of Mazandaran

Abstract

In this paper, we discuss the Bayesian estimation problem for the Rayleigh distribution based on upper record ranked set samples. The Bayes estimators are obtained with respect to two different loss functions. We also obtain the Bayes confidence intervals for the parameter of the Rayleigh distribution. Finally, we present a simulation study for the purpose of numerical comparison.

Keywords: General entropy loss function, Maximum likelihood estimator, Simulation.

1 Introduction

The record ranked set sampling scheme has been introduced recently by [2]. Here, we describe this sampling scheme, briefly, according to [2] as follows: Suppose that we have m independent sequences of continuous random variables. If $R_{i,i}$ denotes the i-th record value in the i-th sequence for $i = 1, ..., m$, then i-th sequence sampling is terminated when $R_{i,i}$ is observed. Then, the only available observations, which are called record ranked set sample (RRSS), include $R_{1,1}, \cdots, R_{m,m}$. These data can be minimal repair times of some reliability systems as mentioned in [2]. The Rayleigh distribution plays a key role in reliability analysis and therefore estimation of its parameter is important. In this paper, we consider the point and interval estimation problem for the Rayleigh distribution based on observed upper RRSSs. Main results are given in Section 2 and a simulation study is presented in Section 3.

[1] m.mirmostafaee@umz.ac.ir
[2] m.aminzadeh1370@yahoo.com

2 Main Results

We say that X has a Rayleigh distribution if its pdf is given by

$$f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad x > 0, \quad \sigma > 0. \tag{1}$$

Let $\boldsymbol{U} = (U_{1,1}, \cdots, U_{m,m})$ be an upper RRSS from the Rayleigh distribution with pdf given in (1), then the likelihood function for the parameter σ given $\boldsymbol{u} = (u_{1,1}, \cdots, u_{m,m})$ is (see [2])

$$\begin{aligned} L(\sigma|\boldsymbol{u}) &= = \prod_{i=1}^{m} \frac{\{-\log(1 - F(u_{i,i}))\}^{i-1}}{(i-1)!} f(u_{i,i}) \\ &= \frac{\exp\left(-\frac{\sum_{i=1}^m u_{i,i}^2}{2\sigma^2}\right)}{\sigma^{m(m+1)} 2^{m(m-1)/2}} \prod_{i=1}^{m} \frac{u_{i,i}^{2i-1}}{(i-1)!}, \end{aligned} \tag{2}$$

where $u_{i,i} > 0$ for $i = 1, ..., m$ and $\sigma > 0$. The maximum likelihood estimator (MLE) of σ is readily obtained to be

$$\widehat{\sigma}_{ML} = \sqrt{\frac{\sum_{i=1}^m U_{i,i}^2}{m(m+1)}}. \tag{3}$$

For the Bayesian estimation, we take the conjugate prior

$$\pi(\sigma) \propto \sigma^{-2b-1} \exp(-a/2\sigma^2), \quad \sigma > 0, \tag{4}$$

where a and b are positive hyperparameters. Note that for $a = b = 0$, we arrive at the non-informative prior. Now, from Equations (2) and (4), the posterior distribution of σ, given $\boldsymbol{u}$, becomes

$$\pi(\sigma|\boldsymbol{u}) = \frac{2w^{b+m(m+1)/2} \exp(-w/\sigma^2)}{\Gamma\big(b + m(m+1)/2\big)\sigma^{2b+m(m+1)+1}}, \tag{5}$$

where $\Gamma(\cdot)$ is the complete gamma function, $W = \dfrac{a + \sum_{i=1}^m U_{i,i}^2}{2}$, and w is the observed value of W.

Let $\widehat{\sigma}$ be an estimator of σ, then the squared error loss (SEL) function is defined as $L_1(\sigma, \widehat{\sigma}) = (\widehat{\sigma} - \sigma)^2$. The Bayes estimator of σ under SEL function, based on RRSS, is the mean of the posterior density (5)

$$\widehat{\sigma}_{BS} = \frac{\Gamma\big(b + [m(m+1) - 1]/2\big)\sqrt{W}}{\Gamma\big(b + m(m+1)/2\big)}.$$

The SEL function is symmetric namely it assigns equivalent dimensions to underestimation and overestimation. But in many real situations, overestimation and underestimation have different consequences. Therefore, we consider a useful asymmetric loss function, called the general entropy loss (GEL) function, introduced by [1], which is defined as

$$L_2(\sigma, \widehat{\sigma}) = (\widehat{\sigma}/\sigma)^q - q\log(\widehat{\sigma}/\sigma) - 1, \quad q \neq 0.$$

The sign and magnitude of parameter q must be determined properly. The positive values of q cause the overestimation to get more serious than underestimation and vice versa. The Bayes estimator of σ under GEL function, based on RRSS, is

$$\widehat{\sigma}_{BG} = \left[E(\sigma^{-q}|\boldsymbol{U})\right]^{-1/q} = \left[\frac{\Gamma\big(b+[m(m+1)+q]/2\big)}{\Gamma\big(b+m(m+1)/2\big)}\right]^{-1/q}\sqrt{W}, \tag{6}$$

provided that $b+[m(m+1)+q]/2>0$.

Next, we want to find Bayesian confidence intervals for σ. From (5), we see that $Q_1 = w/\sigma^2|\boldsymbol{u} \sim G\big(b+m(m+1)/2, 1\big)$, where $G(\lambda_1,\lambda_2)$ stands for the gamma distribution with the shape parameter λ_1 and scale parameter λ_2. Let $\xi_\gamma(\lambda_1,\lambda_2)$ denote the upper γ-th quantile of $G(\lambda_1,\lambda_2)$, that is $P(V>\xi_\gamma(\lambda_1,\lambda_2))=\gamma$ where $V\sim G(\lambda_1,\lambda_2)$. Then, a $100(1-\alpha)\%$ two-sided equi-tailed Bayesian confidence interval (TEB CI) for σ is given by

$$\left(\sqrt{\frac{W}{\xi_{\alpha/2}(b+m(m+1)/2,1)}}, \sqrt{\frac{W}{\xi_{1-\alpha/2}(b+m(m+1)/2,1)}}\right).$$

We also derive the highest posterior density intervals (HPDIs) for σ. For unimodal posterior distributions, the HPDIs are the same as their corresponding shortest credible intervals. Since the posterior pdf of σ given in (5) is unimodal, it can be easily verified that a $100(1-\alpha)\%$ HPDI for σ, given $W=w$, possesses the form (σ_L,σ_U) such that

$$\frac{\Gamma(A(b,m), w/\sigma_U^2, w/\sigma_L^2)}{\Gamma(b+m(m+1)/2)} = 1-\alpha, \quad \text{and} \quad \left(\frac{\sigma_U}{\sigma_L}\right)^{2A(b,m)+1} = e^{w(\sigma_L^{-2}-\sigma_U^{-2})},$$

where $A(b,m) = b+m(m+1)/2$ and $\Gamma(\nu,u_1,u_2) = \int_{u_1}^{u_2} t^{\nu-1}e^{-t}\mathrm{d}t$ is the generalized incomplete gamma function.

3 A simulation study

In this section, we performed a simulation in order to compare the point and interval estimators. In this simulation, we randomly generated $M = 10000$ upper RRSSs of size $m=6$ from the Rayleigh distribution with $\sigma=1$. We considered 3 cases for the prior distribution described as follows:

Case I: Non-informative prior with $a=b=0$.

Case II: Informative prior with prior information $E(\sigma)=1$ =true value, and $Var(\sigma)=2$ and from (4), we have $a=0.9115$ and $b=1.1519$.

Case III: Informative prior with prior information $E(\sigma)=1$ and $Var(\sigma)=0.5$ which corresponds to $a=1.6989$ and $b=1.5663$.

We then obtained the MLEs and the Bayes estimators of σ under SEL and GEL (for $q=-2$, 2) functions, which are denoted by $\widehat{\sigma}_{ML}(i)$, $\widehat{\sigma}_{BS}(i)$ and $\widehat{\sigma}_{BG}(i)$, in the i-th iteration, respectively. The estimated risks (ERs) of the estimators were obtained using the relations $ER_S(\widehat{\sigma}_{BS}) = \frac{1}{M}\sum_{i=1}^{M}[\widehat{\sigma}_{BS}(i)-\sigma]^2$, and $ER_G(\widehat{\sigma}_{BG}) = \frac{1}{M}\sum_{i=1}^{M}[(\widehat{\sigma}_{BG}(i)/\sigma)^q - q\log(\widehat{\sigma}_{BG}(i)/\sigma)-1]$. We calculated the ER of each Bayes estimator according to its own loss function. For the MLEs, we calculated both kinds of ERs, i.e. ER_S and ER_G to compare them with their corresponding Bayes estimators. We also obtained 95% TEB CIs as well as 95% HPDIs for σ and calculated the coverage probabilities (CPs) and the average widths (AWs) of the CIs over 10000 replications. The results are presented in Table 1.

From Table 1, we observe that the ERs of the Bayes estimators, especially the ones obtained under the informative cases, are smaller than the ERs of the corresponding MLEs which reveals the superiority of the Bayesian methods as compared with the likelihood ones. Moreover, Case III contains the smallest ERs and the shortest CIs which is quite reasonable as Case III has the smallest prior variance and therefore is the most informative case. We also observe that the HPDIs are shorter than their corresponding TEB CIs.

Table 1: The results of the simulation.

	ER_S	ER_G		TEB CI		HPDI	
		$q=-2$	$q=2$	AW	CP	AW	CP
MLE	0.0119	0.0260	0.0240	–	–	–	–
Case I	0.0124	0.0248	0.0240	0.4437	0.9500	0.4364	0.9493
Case II	0.0114	0.0242	0.0235	0.4243	0.9484	0.4177	0.9426
Case III	0.0110	0.0232	0.0227	0.4201	0.9510	0.4137	0.9446

References

[1] Calabria, R. and Pulcini, G. (1996), Point estimation under asymmetric loss functions for left-truncated exponential samples, *Communications in Statistics-Theory and Methods*, **25**, 585-600.

[2] Salehi, M. and Ahmadi, J. (2014), Record ranked set sampling scheme, *Metron*, **72**, 351-365.

Estimation for the Exponential-Geometric Distribution Under Progressively Type-II Censoring with Binomial Removals

MirMostafaee, S.M.T.K. [1] and Azizi, E. [2]

Department of Statistics, University of Mazandaran

Abstract

In this paper, we study the estimation problem for the exponential-geometric distribution under type II progressive censoring with binomial removals. The maximum likelihood estimators as well as the asymptotic confidence intervals for the parameters are derived. Finally, a real data example is presented to illustrate the results of the paper.

Keywords: Asymptotic confidence interval, Binomial censoring scheme, Type II progressive censoring.

1 Introduction

The exponential-geometric (EG) distribution was first introduced by [1], whose probability density function (pdf) and cumulative distribution function (cdf), are given by

$$f(x|\theta) = (1-\theta)e^{-x}(1-\theta e^{-x})^{-2}, \quad x>0, \quad \theta>0, \tag{1}$$

and

$$F(x|\theta) = (1-e^{-x})(1-\theta e^{-x})^{-1}, \tag{2}$$

respectively. In what follows, we focus on estimation for this model under type II progressive censoring with binomial removals.

[1] m.mirmostafaee@umz.ac.ir
[2] esmaeelazizi1369@yahoo.com

2 Main results

Let $\boldsymbol{X} = (X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ denote a progressively type II right censored sample of size m extracted from a sample of size n, where lifetimes have an EG distribution with pdf given in (1). At the i-th failure, $R_i = r_i,\ i = 1, ..., m-1$ units are removed randomly from the experiment. When the m-th failure is observed, the remaining $r_m = n - m - \sum_{j=1}^{m-1} r_j$ are all removed. Supposing that the $R = (R_1, ..., R_{m-1})$ is predetermined, the conditional likelihood function becomes (see for example [2])

$$\begin{aligned} L(\theta, \boldsymbol{x}|R = r) &= C \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{r_i} \\ &= C(1-\theta)^n e^{-\sum_{i=1}^{m} x_i(1+r_i)} \prod_{i=1}^{m} (1 - \theta \exp(-x_i))^{-(r_i+2)}, \end{aligned}$$

where C is the normalizing constant and $\boldsymbol{x}$ is the observed vector of $\boldsymbol{X}$. Now assume that R_i's are discrete random variables such that R_1 follows $Bin(n-m, p)$ and $R_i|R_1 = r_1, ..., R_{i-1} = r_{i-1}$ follows $Bin(n - m - \sum_{j=1}^{i-1} r_j, p)$ for $i = 2, ..., m-1$, where $Bin(n, p)$ denotes for the binomial distribution with parameters n and p. Assume further $R = (R_1, ..., R_{m-1})$ and $\boldsymbol{X}$ are independent. Therefore, the joint likelihood function of $\boldsymbol{X}$ and R is

$$\begin{aligned} L(\theta, p; \boldsymbol{x}, r) &= L(\theta, \boldsymbol{x}|R = r) P(R_1 = r_1) \prod_{i=2}^{m-1} P(R_i = r_i) \\ &= A(1-\theta)^n \prod_{i=1}^{m} (1 - \theta \exp(-x_i))^{-(r_i+2)} \\ &\quad \times p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i}, \end{aligned}$$

where $A = C(n-m)! e^{-\sum_{i=1}^{m} x_i(1+r_i)} / \{\prod_{i=1}^{m-1} r_i!(n - m - \sum_{i=1}^{m-1} r_i)!\}$, does not depend on the parameters. Let $\ell(\theta, p) = \log L(\theta, p; x, r)$ be the log-likelihood function. Then the maximum likelihood estimators (MLEs) of the parameters will be obtained by maximizing $\ell(\theta, p)$ with respect to θ and p. Upon differentiating $\ell(\theta, p)$ with respect to p, the MLE of p, denoted as $\widehat{p}_{MLE}$, is obtained to be

$$\widehat{p}_{MLE} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i-1) r_i},$$

and the MLE of θ, denoted as $\widehat{\theta}_{MLE}$, is the solution of the following equation

$$\frac{\partial \ell(\theta, p)}{\partial \theta} = -\frac{n}{1-\theta} + \sum_{i=1}^{m} \frac{(r_i + 2) e^{-x_i}}{1 - \theta e^{-x_i}} = 0.$$

Next, we derive the asymptotic confidence intervals of the unknown parameters $\alpha = (\theta, p)$ based on the idea of large sample approximation. The inverse of the observed information matrix is given by

$$\begin{aligned} \mathbf{I}^{-1}(\alpha) &= \begin{bmatrix} -\frac{\partial^2 \ell(\theta,p)}{\partial \theta^2} & -\frac{\partial^2 \ell(\theta,p)}{\partial \theta \partial p} \\ -\frac{\partial^2 \ell(\theta,p)}{\partial \theta \partial p} & -\frac{\partial^2 \ell(\theta,p)}{\partial p^2} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} Var(\widehat{\theta}_{MLE}) & Cov(\widehat{\theta}_{MLE}, \widehat{p}_{MLE}) \\ Cov(\widehat{\theta}_{MLE}, \widehat{p}_{MLE}) & Var(\widehat{p}_{MLE}) \end{bmatrix}. \end{aligned}$$

The elements of $\mathbf{I}^{-1}(\alpha)$ are

$$-\frac{\partial^2 \ell(\theta,p)}{\partial\theta^2} = \frac{n}{(1-\theta)^2} - \sum_{i=1}^{m} \frac{(r_i+2)e^{-2x_i}}{(1-\theta e^{-x_i})^2}, \qquad -\frac{\partial^2 \ell(\theta,p)}{\partial\theta\partial p} = 0,$$

$$-\frac{\partial^2 \ell(\theta,p)}{\partial p^2} = \frac{\sum_{i=1}^{m} r_i}{p^2} + \frac{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}{(1-p)^2}.$$

Under the regularity conditions that are fulfilled for the parameters (see [3], pp. 461-463), the asymptotic joint distribution of $(\widehat{\theta}, \widehat{p})$, as $n \to \infty$, is a 2-variate normal distribution with mean (θ, p) and variance-covariance $\mathbf{I}^{-1}(\alpha)$. Unknown parameters which may appear in the elements of $\mathbf{I}^{-1}(\alpha)$ may be substituted by their corresponding MLEs. Therefore, the asymptotic $100(1-\gamma)\%$ two-sided equi-tailed confidence intervals (TE CIs) for the parameters θ and p, respectively, are given by

$$\widehat{\theta}_{MLE} \pm z_{\gamma/2}\sqrt{\widehat{Var(\widehat{\theta}_{MLE})}}, \qquad \text{and} \qquad \widehat{p}_{MLE} \pm z_{\gamma/2}\sqrt{\widehat{Var(\widehat{p}_{MLE})}},$$

where $z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.

3 A real data example

In this section, we consider a real data example, discussed by [4], regarding $n = 23$ deep-groove ball bearing failure times. The data are:
0.1788, 0.2892, 0.3300, 0.4152, 0.4212, 0.4560, 0.4848, 0.5184, 0.5196, 0.5412, 0.5556, 0.6780, 0.6864 ,0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 1.0512, 1.0584, 1.2792, 1.2804, 1.7340.
These observations are the number of revolutions (in hundred of millions) to failure for each ball bearing. The Kolmogorov-Smirnov test showed that the EG distribution with $\widehat{\theta} = 0.0677$ is acceptable for these data (p-value>0.165). Here, we analyze the data from the perspective of progressive type II censoring with binomial removals. We take $m = 18$ and use several values of p to generate different removal schemes. These removal schemes are presented in Table 1. For each scheme, we randomly extracted a progressively type II censored sample with binomial removals from the ball bearing data and then obtained the MLEs as well as the asymptotic 95% TE CIs for θ and p based on the extracted samples. The results are presented in Table 2. Note that the CIs for θ, marked by *, are the ones whose lower bounds are obtained less than zero and therefore are replaced with zero as $\theta > 0$. In addition, the CI for p, marked by **, is the one whose upper bound is obtained more than one and therefore is replaced with one.

Table 1: The censoring schemes.

Number	p	scheme
1	0.2	1 1 1 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0
2	0.5	2 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3	0.7	2 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
4	0.9	4 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Table 2: The MLEs and the asymptotic 95% CIs for the parameters based on the extracted samples.

scheme	$\widehat{p}_{MLE}$	CI(p)	$\widehat{\theta}_{MLE}$	CI(θ)
1	0.2631579	(0.06515356, 0.4611622)	0.03888371	(0*, 0.6644923)
2	0.4545455	(0.1602879, 0.748803)	0.002534581	(0*, 0.6577489)
3	0.625	(0.2895199, 0.9604801)	0.03231922	(0*, 0.6728525)
4	0.8333333	(0.5351288, 1**)	0.01659866	(0*, 0.6601827)

References

[1] Adamidis K. and Loukas, S. (1998), A lifetime distribution with decreasing failure rate, *Statistics and Probability Letters*, **39**, 35-42.

[2] Cohen, A.C. (1963), Progressively censored samples in life testing, *Technometrics*, **5**(3), 327-339.

[3] Lehmann, L.E. and Casella, G. (1998), *Theory of Point Estimation*, 2nd ed., Springer, New York.

[4] Leiblein, J. and Zelen, M. (1956), Statistical investigation of the fatigue life of deep-groove ball bearings, *Journal of Research of the National Bureau of Standards*, **57**(5), 273-316.

Estimation for the Weighted Exponential Distribution Using the Probability Weighted Moments Method

MirMostafaee, S.M.T.K. [1] and Khoshkhoo Amiri, Z. [2]

Department of Statistics, University of Mazandaran

Abstract

In this paper, we focus on the problem of estimation for the scale and shape parameters of the weighted exponential distribution. The probability weighted moments method has been developed for estimating the parameters. A real data example ends the paper.

Keywords: Maximum likelihood estimator, Probability weighted moment method, Weighted exponential distribution.

1 Introduction

The weighted exponential (WE) distribution was first introduced by [3] and has the following probability density function (pdf)

$$f(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\lambda\alpha x}), \quad x>0, \quad \alpha>0, \quad \lambda>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x) = 1-\frac{\alpha+1}{\alpha}e^{-\lambda x}+\frac{1}{\alpha}e^{-\lambda(1+\alpha)x}, \qquad x>0.$$

The WE distribution can be applicable in reliability and therefore estimation of its parameters are important in this field. Recently, Dey et al. [1] investigated different methods of estimation for this distribution, including the moment, maximum likelihood (ML), weighted least-squares and percentile methods. But they did not consider one of the well-known methods known as the probability weighted moments (PWM) method. In what follows, we review the procedure of finding the moment and ML estimators of the parameters and then discuss how to obtain PWM estimators. Section 2 contains the main results and a real data example is given in Section 3.

[1]m.mirmostafaee@umz.ac.i

[2]z.khoshkhoo@stu.umz.ac.ir

2 Main Results

Let $X_1, \cdots, X_n$ be a random sample of size n from the WE distribution with pdf given in (1). In this section, we discuss the moment, ML and PWM methods for estimation of the parameters.

2.1 Moment Estimation

In this method, the estimators of the parameters are obtained by equating the population moments with the sample moments. For the WE distribution, the moment estimators (MEs) of the parameters are (see [3])

$$\widehat{\alpha}_{ME} = \frac{-(\overline{X}^2 - 2S^2) + \sqrt{(\overline{X}^2 - 2S^2)^2 - 2(\overline{X}^2 - S^2)(\overline{X}^2 - 2S^2)}}{\overline{X}^2 - S^2},$$

and $\widehat{\lambda}_{ME} = \frac{1}{\overline{X}}\left(1 + \frac{1}{1+\widehat{\alpha}_{ME}}\right)$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean and $S^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2$ is the sample variance. It can be proved that the MEs exist and are feasible if and only if $S^2 < \overline{X}^2 < 2S^2$, see [3].

2.2 Maximum Likelihood Estimation

The log likelihood function for the WE distribution, given an observed random sample of size n, is given by
$\ell(\alpha; \lambda, x) = n\ln(\alpha+1) - n\ln\alpha + n\ln\lambda - \lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln(1 - e^{-\lambda\alpha x_i})$.

The ML estimators will be obtained by maximization of the log likelihood function with respect to the parameters. Upon differentiating the log likelihood function with respect to the parameters and equating them with zero, we have

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha+1} - \frac{n}{\alpha} + \lambda\sum_{i=1}^{n}\frac{x_i e^{-\lambda\alpha x_i}}{(1 - e^{-\lambda\alpha x_i})} = 0, \\
\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \alpha\sum_{i=1}^{n}\frac{x_i e^{-\lambda\alpha x_i}}{1 - e^{-\lambda\alpha x_i}} = 0.
\end{aligned}$$

Numerical techniques may be applied to solve the equations.

2.3 Probability weighted moments method

The PWM method was first introduced by [2]. For an arbitrary random variable X with cdf $F(x)$, the probability weighted moment of order (l, k, r) is defined as

$$M_{l,k,r} = E[X^k\{F(X)\}^k\{1 - F(X)\}^r],$$

where l, k and r are real numbers. Clearly, the quantities $M_{l,0,0}, l = 1, 2, ...$ are the usual noncentral moments. In the context of estimation, it is preferable to use either $M_{1,k,0}, k = 0, 1, 2, ...$ or $M_{1,0,r}, r = 0, 1, 2, ...$ depending on the structure of the cdf of X, see [4] for more related details. Landwehr et al. [5] emphasized that an unbiased estimator of $M_r \equiv M_{1,0,r}$, when r is a nonnegative integer number, based on a random sample of size n, is

$$\widehat{M_r} = \frac{1}{n}\sum_{i=1}^{n} X_i \binom{n-i}{r} \Big/ \binom{n-1}{r}.$$

Therefore, one can obtain the PWM estimators of the unknown parameters by equating M_r with $\widehat{M_r}$ for $r = 0, 1, ..., s$ where $s+1$ is the number of parameters. For the WE distribution, $M_0 = E(X) = \lambda^{-1}(1 + \frac{1}{1+\alpha})$ and

$$\begin{aligned} M_1 &= E[X\{1-F(X)\}] = \int_0^\infty x\left(\frac{\alpha+1}{\alpha}e^{-\lambda x} - \frac{1}{\alpha}e^{-\lambda(\alpha+1)x}\right) \\ &\quad \times \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\lambda\alpha x})\mathrm{d}x \\ &= \frac{\alpha+1}{\lambda\alpha^2}\left(\frac{\alpha+1}{4} - \frac{1}{\alpha+2} + \frac{1}{4(\alpha+1)^2}\right). \end{aligned}$$

Thus, we can obtain the PWM estimators of λ and α by solving the equations: $M_0 = \overline{X}$ and $M_1 = \frac{1}{n(n-1)}\sum_{i=1}^{n}(n-i)X_i$, simultaneously. From these equations, we can see that, for a given α, the PWM estimator of λ is

$$\widehat{\lambda}_{PWM}(\alpha) = \frac{1}{\overline{X}}\left(1 + \frac{1}{1+\alpha}\right), \quad (2)$$

and the PWM estimator of α can be obtained as a solution of the following fixed-point type equation

$$\frac{(\alpha+1)^2}{(\alpha+2)\alpha^2}\left(\frac{\alpha+1}{4} - \frac{1}{\alpha+2} + \frac{1}{4(\alpha+1)^2}\right) = \frac{1}{n(n-1)\overline{X}}\sum_{i=1}^{n}(n-i)X_i.$$

Once we get the PWM estimator of α, the PWM estimator of λ can be obtained from (2).

3 A real data example

Here, we consider a real data set, reported by [3, page 632], which is the marks of the slow pace students in Mathematics in the final examination in 2003. For these data, the Moment, ML and the PWM estimators of the parameters are: $\widehat{\alpha}_{ME} = 0.4384$, $\widehat{\lambda}_{ME} = 0.0655$, $\widehat{\alpha}_{MLE} = 0.2919$, $\widehat{\lambda}_{MLE} = 0.0685$, $\widehat{\alpha}_{PWM} = 0.0007764$, $\widehat{\lambda}_{PWM} = 0.0772$.

The following codes in R 3.1.2 were used to find the PWM estimators of the parameters.

```
library(nleqslv)
x=c(29,25,50,15,13,27,15,18,7,7,8,19,12,18,5,21,15,86,21,15,14,
39,15,14,70,44,6,23,58,19,50,23,11,6,34,18,28,34,12,37,4,60,20,
23,40,65,19,31)
z=c()
for(i in 1:n)  z[i]=(n-i)*x[i]
MODEL=function(u){
v=numeric(1)
v[1]=(u[1]+1)^2/(u[1])^2/(u[1]+2)*( (u[1]+1)/4-1/(u[1]+2)
+1/4/(u[1]+1)^2)-1/sum(x)/(n-1)*sum(z)
v
```

```
}
RES=nleqslv(c(2),MODEL)
alph=RES$x[1]
lamb=1/mean(x)*(1+1/(alph+1))
```

References

[1] Dey, S., Ali, S. and Park, C. (2015), Weighted exponential distribution: properties and different methods of estimation, *Journal of Statistical Computation and Simulation*, DOI: 10.1080/00949655.2014.992346.

[2] Greenwood, J.A., Landwehr, J.M., Matalas, N.C. and Wallis, J.R. (1979), Probability weighted moments- Definition and relation to parameters of several distributions expressable in inverse form, *Water Resources Research*, **15**, 1049-1054.

[3] Gupta, R.D. and Kundu D. (2009), A new class of weighted exponential distributions, *Statistics*, **43**, 621-634.

[4] Hosking, J.R.M., Wallis, J.R. and Wood, E.F. (1985), Estimation of the generalized extreme-value distribution by the method of probability-weighted moments, *Technometrics*, **27**, 251-261.

[5] Landwehr, J.M., Matalas, N.C. and Wallis, J.R. (1979), Probability weighted moments compared with some traditional techniques in estimating Gumbel parameters and quantiles, *Water Resources Research*, **15**, 1055-1064.

Survival Modeling of Spatially Correlated Data

Mohammadzadeh, M. [1], Motarjem, K. [2] and Abyar, A. [3]

Department of Statistics, Tarbiat Modares University

Abstract

Identifying risk sources of survival data are given special emphasis in survival analysis. Identifiable risk factors can be modeled by available covariates using some models like Cox proportional hazards model. However some risk factors are often unidentifiable or immeasurable. The spatial correlation of data is one of these factors that is rarely noticed. In this paper a spatial survival model is introduced for such data. A simulation study is performed to show the high performance of the model parameter estimations for the proposed model. Results validate our approach.

Keywords: Proportional hazards model, Unknown risk factors, Spatial random effect, Spatial survival model.

1 Introduction

Survival models are usually used to analyze the realizations of a response variable which is the waiting time until the occurrence of a well-defined event. Suppose that the observations are censored, in the sense that for some units the event of interest has not occurred at the time the data are analyzed, and also there are some explanatory variables whose effects on the waiting time we wish to assess.

In many applications, there are some unknown risk factors effecting survival times. For instance, gender, age, race, level of welfare of children and living environments can affect on duration of seizures due to asthma in asthmatic children. While the location of children lodging can also be one of the risk factors. The time of seizures for children living in downtown areas with more polluted air, is greater than for whom living in areas with

[1]mohsen-m@modres.ac.ir
[2]kmotarjem@modares.ac.ir
[3]a.abyar@modares.ac.ir

less pollution. Due to various concentration of air pollution in different parts of a city, the survival times should be modeled using the location of children lodging.

The most common model for fitting the independent survival data is the Cox proportional hazard function. But in many cases data are correlated. Using this model for correlated survival data, may lead to inappropriate analysis. This paper is aimed to show that the regular state of art modeling methods and analyzing the spatial survival data may not be applicable due to the correlated nature of spatial data.

The Cox proportional hazard function is defined as $h(t|X) = h_0(t)e^{\beta' X}$ where X is a $q \times 1$ vector of covariates, β denotes the vector of regression coefficient parameters and $h_0(\cdot)$ is the baseline hazard function that describes the risk for individuals with $X = 0$. This model does not include the unknown risk factors. Thus, the frailty model which is a generalization of the proportional hazards model and includes unknown risk factor as a random variable is introduced by [2]. This model is given by $h(t|X) = Wh_0(t)\exp(\beta' X)$ where W is a random variable with positive support.

Let $D_1, \ldots, D_m$ are m geographic regions (e.g., census blocks) in a zone or a city and s_j denotes a representative position or the center of the region D_j. Suppose in each region, n subjects are followed until failure or censoring, whichever comes first. For each individual $i = 1, \ldots, n$, along with the survival time T_{ij} of the ith subject in the jth region D_j, a length-q covariate vector X_{ij} are also observed. In the following exposition, the covariate X_{ij} is assumed to be time independent, although, it should be straightforward to extend the results to accommodate time-dependent covariates. Suppose the random field $Z(s_j)$ denotes the random effect of the region D_j. The model proposed by Li and Ryan [3] specifies that, conditional on the covariates X_{ij} and the region-specific random effect $Z(s)$, the survival time T_{ij} is independent and has the following spatial hazard function

$$h(s_j, t_{ij}|X_{ij}, Z(s_j)) = h_0(t_{ij})\exp(\beta' X_{ij} + Z(s_j)) \tag{1}$$

The model (1), which is introduced for lattice data, may not be able to reflect all correlation information of data. Thus we consider the geographical coordinates of each subject instead of region representative. To accomplish this goal and using the model (1) for a set of $N = nm$ geostatistical data, the spatial hazard function would be derived as

$$h(s_k, t_k|X_k, Z(s_i)) = h_0(t_k)\exp(\beta' X_k + Z(s_k)), \quad k = 1, \ldots, N \tag{2}$$

Using model (2) the likelihood function is obtained as bellow,

$$L(\beta, \theta) = \int \prod_{k=1}^{N} \{\frac{\exp(\beta' X_k + Z(s_k))}{\sum_{k'=1}^{N} y_{k'}(t_k)\exp(\beta' X_{k'} + Z(s_{k'}))}\}^{\delta_k} dF_\theta(Z(s_1), \ldots, Z(s_N))$$

where θ is the vector of spatial covariance parameters,

$$y_{k'}(t_k) = \begin{cases} 1, & t_{k'} \leq t_k \\ 0, & t_{k'} > t_k \end{cases}$$

is an indicator function illustrating the subjects which did not fail until time t_i and δ_i shows the censoring of the ith subject. In next section the random effects are modeled by a spatial covariogram. Also a simulation study is performed to compare the precision of parameter estimations for different proportional hazards, frailties and spatial survival data.

2 Simulation Study

In order to investigate the performance of our model, a spatial survival data set including 49 survival times located in a 7×7 square is simulated. Bender *et al.* [1] proposed a method to generate survival time. Here, we generalize this method to generate spatial survival data. Consider the Gaussian covariogram function,

$$C(d) = \sigma^2 \exp(-\frac{d^2}{a^2}), \quad \sigma > 0, \ a > 0$$

where d is the distance of two locations, σ^2 is the sill and a is the range of the random field. In this study, it is assumed that $\sigma^2 = 1$ and $a = 1$. The survival times are modeled by (1) where $Z(s)$ is a Gaussian random field at site s, X is a covariate generated from standard Normal distribution and β is the regression coefficient, assumed to be equal to 1. The simulated data are first censored, then applied by proportional hazards rate, frailty and spatial survival models. The results of Mean Square Error (MSE) and Mean Absolute Percentage Bias (MAPB) of parameter estimators are reported in Tables 1 and 2.

Table 1: Estimated β and errors for proportional hazards and frailty models

		Iteration					
	Percentage	100			500		
Model	Censor	Estimate	MAPB	MSE	Estimate	MAPB	MSE
Proportional	20	0.719	28.710	0.120	0.749	25.300	0.111
Hazards	80	0.871	19.572	0.144	0.900	23.866	0.145
Frailty	20	0.864	18.546	0.080	0.898	21.219	0.086
	80	0.932	19.680	0.151	0.971	24.968	0.170

Table 2: Estimated parameters and errors for spatial survival model

		Iteration					
Percentage		100			500		
Censor	Parameter	Estimate	MAPB	MSE	Estimate	MAPB	MSE
	β	1.063	6.268	0.075	1.107	10.728	0.048
20	σ^2	0.898	10.157	0.120	0.987	1.278	0.054
	a	1.091	9.148	0.072	0.998	0.183	0.041
	β	1.195	19.516	0.120	1.188	18.777	0.071
80	σ^2	1.149	14.887	0.123	1.070	7.011	0.101
	a	1.089	8.864	0.145	1.148	14.828	0.077

The results in Table 1 show that by increasing the censoring percentage of data, the precision of the parameter estimates is reduced. On the other hand, the regression parameter in frailty model is estimated with more accuracy than the proportional hazard model. The results in Table 2 show that considering the spatial correlation of data in the spatial survival model provides higher accuracy of parameter estimations than the other two models.

3 Discussion and Results

In this paper, spatial survival model for analysis of spatially correlated survival data is introduced. A simulation study is carried out comparing the performances of proportional hazards, frailty models with our model. We showed that the frailty model due to considering unknown risk factors is more accurate than proportional hazards model. Detection of spatial correlation of the survival data can be a potential subject of further study.

References

[1] Bender, R., Augustin, T. and Blettner, M. (2005), Generating survival times to simulate Cox proportional hazards models, *Statistics in Medicine*, 24, 1713-1723.

[2] Clayton, D.G. and Cuzik, J. (1985). Multivariate generalization of the proportional hazards model, Journal of the Royal Statistical Society, Series A, 148, 82-117.

[3] Li, Y. and Ryan, L. (2002), Modeling spatial survival data using semiparametric frailty models, *Biometrics*, 58, 287-297.

Estimation of $P(X > Y)$ Using Imprecise Data in the Lindley Distribution

Pak, A. [1]

Department of Computer Sciences, Shahrekord University

Abstract

Classical estimation procedures of the stress-strength parameter $R = Pr(X > Y)$ are based on precise data. However, in real world situations, some collected data might be imprecise and are represented in the form of fuzzy numbers. In this paper, we obtain the maximum likelihood estimation of the parameter R when X and Y are independent Lindley random variables, and the available data are reported in the form of fuzzy numbers. A Monte Carlo simulation study is carried out in order to assess the accuracy of the proposed method.

Keywords: Stress-Strength model, Fuzzy data analysis, Maximum likelihood estimation.

1 Introduction

Extensive research has been conducted on the stressstrength model. This model involves two independent random variables X and Y, and the parameter of interest is the probability $R = P(X > Y)$. A comprehensive account of this topic is given by Kotz et al. (2003). The developments in this field covered a variety of data types including complete data, censored data as well as data with explanatory variables. However, in real world situations, the results of an experimental performance can not always be recorded or measured precisely, but each observable event may only be identified with a fuzzy subset of the sample space. Our aim in this paper is to develop an inferential procedure for the stress-strength model in the situation where the stress measurements and the strength measurements are both in terms of fuzzy numbers. We will construct maximum likelihood estimation for the stress-strength reliability assuming two independent samples from Lindley distribution. In Section 2, We first introduce a generalized likelihood function based

[1] abbas.pak1982@gmail.com

on fuzzy data and then discuss the maximum likelihood estimation of the parameter R. A Monte Carlo simulation study is presented in Section 3, in order to assess the accuracy of the proposed method. For a review about the main definitions of fuzzy sets see Pak et al. (2014) and the references therein.

We use the following notation. A Lindley distribution with the parameter θ, will be denoted by $Lindley(\theta)$ and the corresponding probability density function is as follows;

$$f(x;\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}; \qquad x>0;\ \theta>0. \tag{1}$$

2 Maximum likelihood estimation

Let the strength X and stress Y follow $Lindley(\theta_1)$ and $Lindley(\theta_2)$, respectively, and they are independent. Then, it can be easily shown that

$$\begin{aligned} R &= Pr(Y<X) \\ &= 1-\frac{\theta_1^2[\theta_1^2(\theta_1+1)+\theta_2(\theta_1+1)(\theta_1+3)+\theta_2^2(2\theta_2+3)+\theta_2^2]}{(\theta_1+1)(\theta_2+1)(\theta_1+\theta_2)^3}. \end{aligned} \tag{2}$$

Suppose that partial information about the stress and strength are available in the form of fuzzy numbers $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ with the Borel measurable membership functions $\mu_{\tilde{\boldsymbol{x}}}(\boldsymbol{x})$ and $\mu_{\tilde{\boldsymbol{y}}}(\boldsymbol{y})$. Then, the corresponding observed-data log likelihood function can be obtained as:

$$\begin{aligned} L_O(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}};\theta_1,\theta_2) &= n\log\left(\frac{\theta_1^2}{1+\theta_1}\right)+\sum_{i=1}^{n}\log\int(1+x)e^{-\theta_1 x}\mu_{\tilde{x}_i}(x)dx \\ &+ m\log\left(\frac{\theta_2^2}{1+\theta_2}\right)+\sum_{j=1}^{m}\log\int(1+y)e^{-\theta_2 y}\mu_{\tilde{y}_j}(y)dy. \end{aligned}$$

To compute the maximum likelihood estimate (MLE) of R, we need to compute the MLEs of θ_1 and θ_2, say $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively. The MLE $\hat{R}$ of R can then be obtained by substituting $\hat{\theta}_k$ in place of θ_k, in (2.1) for $k=1$ and 2.
Since the observed fuzzy data $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ can be viewed as incomplete specifications of the complete data vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, the EM algorithm is applicable to obtain the MLEs of the unknown parameters.

To perform the E-step of the algorithm, we need to compute the conditional expectation of the complete-data log-likelihood function conditionally on the observed data $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ as follows:

$$\begin{aligned} & n\log\left(\frac{\theta_1^2}{1+\theta_1}\right)+m\log\left(\frac{\theta_2^2}{1+\theta_2}\right) \\ - & \theta_1\sum_{i=1}^{n}E_{\theta_1^{(h)}}(X_i\mid\tilde{x}_i)-\theta_2\sum_{j=1}^{m}E_{\theta_2^{(h)}}(Y_j\mid\tilde{y}_j) \end{aligned} \tag{3}$$

where

$$E_{\theta_1^{(h)}}(X_i\mid\tilde{x}_i)=\frac{\int x(1+x)e^{-\theta_1^{(h)}x}\mu_{\tilde{x}_i}(x)dx}{\int(1+x)e^{-\theta_1^{(h)}x}\mu_{\tilde{x}_i}(x)dx}, \qquad i=1,...,n,$$

$$E_{\theta_2^{(h)}}(Y_j \mid \tilde{y}_j) = \frac{\int y(1+y)e^{-\theta_2^{(h)}y}\mu_{\tilde{y}_j}(y)dy}{\int (1+y)e^{-\theta_2^{(h)}y}\mu_{\tilde{y}_j}(y)dy}, \quad j = 1, ..., m.$$

The M-step of the algorithm involves maximizing (2.2) with respect to θ_1 and θ_2, which yields

$$\begin{aligned} \theta_1^{(h+1)} &= \frac{1}{2}(\alpha_h - 1) + \left[(1-\alpha_h)^2 + 8\alpha_h\right], \\ \theta_2^{(h+1)} &= \frac{1}{2}(\beta_h - 1) + \left[(1-\beta_h)^2 + 8\beta_h\right] \end{aligned}$$

where

$$\alpha_h = \frac{n}{\sum_{i=1}^{n} E_{\theta_1^{(h)}}(X_i \mid \tilde{x}_i)}, \qquad \beta_h = \frac{m}{\sum_{j=1}^{m} E_{\theta_2^{(h)}}(Y_j \mid \tilde{y}_j)}.$$

The MLEs of θ_1 and θ_2 can be obtained by repeating the E-step and M-step until convergence occurs.

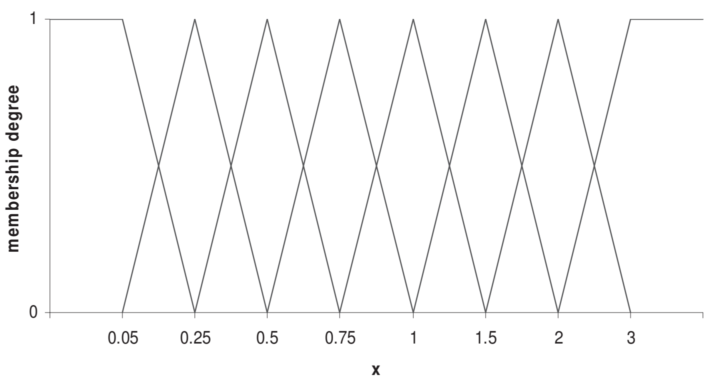


Figure 1: Fuzzy information system used to encode the simulated data

3 Simulation study

In order to assess the accuracy of the proposed method, we have carried out a Monte Carlo simulation study. First, for different sample sizes and a set of parameter values, namely $(\theta_1, \theta_2) = (1.0, 2.0)$, we have generated random samples from Lindley distribution. Then, each realization of the random samples was fuzzified using the fuzzy information system shown in Fig.1 and the estimate of the parameters θ_1, θ_2 and R for the fuzzy samples were computed using the maximum likelihood procedure. The average values (AV) and mean squared errors (MSE) of the ML estimates over 1000 replications are presented in Table 1. From the experiments, we found that the performance of the ML estimates are quite satisfactory and as the sample size increases, the MSEs of the estimates decrease as expected.

Table 1: AVs and MSEs of the ML estimates $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{R}$ for different sample sizes.

(n,m)	$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{R}$	
	AV	MSE	AV	MSE	AV	MSE
(20,20)	1.1931	0.0827	2.2180	0.1721	0.7025	0.0136
(20,30)	1.1827	0.0640	2.2039	0.1432	0.7003	0.0113
(30,20)	1.1838	0.0644	2.2057	0.1454	0.6891	0.0081
(30,30)	1.1210	0.0492	2.1863	0.1197	0.6678	0.0069
(30,50)	1.0731	0.0313	2.1625	0.0893	0.6653	0.0052
(50,30)	1.0690	0.0307	2.1597	0.0822	0.6672	0.0057
(50,50)	1.0248	0.0238	2.1139	0.0517	0.6319	0.0031

References

[1] S. Kotz, Y. Lumelskii, and M. Pensky, The Stress-Strength Model and its Generalizations: Theory and Applications. Singapore: World scientific, 2003.

[2] A., Pak, G.H., Parham, and M., Saraj, Inferences on the Competing Risk Reliability Problem for Exponential Distribution Based on Fuzzy Data. IEEE Transactions on Reliability, 63(1), 2014, 1-10.

Stress-strength system with non-identical exponentiated exponential distribution

Pakdaman, Z. [1] and Ahmadi, J. [2]

Department of Statistics, Ferdowsi University of Mashhad

Abstract

A multicomponent stress-strength system is considered, while the stress and the strength system have non-identical exponentiated exponential distributions with different parameters. The estimation of stress-strength reliability parameter is studied.

Keywords: Stress-strength reliability, Uniformly minimum variance unbiased estimator, Maximum likelihood estimator

1 Introduction

The problem of increasing reliability of any system is now a well-recognized and rapidly developing branch of engineering. Stress-strength reliability the probability that the random variable X (stress) is exceeded by its strength which is a realization of a random variable Y which is equal to $R := P(X < Y)$. The problem of estimation of R has been discussed in the literature extensively. Multicomponent stress-strength reliability also has been studied by several authors, see for examples, Bhattacharyya and Johnson (1974), Pandey et al. (1992) and Eryilmaz (2008b). Nguimkeu et al. (2014) proposed a procedure to obtain accurate confidence intervals for the stress-strength reliability $R = P(X > Y)$ when (X, Y) is a bivariate normal distribution with unknown means and covariance matrix. Cha and Finkelstein (2015) studied a dynamic stress-strength model under external shocks.

[1] pakdaman.zohreh@stu.um.ac.ir
[2] ahmadi-j@um.ac.ir

2 Main aim of the paper

In the present paper, we consider the following multicomponent stress-strength:

- A parallel system of n_1 components having stress following n_1 independently and non-identically distributed random variables X_i for $i = 1, \ldots, n_1$ with cdf $F_i(x) = [1 - \exp(-\gamma x)]^{\alpha(i)}$, where γ and $\alpha(i)$ are positive constants.

- The strengths of the components are independent but non-identical random variables Y_j for $j = 1, ..., n_2$ with cdf $G_j(y) = [1-\exp(-\beta y)]^{\nu(j)}$, where β and $\nu(j)$ are positive constants.

The aforementioned cdfs F_i and G_j are known as exponentiated exponential distribution in the literature, see for example Gupta and Kundu (2001). As one can see, we can assume that the stress system consist n_1 parallel systems where the ith system contains $\alpha(i)$ parallel components with independent and identical cdf $F(x) = 1 - \exp(-\gamma x)$, for $i = 1, \ldots, n_1$. Under these assumptions, we find

$$R = P\left(\max_{1\leq j\leq n_2} Y_j > \max_{1\leq i\leq n_1} X_i\right) = P(W > Z) = \int_0^\infty F_Z(w) g_W(w)\, dw, \tag{1}$$

where $W = \max_{1\leq j\leq n_2} Y_j$ and $Z = \max_{1\leq i\leq n_1} X_i$. Then, we have

$$f_Z(z) = \gamma e^{-\gamma z} \sum_{i=1}^{n_1} \alpha(i)(1 - e^{-\gamma z})^{\sum_{i=1}^{n_1}\alpha(i)-1} \tag{2}$$

and

$$g_W(w) = \beta e^{-\beta w} \sum_{j=1}^{n_2} \nu(j)(1 - e^{-\beta w})^{\sum_{j=1}^{n_2}\nu(j)-1}. \tag{3}$$

By substituting (2) and (3) into (1), and doing some calculations, we obtain

$$R = \int_0^1 \left[1 - \left(1 - u^{\frac{1}{\sum_{j=1}^{n_2}\nu(j)}}\right)^{\frac{\gamma}{\beta}}\right]^{\sum_{i=1}^{n_1}\alpha(i)} du. \tag{4}$$

In what follows, we will study the estimation of R in (4).

3 Estimation of R

We obtain two common point estimators of R, namely MLE and UMVUE.

MLE: Let $Z_1, \ldots, Z_n$ be a random sample of size n from Z with pdf in (2) and $W_1, \ldots, W_m$ be a random sample of size m from each distributed as W with pdf in (3). Using the invariance property of MLE, the MLE of R is given by

$$\widehat{R}_M = \int_0^1 \left[1 - \left(1 - x^{\frac{U}{m}}\right)^{\frac{\gamma}{\beta}}\right]^{\frac{n}{T}} dx, \tag{5}$$

where $U = -\sum_{s=1}^{m} \log(1 - e^{-\beta W_s})$ and $T = -\sum_{r=1}^{n} \log(1 - e^{-\gamma Z_r})$.

UMVUE: To obtain the UMVUE of R noting that U and T are complete sufficient statistics for $\sum_{j=1}^{n_2} \nu(j)$ and $\sum_{i=1}^{n_1} \alpha(i)$ and are independent and each distributed as $\Gamma\left(m, \sum_{j=1}^{n_2} \nu(j)\right)$ and $\Gamma\left(n, \sum_{i=1}^{n_1} \alpha(i)\right)$, respectively. By doing some calculations, we have

$$\begin{aligned}
\widehat{R}_U &= \int_{-\frac{1}{\gamma}\ln(1-e^{-t})}^{\infty} \frac{(m-1)\beta e^{-\beta w_1}}{u(1-e^{-\beta w_1})} \left(1 + \frac{\ln(1-e^{-\beta w_1})}{u}\right)^{m-2} \left(1 + \frac{\ln(1-e^{-\gamma w_1})}{t}\right)^{n-1} dw_1 \\
&= (m-1)\sum_{r=0}^{n-1} \binom{n-1}{r} \frac{1}{t^r} \int_{1+\frac{1}{u}\ln\left(1-(1-e^{-t})^{\frac{\gamma}{\beta}}\right)}^{1} s^{m-2} \left(\ln(1-(1-e^{u(s-1)})^{\frac{\gamma}{\beta}})\right)^r ds. \qquad (6)
\end{aligned}$$

if $-\frac{1}{\gamma}\ln(1-e^{-t}) \geq -\frac{1}{\beta}\ln(1-e^{-u})$ and

$$\begin{aligned}
\widehat{R}_U &= \int_{-\frac{1}{\beta}\ln(1-e^{-u})}^{\infty} \frac{(m-1)\beta e^{-\beta w_1}}{u(1-e^{-\beta w_1})} \left(1 + \frac{\ln(1-e^{-\beta w_1})}{u}\right)^{m-2} \left(1 + \frac{\ln(1-e^{-\gamma w_1})}{t}\right)^{n-1} dw_1 \\
&= (m-1)\sum_{r=0}^{n-1} \binom{n-1}{r} \frac{1}{t^r} \int_{0}^{1} s^{m-2} \left(\ln(1-(1-e^{u(s-1)})^{\frac{\gamma}{\beta}})\right)^r ds, \qquad (7)
\end{aligned}$$

if $-\frac{1}{\gamma}\ln(1-e^{-t}) < -\frac{1}{\beta}\ln(1-e^{-u})$.

The performance of MLE and UMVUE are compared in the next section, by using the mean squared error (MSE), through generated many sample sizes by using simulation technique.

4 Numerical studies and conclusions

Our goal in this section is to compare the presented estimators, numerically when R changes from 0.01 to 0.99. For this purpose, we assume that $\gamma = 2$, $n_1 = n_2 = 5$, $\alpha(i) = (1+\theta_1)^{i-1}$ for $i = 1, \ldots, n_1$, $\nu(j) = (1+\theta_2)^{j-1}$ for $j = 1, \ldots, n_2$ and values (5,5), (5,10), (10,5),(10,10) for (m, n). we consider three different values for (θ_1, θ_2) as (1,1) and (1, 3). For each of them, we obtain MSE and bias of UMVU and ML estimators. The results are displayed in Figures 1 and 2.

References

[1] Bhattacharyya, G. K. and Johnson, R. A.(1974), Estimation of reliability in a multicomponent stress-strength model, *Journal of the American Statistical Association*, **69**, 966–970.

[2] Cha, J. H., and Finkelstein, M. (2015), A dynamic stress-strength model with stochastically decreasing strength. *Metrika*, DOI 10.1007/s00184-015-0528-x.

[3] Gupta, R. D. and Kundu, D. (2001), Exponentiated exponential family: an alternative to gamma and Weibull, *Biometrical Journal*, **33** 117–130.

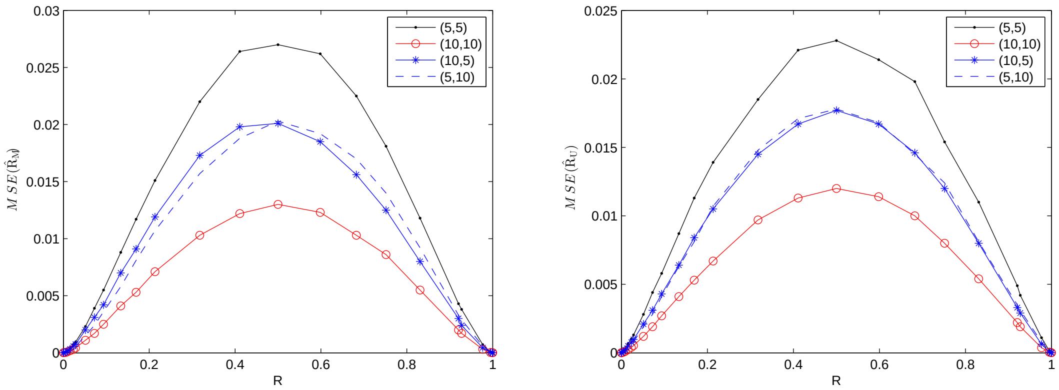


Figure 1: MSE($\hat{R}_M$) and MSE($\hat{R}_U$) for $\theta_1 = \theta_2 = 1$.

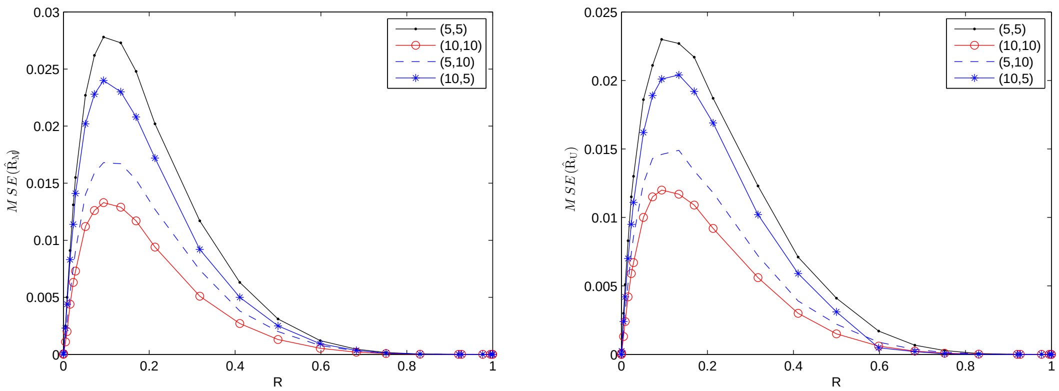


Figure 2: MSE($\hat{R}_M$) and MSE($\hat{R}_U$) for $\theta_1 = 1, \theta_2 = 3$.

[4] Nguimkeu, P., Rekkas, M., and Wong, A. (2014), Interval Estimation for the Stress-Strength Reliability with Bivariate Normal Variables. Open Journal of Statistics, 4(08), 630.

[5] Pandey, M., Uddin, M. B. and Ferdous, J. (1992), Reliability estimation of an s-out-of-k system with non-identical component strengths: the Weibull case, *Reliability Engineering and System Safety*, **36**, 109–116.

A New Investigation About Parallel $(2, n-2)$ System Using FGM Copula

Parsa, M. [1] and Jabbari, H. [2]

Depertment of Statistics, Ferdowsi University of Mashhad

Abstract

Redundancy is a highly used technique to increase the systems lifetimes and availability. Recently, employing standby units in systems has received great attention. Here, parallel system of n units and two-unit parallel system supported by $(n-2)$ cold standbys are considered where units lifetimes are assumed to be dependent in terms of Farlie-Gumbel-Morgenstern copula structure. Applicable formulas of mean time to system failure are given and the impact of dependence parameter on systems lifetimes are investigated.

Keywords: Cold standby FGM copula Mean time to system failure Parallel system.

1 Introduction

Parallel systems are known as the first redundant systems, later application of standby units extended redundant models. In recent studies correlated units lifetimes are also assumed. Papageorgiou and Kokolakis [3] evaluated reliability of two-unit parallel system supported by $(n-2)$ standbys. Papageorgio and Kokolakis [4] extended the main results of their previous study and developed the system reliability and mean time to system failure (MTSF). Eryilmaz and Tank [1] employed copula function to model the reliability and MTSF of a series system with a single cold standby unit. Here, we use copula function to express the reliability and MTSF of two-unit parallel system supported by $(n-2)$ cold standbys which is briefly expressed as parallel $(2, n-2)$ system. The results are specifically given in terms of Farlie-Gumbel-Morgenstern (FGM) copula.

[1] parsa.motahareh@stu-mail.um.ac.ir
[2]

2 Parallel system with dependent units

Assuming $T_1, T_2, \ldots, T_n$ represent the lifetimes of n units of the parallel system and $T_{i:n}$, $i = 1, 2, \ldots, n$, denotes the ith ordered unit lifetime, $T_{n:n}$ implies the parallel system lifetime, *i.e.*, $T_{P,n} = T_{n:n}$. We expect that, failure of one unit leads more pressure to active ones, therefore, the sooner one unit fails the second unit failure occurs earlier, and this process is expected to be continued. Based on this issue, units lifetimes are considered to be positively dependent. The following proposition presents the reliability of a parallel system with dependent units lifetimes. The reliability of a parallel system containing n units with dependent lifetimes is

$$R_{P,n}(t) = P(T_{P,n} > t) = 1 - C(F_1(t), F_2(t), \ldots, F_n(t)),$$

where $F_i(t)$, $i = 1, 2, \ldots, n$ is the df of ith unit lifetime and C is a n-copula family which models the appropriate positive dependence structure of units lifetimes.

The next proposition delivers the general form of reliability and MTSF of parallel system while the relationship between units lifetimes is modelled via FGM n-copula which has been introduced by Johnson and Kotz [2]. Suppose that lifetimes of n units in a parallel system are identically distributed and the units lifetimes can be modelled via FGM n-copula with the same non-negative dependence parameters. Then, reliability function of the system is,

$$R_{P,n}(t) = 1 - F^n(t)\left[1 + \alpha \sum_{i=2}^{n} \binom{n}{i} \bar{F}^i(t)\right], \quad 0 \le \alpha \le 1.$$

The following example illustrates the reliability and MTSF of parallel system containing different numbers of units.

Example 1. *Let $F(t) = 1 - e^{-\lambda t}$, $\lambda, t > 0$. Then under the assumptions of Proposition 2, one easily concludes the following formulas.*

(i) If $n = 2$; $MTSF_{P,2} = \int_0^\infty R_{P,2}(t)dt = \frac{1}{\lambda}(\frac{3}{2} - \frac{\alpha}{12})$.

(ii) If $n = 3$; $MTSF_{P,3} = \int_0^\infty R_{P,3}(t)dt = \frac{1}{\lambda}(\frac{11}{6} - \frac{\alpha}{6})$.

(iii) If $n = 4$; $MTSF_{P,4} = \int_0^\infty R_{P,4}(t)dt = \frac{1}{\lambda}(\frac{25}{12} - \frac{29\alpha}{120})$.

The trend of MTSFs are decreasing in terms of dependence parameter α. Figure 1 shows the trend of MTSFs for different possible values of dependence parameter α while $\lambda = 0.1$ in marginal df.

3 Parallel (2,n-2) system with dependent units

Consider a parallel $(2, n-2)$ system which in fact is a two-unit parallel system supported by $(n-2)$ cold standbys where $n \ge 3$, is a fixed number of non-repairable units. The failed active unit is replaced upon its failure instantaneously by one of the standbys. This process is continued until all the standbys are used in the system and the system operates if at least one unit is active. Assume that T_i, $i = 1, 2$, represents the lifetime of the ith initial active unit and S_j, $j = 1, \cdots, n-2$, represents the jth cold standby lifetime in the system. The next trivial proposition investigates this system lifetime. Parallel $(2, n-2)$ system lifetime, $T_{P,(2,n-2)}$, for some n is expressed as follows.

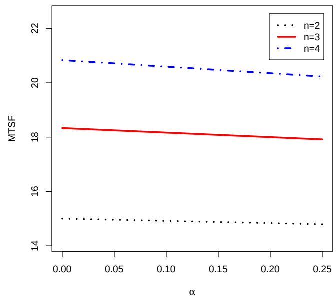


Figure 1: MTSF trend of parallel system for different possible values of dependence parameter α.

Let $n = 3$; $\quad T_{P,(2,1)} = \min(T_1, T_2) + \max(T_1^*, S_1)$, where T_1^* is the residual life of T_1 after the first unit is failed.

Let $n = 4$; $\quad T_{P,(2,2)} = \min(T_1, T_2) + \min(T_1^*, S_1) + p\max(T_1^{**}, S_2) + (1-p)\max(S_1^*, S_2)$, where S_1^* is defined similarly as T_1^*; and T_1^{**} is the residual life of T_1^*.

Let C be a copula that models the joint distribution of units lifetimes. Then we have

$$MTSF_{P,(2,1)} = 2\int_0^\infty \bar{F}(t)dt + \int_0^\infty C(F(t), F(t))dt - \int_0^\infty C(F^*(t), F(t))dt,$$

and

$$\begin{aligned} MTSF_{P,(2,2)} = 3\int_0^\infty \bar{F}(t)dt - \int_0^\infty F^*(t)dt + \int_0^\infty C(F(t), F(t))dt \\ + p\int_0^\infty \big[C(F^*(t), F(t)) - C(F^{**}(t), F(t))\big]dt. \end{aligned}$$

The following example presents the MTSFs of parallel $(2, n-2)$ system for some n and specific marginal distribution.

Example 2. *Let $F(t) = 1 - e^{-\lambda t}$, $\lambda, t > 0$, and dependence structure of units lifetimes is modelled by FGM 2-copula. Hence, the corresponding MTSF of parallel $(2, n-2)$ system is attained as follows.*

(i) If $n = 3$; $\quad MTSF_{P,(2,1)} = \frac{1}{\lambda}(2 - \frac{\alpha}{9} - \frac{\alpha^2}{180} - \frac{22\alpha^3}{945})$.

(ii) If $n = 4$; $\quad MTSF_{P,(2,2)} = \frac{1}{22680\lambda(\alpha^2+5\alpha-45)^2}\Big[114817500 - alpha(29342250 + 3402000p) - \alpha^2(2835000 + 1256850p) + \alpha^3(689850 + 38430p) + \alpha^4(37800 - 28620p) - \alpha^5(1890 + 951p) + 120p\alpha^6 + 31p\alpha^7\Big]$.

The MTSFs are decreasing in terms of dependence parameter α. For better intuition, see Figure 2 where $\lambda = 0.1$ in marginal df.

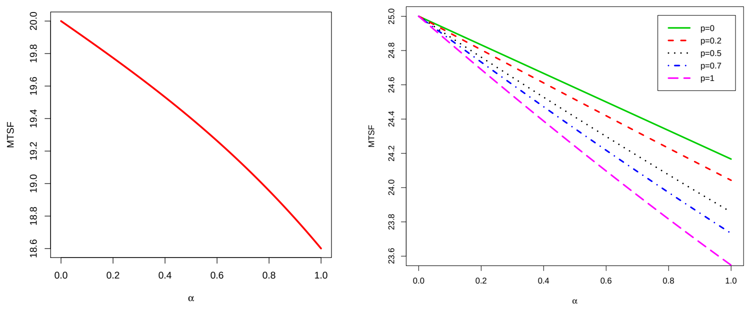


Figure 2: MTSF trend of parallel $(2, 1)$ system (left) and parallel $(2, 2)$ system (right) for different possible values of dependence parameter α and probabilities p.

References

[1] Eryilmaz, S. and Tank, F. (2012) On reliability analysis of a two-dependent-unit series system with a standby unit, *Applied Mathematics and Computation*, **218**, 7792-7797.

[2] Johnson, N.L. and Kotz, S. (1972), *Distributions in Statistics: Continuous multivariate distributions*, Wiley, New York.

[3] Papageorgiou, E. and Kokolakis, G. (2004), A two-unit parallel system supported by $(n-2)$ cold standbys – Analytic and simulation approach, *European Journal of Operational Research*, **176**, 1-12.

[4] Papageorgiou, E. and Kokolakis, G. (2007), A two-unit parallel system supported by $(n-2)$ standbys with general and non-identical lifetimes, *International Journal of System Science*, **35**, 1016-1032.

On Mean Residual Life Ordering Among Weighted-k-out-of-n Systems

Rahmani, R. [1] , Izadi, M. [2] and Khaledi, B. [3]

Department of Statistics, Razi University

Abstract

Consider a system consisting of n binary components with different contributions (weights) on determining the state of the system. The system is known as weighted-k-out-of-n system when it works iff the total weight of working components are greater than a pre-specified value k. Suppose that this system has the property that, with probability 1, operates as long as at least $n-s+1$ components operate ($s \leq n$). In this paper, we compare two such systems with respect to their mean residual life function under the condition that $n-r+1$ components ($r \leq s$) of the systems are working at time t.

Keywords: Weighted-k-out-of-n system, Mean residual life, Usual stochastic order.

1 Introduction

Consider a system consisting of n binary components with different contributions on determining the state of the system. Let w_i, $i = 1, \ldots, n$, be the positive weight of the component i. The system is known as weighted-k-out-of-n system when it works iff the total weight of working components are greater than a pre-specified value k, that is, $\sum_{i=1}^{n} w_i X_i \geq k$ where X_i is the state of the component i, $i = 1, \ldots, n$. The weighted-k-out-of-n system was introduced by Wu and Chen (1994) and studied by many researchers including Higashiyama (2001), Chen and Yang (2005), Samaniego and Shaked (2008) and Eryilmaz and Bozbulut (2014).

[1] rahmani.stat@gmail.com
[2] m.izadi@razi.ac.ir
[3] bkhaledi@hotmail.com

One of the most important characteristics of a system in the reliability theory is the mean residual life (MRL) function. Let T be the random lifetime of a system with survival function $\bar{F}$. Then, the MRL function of the system at time t is given by

$$m(t) = E(T - t|T > t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} \quad , \quad t > 0.$$

We refer the reader to Kotz and Shanbhag (1980), Guess and Proschan (1988), Shaked and Shanthikumar (2007) and Asadi and Goliforushani (2008) for some results regarding the MRL function.

Now, consider a weighted-k-out-of-n system consisting of n components with lifetimes $T_1, \dots, T_n$ and the weight vector $\mathbf{w} = (w_1, \dots, w_n)$. Suppose that this system has the property that, with probability 1, operates as long as at least $n-s+1$ components operate $(s \leq n)$. We denote this sytem by s-weighted-k-out-of-n system. Under the condition that at time t at least $(n - r + 1)$ components $(r \leq s)$ are alive, the residual life of the system is

$$(T_{\mathbf{w}} - t|T_{(r)} > t), \quad r = 1, \dots, s$$

where $T_{\mathbf{w}}$ is the lifetime of the system and $T_{(r)}$ is the rth order statistics of $T_1, \dots, T_n$. The MRL function of the above system can be defined as

$$m_{\mathbf{w}}^{r,s}(t) = E[T_{\mathbf{w}} - t|T_{(r)} > t]. \tag{1}$$

In this paper, we are interested in the comparison of such weighted-k-out-of-n systems (described above) with respect to their mean residual life function defined in (1).

We end this section by recalling the signature vector of a system and the usual stochastic order that will be use later in the paper. Consider a system with lifetime T whose component lifetimes $T_1, \dots, T_n$ are independent and identically distributed. Samaniego (1985) defined the signature vector of the system as a probability vector $\mathbf{q} = (q_1, \dots, q_n)$ with

$$q_i = P\{T = T_{(i)}\}, \quad i = 1, \dots, n.$$

Let X and Y be two random variables with survival function $\bar{F}$ and $\bar{G}$, respectively. X is said to be less than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in R$.

2 Main results

Let $T_{\mathbf{w}}$ and $T_{\mathbf{w}'}$ be the lifetime of two weighted-k-out-of-n systems with independent and identically distributed component lifetimes T_1, ..., T_n and $T'_1, \dots, T'_n$, weight vectors $\mathbf{w} = (w_1, \dots, w_n)$ and $\mathbf{w}' = (w'_1, \dots, w'_n)$ and signature vectors $\mathbf{q} = (q_1, \dots, q_n)$ and $\mathbf{q}' = (q'_1, \dots, q'_n)$, respectively.

Our first result is in the following theorem.

Theorem 1. *Consider two weighted-k-out-of-n systems as above. If* $\mathbf{w} \leq \mathbf{w}'$*, i.e.* $w_i \leq w'_i, i = 1, \dots, n$*, then* $\mathbf{q} \leq_{st} \mathbf{q}'$.

Proof. It is enough to show that $\sum_{i=j}^{n} q_i \leq \sum_{i=j}^{n} q'_i$, for $j = 1, \ldots, n$.

$$\begin{aligned}
\sum_{i=j}^{n} q_i &= P(T_{\mathbf{w}} \geq T_{(j)}) \\
&= \sum_{\{(i_1,\ldots,i_n);\sum_{h=j}^{n} w_{i_h} \geq k\}} P(T_{i_1} \leq \cdots \leq T_{i_n}) \\
&\leq \sum_{\{(i_1,\ldots,i_n);\sum_{h=j}^{n} w'_{i_h} \geq k\}} P(T_{i_1} \leq \cdots \leq T_{i_n}) \\
&= \sum_{\{(i_1,\ldots,i_n);\sum_{h=j}^{n} w'_{i_h} \geq k\}} P(T'_{i_1} \leq \cdots \leq T'_{i_n}) \\
&= P(T_{\mathbf{w}'} \geq T'_{(j)}) \\
&= \sum_{i=j}^{n} q'_i.
\end{aligned}$$

Now, consider an s-weighted-k-out-of-n system (introduced in Section 1) with independent and identically distributed component lifetimes $T_1, \ldots, T_n$. It is obvious that such a system has the signature vector of the form $\mathbf{q} = (0, \ldots, 0, q_s, q_{s+1}, \ldots, q_n)$. If $T_1, \ldots, T_n$ are distributed according to a common continuous distribution F, then

$$m_{\mathbf{w}}^{r,s}(t) = \sum_{i=r}^{n} \sum_{u=0}^{r-1} \sum_{v=u}^{i-1} q_i \binom{n}{u} \binom{n-u}{v-u} \frac{\int_t^{\infty} F^u(t)(F(x) - F(t))^{v-u} \bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1} \binom{n}{u} F^u(t) \bar{F}^{n-u}(t)} \tag{2}$$

Theorem 2. *Consider two s-weighted-k-out-of-n systems with weight vectors* $\mathbf{w}$ *and* $\mathbf{w}'$, *signature vectors* $\mathbf{q}$ *and* $\mathbf{q}'$, *both based on components with independent and identical lifetimes with common distribution F. Let* $m_{\mathbf{w}}^{r,s}(t)$ *and* $m_{\mathbf{w}'}^{r,s}(t)$ *be their respective mean residual life functions defined in (1). If* $\mathbf{q} \leq_{st} \mathbf{q}'$, *then* $m_{\mathbf{w}}^{r,s}(t) \leq m_{\mathbf{w}'}^{r,s}(t)$.

Proof. By interchanging the order of the summations in (2), we have that,

$$\begin{aligned}
m_{\mathbf{w}}^{r,s}(t) = \\
&\sum_{u=0}^{r-1} \sum_{v=u}^{r-1} \Big(\sum_{i=r}^{n} q_i \Big) \binom{n}{u} \binom{n-u}{v-u} \frac{\int_t^{\infty} F^u(t)(F(x) - F(t))^{v-u} \bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1} \binom{n}{u} F^u(t) \bar{F}^{n-u}(t)} \\
&+ \sum_{u=0}^{r-1} \sum_{v=r}^{n-1} \Big(\sum_{i=v+1}^{n} q_i \Big) \binom{n}{u} \binom{n-u}{v-u} \frac{\int_t^{\infty} F^u(t)(F(x) - F(t))^{v-u} \bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1} \binom{n}{u} F^u(t) \bar{F}^{n-u}(t)} \\
&\leq \sum_{u=0}^{r-1} \sum_{v=u}^{r-1} \Big(\sum_{i=r}^{n} q'_i \Big) \binom{n}{u} \binom{n-u}{v-u} \frac{\int_t^{\infty} F^u(t)(F(x) - F(t))^{v-u} \bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1} \binom{n}{u} F^u(t) \bar{F}^{n-u}(t)} \\
&+ \sum_{u=0}^{r-1} \sum_{v=r}^{n-1} \Big(\sum_{i=v+1}^{n} q'_i \Big) \binom{n}{u} \binom{n-u}{v-u} \frac{\int_t^{\infty} F^u(t)(F(x) - F(t))^{v-u} \bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1} \binom{n}{u} F^u(t) \bar{F}^{n-u}(t)} \\
= m_{\mathbf{w}'}^{r,s}(t).
\end{aligned}$$

The inequality follows from the assumption $\mathbf{q} \leq_{st} \mathbf{q}'$.

The following corollary follows from Theorems 1 and 2.

Corollary 1. Consider two s-weighted-k-out-of-n systems given in Theorem 2. If $\mathbf{w} \leq \mathbf{w}'$, then $m^{r,s}_{\mathbf{w}}(t) \leq m^{r,s}_{\mathbf{w}'}(t)$.

References

[1] Asadi, M. and Goliforushani, S. (2008), On the mean residual life function of coherent systems. *IEEE Transactions on Reliability*, **57**, 574–580.

[2] Chen, Y. and Yang, Q. (2005), Reliability of two-stage weighted k-out-of-n systems with components in common. *IEEE Transactions on Reliability*, **54**, 431–440.

[3] Eryilmaz, S. and Bozbulut, A.R. (2014), Computing marginal and joint Birnbaum, and Barlow-Proschan importances in weighted-k-out-of-n:G systems. *Computers & Industrial Engineering*, **72**, 255–260.

[4] Guess, F. and Proschan, F. (1988), Mean residual life, theory and applications, In: Krishnaiah, P.R., Rao, C.R.

[5] Higashiyama, Y. (2001), A factored reliability formula for weighted-k-out-of-n System. *Asia-Pacific Journal of Operational Research*, **18**, 61–66.

[6] Kotz, S. and Shanbhag, D.N. (1980), Some new approaches to probability distributions. *Advanced Applied Probability*, **12**, 903–921.

[7] Samaniego, F.J. (1985), On the closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability*, **34**, 69-72.

[8] Samaniego, F.J. and Shaked, M. (2008), Systems with weighted components. *Statistics and Probability Letters*, **78**, 815–823.

[9] Shaked, M. and Shanthikumar, J.G. (1994), Stochastic Orders and their applications. Academic Press, Boston.

[10] Wu, J.S. and Chen, R.J. (1994), An algorithm for computing the reliability of a weighted-k-out-of-n system. *IEEE Transactions on Reliability*, **43**, 327–328.

Use Weibull Distribution in Accelerated Life Testing for Computing MTTF Under Normal Operating Conditions

Ramezani, R. [1]

Department of Statistics, University of Damghan

Abstract

The intensity of the global competition for the development of new products in a short time. Testing under normal operating conditions for compute reliability quatities, requires a very long time. This has led to the development of accelerated life testing (ALT). In this article, We compute MTTF of Bourdon tubes (used as a part of pressure sensors in avionics) in stress conditiion. The failure is leak in the tube. Base on Anderson-Darling test Weibull distribution is appropriate for fitting data under stress condition. We determine MTTF of Bourdon tubes in operating condition base on arrhenius model and mean of Weibull distributions.

Keywords: Acceleration test, Arrheinus model, Mean time to failure , Anderson-Darling test, Weibull distribution.

1 Introduction

The intensity of the global competition for the development of new products in a short time has motivated the development of new methods. Testing under normal operating conditions requires a very long time. This has led to the development of accelerated life testing (ALT), where units are subjected to a more severe environment (increased or decreased stress levels) than the normal operating environment so that failures can be induced in a short period of test time. Information obtained under accelerated conditions is then used in estimate the characteristics of life distributions at normal operating conditions.

[1] r_ramezani@du.ac.ir

2 Design of Accelerated Life Testing Plans

A detailed test plan is usually designed before conducting an accelerated life test. The plan requires determination of the type of stress, methods of applying stress, stress levels, the number of units to be tested at each stress level, and an applicable accelerated life testing model that relates the failure times at accelerated conditions to those at normal conditions. Stress in ALT can be applied in various ways as decrease, increase, constant or Synthetic. In [2] provide extensive tables and practical guidelines for planning an ALT. In [1] introduce theoritcal discssion on ALT. Three department of sumsung company result of them research in case one pump in washing machine represent in [3].
Assumed that the components are tested at different accelerated stress levels s1, s2, . . . , sn. The failure times at each stress level are then used to determine the most appropriate failure time probability distribution, along with its parameters. Under the parametric statistics-based model assumptions, the failure times at different stress levels are linearly related to each other. Thus

$$t_o = C_A t_s$$

where t_o is the failure time under operating conditions, t_s is the failure time under stress conditions, and C_A is the acceleration factor.

3 Acceleration Model for the Weibull Model

The relationships between the failure time distributions at the accelerated and normal conditions base on weibull distribution can be derived the following:

$$R_s(t) = e^{-(t/\beta_s)^{\alpha_s}} \qquad t \geq 0, \alpha_s \geq 0, \beta > 0$$

where α_s is the shape parameter of the Weibull distribution under stress conditions and β_s is the scale parameter under stress conditions. The CDF under normal operating conditions is

$$R_o(t) = R_s(\frac{t}{C_A}) = e^{-(t/C_A\beta_s)^{\alpha_s}} = e^{-(t/\beta_o)^{\alpha_o}}$$

The underlying linearity assumption $\alpha_s = \alpha_o$, and $\beta_o = C_A\beta_s$. If the shape parameters at different stress levels are significantly different, then either the assumption of true linear acceleration is invalid or the Weibull distribution is inappropriate to use for analysis of such data. Let $\alpha_s = \alpha_o = \alpha \geq 1$. Then the probability density function under normal operating conditions is

$$f_o(t) = (\frac{1}{C_A})f_s(\frac{t}{C_A}) = \frac{\alpha}{C_A\beta_s}(\frac{t}{C_A\beta_s})^{\alpha-1}e^{-(t/C_A\beta_s)^{\alpha}}, \qquad t \geq 0, \beta_s \geq 0$$

The MTTF under normal operating conditions is

$$MTTF_o = \beta_o\Gamma(1 + \frac{1}{\alpha})$$

The failure rate under normal operating conditions is

$$h_o(t) = (\frac{1}{C_A})h_s(\frac{t}{C_A}) = \frac{\alpha}{C_A\beta_s}(\frac{t}{C_A\beta_s})^{\alpha-1} = \frac{h_s(t)}{C_A^{\alpha}}$$

4 Case study

A manufacturer of Bourdon tubes (used as a part of pressure sensors in avionics) wishes to determine its MTTF. The manufacturer defines the failure as a leak in the tube. The tubes are manufactured from 18 Ni (250) maraging steel and operate with dry 99.9fluid as the internal working agent. Tubes fail as a result of hydrogen embrittlement arising from the pitting corrosion attack. Because of the criticality of these tubes, the manufacturer decides to conduct ALT by subjecting them to different levels of pressures and determining the time for a leak to occur. The units are continuously examined using an ultrasound method for detecting leaks, indicating failure of the tube. Units are subjected to three stress levels of gas pressures and the times for tubes to show leak are recorded. Determine the mean lives and plot the reliability functions for design pressures of 80 and 90 psi.
Solution. The result of fit the failure times to Weibull distributions shows in table 1. Base on Anderson-Darling test at level 0f 0.05 error, Weibull distribution is good for fit failure time at every 3 level of stress pressure. P-Value at every 3 level in this test are greater of 0.25.

Table 1: Parameters estimation of weibull distribution in different pressure

parameter	100 psi	120 psi	140 psi
α	2.87	2.67	2.52
β	10392	5375	943

Table 2: Mean of time failure in different pressure

	100 psi	120 psi	140 psi
mean	9236	4777	838

Since $\alpha_1 = \alpha_2 = \alpha_3 \cong 2.65$, then the Weibull model is appropriate to describe the relationship between failure times under accelerated conditions and normal operating conditions. We determine the mean time of the population fails as

$$t = \beta\Gamma(1 + \frac{1}{\alpha})$$

The mean of life time at every 3 level of stress are shown in table 2.
The relationship between the failure time t and the applied pressure P can be assumed to be similar to the Arrhenius model; thus

$$t = ke^{c/P}$$

where k and c are constants. By making a logarithmic transformation, the above expression can be written as

$$ln(t) = ln(k) + \frac{c}{P}$$

Using a linear regression model, we obtain $k = 3.391$ and $c = 811.400$.The estimated mean at 80 psi and 90 psi are 86131 h and 27980 h respectively. The corresponding acceleration factors are 9.33 and 3.03. The failure rates under normal operating conditions are

$$h_{80}(t) = \frac{2.65}{1.63847 \times 10^{13}} t^{1.65}, \qquad h_{90}(t) = \frac{2.65}{8.31909 \times 10^{11}} t^{1.65}$$

The MTTFs for 80 and 90 psi are calculated as

$$MTTF_{80} = \beta\Gamma(1 + \frac{1}{\alpha}) = (1.63847 \times 10^{13})^{1/2.65}\Gamma(1 + \frac{1}{2.65}) = 85807h$$

and

$$MTTF_{90} = 31488 \times 0.885 = 27867h$$

References

[1] Hoang, P. (2003), *Handbook of reliability engineering*, Springer.

[2] Meeker, W. and Hahn, G. (1985), How to plan an accelerated life test-some practical guidelines. *Statistical techniques*, **10**.

[3] Sang-Jun, P., Sang-Deuk, P., Kwang-Suck, K. and Ji-Hyun, C. (2006), Reliability evaluation for the pump assembly using an accelerated test, *International Journal of Pressure Vessels and Piping*, **83**, 283–286.

On Properties of Progressively Type-II Censored Conditionally N-Ordered Statistics Arising from a Non-Identical and Dependent Random Vector

Rezapour, M. [1]

Department of Statistics, Shahid Bahonar University of Kerman

Abstract

In this paper, we investigate progressively Type-II censored conditionally N-ordered statistics arising from a system with identical as well as non-identical but dependent components, jointly distributed according to an Archimedean copula with completely monotone generator (PCCOSDNARCM-N). Our results generalized the results in Bairamov (2006) and is more flexible than those in practice, because of considering the dependency between components that is a common fact for real data.

Keywords: Archimedean copula, Order statistics, Progressive censoring, Progressively Type-II censored order statistics, Reliability systems.

1 Introduction

An experimenter may wish to reduce the size of a life test after having gained often critical early knowledge, while still obtaining information on later failures. The items removed make space for other experiments and reduce costs. Since in the real life we are face with dependent and non-identical data, we consider progressively Type-II right censored order statistics (PCOS-II) from a vector with a copula as the joint distribution function. In this case the marginal distributions are arbitrary and we are free to consider any desirable univariate distribution. Therefore, the marginal distribution of PCOS-II order statistics arising from dependent and non-identical sample according to copulas are applicable in real life. When we have a parallel system, the lifetime of the system equals the maximum of the lifetimes of the components. If we just record the lifetime of the system we ignore the information due to the components which can be used to obtain a

[1] mohsenrzp@gmail.com

more precise lifetime analysis. Hence a progressively Type-II right censored conditionally N-ordered statistics (PCCOS-N) sampling scheme is recommended. In this paper, we shall consider the PCCOS-N sampling scheme for systems that are neither identical nor independent. For example a common shock may affect the efficiency of the components or some stressful environment may lead dependency between components. We shall consider the dependency between systems using Archimedean copulas with completely monotone generators. These copulas have many desirable properties, for example,

- The class of Archimedean copulas with completely monotone generators contain several other such known copulas, see for example P.375 and 376 and 377 Joe [3].
- This family of copulas is MTP2, i.e. it has positive dependence property, (see e. g. [4] Muller and Scarsini, 2005), which is a suitable for lifetime data.
- Some goodness of fit tests exists for Archimedean copulas. In fact, we can consider which Archimedean copulas with completely monotone generator is best fit to a data set. (see e.g. [2]).

Recall that a function $\psi : \mathbb{R}_+ \to [0,1]$ is said to be *d-monotone* if $(-1)^k\psi^{(k)} \geq 0$ for $k \in \{1,\ldots,d-2\}$ and $(-1)^{d-2}\psi^{(d-2)}$ is a decreasing and convex function, where $\psi^{(k)}$, the k-th derivative of the function ψ, exists for $k = 1,2,\ldots,d-2$. If a function is d-monotone for all $d \in \mathbb{N}$, then it is said to be *completely monotone*. If a copula C_ψ has the form

$$C_\psi(u_1,\ldots,u_n) = \psi\left(\sum_{i=1}^{n}\psi^{-1}(u_j)\right), \tag{1}$$

where $\psi : \mathbb{R}_+ \to [0,1]$ is an n-monotone ($n \geq 2$) function such that $\psi(0) = 1$ and $\lim_{x\to\infty}\psi(x) = 0$, it is called an Archimedean copula with generator function ψ (see [3, 6, 5]). In this work, we concentrate on Archimedean copulas with completely monotone generator function. This family of copulas have applications in reliability theory in [7, 8]. Let $G(u) = \exp\{-\psi^{-1}(u)\}$, $u \in [0,1]$, and M_ψ be a distribution function with Laplace transform ψ. Then, an equivalent representation for (1) is given by

$$C_\psi(u_1,\ldots,u_n) = \int_0^\infty \prod_{i=1}^{n} G^\alpha(u_i)dM_\psi(\alpha). \tag{2}$$

This representation is the key to the ensuing developments. Furthermore, we assume that ψ is strictly increasing and its inverse function ψ^{-1} is differentiable. Now, let us consider the PCCOS-N arising from either independent or dependent random vectors $\mathbf{X}_1,\ldots,\mathbf{X}_n$. We assume that for $i = 1,\ldots,N$, $\mathbf{X}_i = (X_i^1, X_i^2,\ldots,X_i^n)$ are absolutely continuous, iid random vectors and let $T(\mathbf{X}_i)$ be the life time of the vector $\mathbf{X}^{\mathbf{R}}_{1:m:N}$. Then $T(\cdot)$ is a measurable $\mathbb{R}^p \setminus \mathbb{R}$ function. Under the PCCOS-N sampling scheme, $\mathbf{X}_1,\ldots,\mathbf{X}_N$ are place on a life test. The first system to fail will be denoted by $\mathbf{X}^{\mathbf{R}}_{1:m:N}$. We have $T(\mathbf{X}^{\mathbf{R}}_{1:m:N}) = \min(T(\mathbf{X}_1),\ldots,T(\mathbf{X}_N))$. We now remove R_1 system at random from the surviving set $\{\mathbf{X}_1,\ldots,\mathbf{X}_N\} \setminus \mathbf{X}^{\mathbf{R}}_{1:m:N}$. After the ith failure, occurring at $N(\mathbf{X}^{\mathbf{R}}_{i:m:N})$, R_i surviving systems are removed at random. The procedure terminates at the mth step, where $R_1 + \cdots + R_m + m = N$. Bairamov [1] showed that the joint pdf of the first r PCCOS-N, $r = 1,2,\ldots,m$, can be represented as follows

$$\begin{aligned} &f_{\mathbf{X}^{\mathbf{R}}_{1:m:N},\ldots,\mathbf{X}^{\mathbf{R}}_{r:m:N}}(\mathbf{x}_1,\ldots,\mathbf{x}_r) \\ &\quad= \Big(\prod_{j=1}^{r}\gamma_j\Big) f(\mathbf{x}_r)\Big(\bar{H}(\mathbf{x}_r)\Big)^{\gamma_r-1}\prod_{j=1}^{r-1} f(\mathbf{x}_j)\Big(\bar{H}(\mathbf{x}_j)\Big)^{R_j}, \end{aligned} \tag{3}$$

where $\gamma_j = n - \sum_{v=1}^{j-1}(R_v+1) = \sum_{v=j}^{m}(R_v+1)$, $1 \leq v \leq m$, $\gamma_1 = n$, and $H(\mathbf{x}) = P\{T(\mathbf{X}) \leq T(\mathbf{x})\}$. Here, we consider progressively Type-II censored conditionally N-ordered statistics arising from a system with identical as well as non-identical but dependent components, jointly distributed according to an Archimedean copula with completely monotone generator
(PCCOSDNARCM-N).

2 Main results

In this section, we obtain the joint and marginal density function of PCCOSDNARCH-N arising from a random vector $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_N)$ with joint survival function

$$P(\mathbf{X} > \mathbf{x}) = \int_0^\infty \prod_{i=1}^{N} G^\alpha(\bar{F}_i(\mathbf{x}_i)) dM_\psi(\alpha)\,, \tag{4}$$

where $\bar{F}_i(\mathbf{x}) = P(\mathbf{X}_i > \mathbf{x})$ is the joint survival function of $\mathbf{X}_i$, $i = 1, \ldots, n$. For this purpose, let $\bar{H}_i(\mathbf{x}) = P\{T(\mathbf{X}_i) > T(\mathbf{x})\}$, then for $k = 1, \ldots, n-1$ the following identity is obvious.

$$\begin{aligned} P(T(\mathbf{X}_1) > T(\mathbf{x}_1), \ldots, T(\mathbf{X}_k) > T(\mathbf{x}_k), \mathbf{X}_{k+1} > \mathbf{x}_{k+1}, \ldots, \mathbf{X}_n > \mathbf{x}_n) \\ = \psi\left(\sum_{i=1}^{k} \psi^{-1}(\bar{H}_i(\mathbf{x}_i)) + \sum_{i=k+1}^{n} \psi^{-1}(\bar{F}_i(\mathbf{x}_i))\right). \end{aligned} \tag{5}$$

We can obtain the joint density function of $\mathbf{X}^{\mathbf{R}}_{1:m:N}, \ldots, \mathbf{X}^{\mathbf{R}}_{m:m:N}$ using the following theorem.

Theorem 1. *For $n \in \mathbb{N}$, let S_n be the set of all permutations π of $(1, 2, \ldots, N)$. For brevity, let $\rho_r = R_1 + \cdots + R_r, 1 \leq r \leq m$, with $\rho_0 = 0$ and $\rho_m = N - m$. Then, the joint density of $\mathbf{X}^{\mathbf{R}}_{1:m:N}, \ldots, \mathbf{X}^{\mathbf{R}}_{m:m:N}$ is given by*

$$\begin{aligned} f_{\mathbf{X}^{\mathbf{R}}}(\mathbf{t}_1, \ldots, \mathbf{t}_m) &= \int_0^\infty \frac{1}{(N-1)!}\Big(\prod_{j=2}^{m} \gamma_j\Big) \sum_{\pi \in S_n} \prod_{j=1}^{m} f_{\pi(j)}(\mathbf{t}_j, \alpha) \\ &\quad \times \Big(\prod_{r=m+\rho_{j-1}+1}^{m+\rho_j} \bar{G}_{\pi(r)}(\mathbf{t}_j, \alpha)\Big) dM_\psi(\alpha)\,, \end{aligned} \tag{6}$$

where $T(\mathbf{t}_1) \leq \cdots \leq T(\mathbf{t}_m)$ and $\pi(i)$ is the i-th component of the permutation vector $\pi \in S_n, 1 \leq i \leq N$ and $\gamma_j = \sum_{i=j}^{m}(R_i + 1)$.

References

[1] Bairamov, I. (2006). Progressive Type II censored order statistics for multivariate observations. *J. Mult. Anal.* **97**, 797-809.

[2] Fermanian J.-D., Radulovic D., Wegkamp M. (2004), Weak convergence of empirical copula processes, *Bernoulli* 10, 5 847-860.

[3] Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman and Hall, London.

[4] Müller A. and Scarsini M. (2005), Archimedean copula and positive dependence, *J. Mult. Anal.* 28, 1 - 12.

[5] McNeil, A.J. and Nèslehovà, J. (2009). Multivariate Archimedean copulas, d-monotone functions and L1-norm symmetric distributions. *Ann. Stat.* 37, 3059-3097.

[6] Nelsen R.B. (2006). *An Introduction to Copulas*. Springer, New York.

[7] Rezapour, M., Alamatsaz, M. H. and Balakrishnan, N. (2013). On properties of dependent general progressively Type-II censored order statistics. *Metrika*, **76**, 909-917.

[8] Rezapour, M., Alamatsaz, M. H., Cramer, E. and Balakrishnan, N. (2013). On properties of progressively Type-II censored order statistics arising from dependent and non-identical random variables. *Stat. Meth.* **10**, 58-71.

Characterization of Bivariate Distribution by Mean Residual Life and Quantile Residual Life

Shafaei-Noughabi, M. [1]

Department of Mathematics and Statistics, University of Gonnabad

Abstract

Nair and Nair (1989) showed that bivariate mean residual life function characterizes the distribution uniquely. The subject of this paper is to verify how closely the bivariate quantile residual life function determines the distribution. It has been shown that like univariate case, two suitable bivariate quantile residual life can characterize the underlying distribution uniquely.

Keywords: Bivariate distribution function, Bivariate α-quantile residual life, Bivariate mean residual life

1 Introduction

When we deal with dependent components, extending reliability concepts to bivariate and multivariate seems inevitable. Same shocks on the components or excessive load survivors bear after their partners fail may cause their dependency. Many authors have introduced and studied bivariate or multivariate reliability concepts, e.g., Basu (1971) and Johnson and Kotz (1973) considered different versions of multivariate failure rate functions, Nair and Nair (1989) studied the mean residual life (MRL) concept for two possibly dependent components, and Roy (1994) studied multivariate aging classes and derived the chain of implications between them.

It is well-known that the MRL function determines the distribution function uniquely in the univariate case. Nair and Nair (1989) proved that the BMRL function uniquely determines the distribution function. Gupta and Longford (1984) determined the class of all distribution function F with the α-quantile residual life (α-QRL) function $q_\alpha(t)$. Song and Cho (1995) showed that in the class of continuous and strictly increasing distributions, two α-QRL functions $q_\alpha(t)$ and $q_\beta(t)$ which $\frac{\ln \bar{\alpha}}{\ln \bar{\beta}}$ is irrational characterize the distribution function uniquely. Lin (2009) proved the result in the broader class of continuous models.

[1] mohamad.shafaee@gmail.com

2 The bivariate mean residual life

Assume the non-negative random vector $\mathbf{X} = (X_1, X_2)$ be lifetimes of two possibly dependent components. Let $\mathbf{X}$ follows absolutely continuous distribution F in the first quadrant of R^2, $Q = \{(x_1, x_2); x_i \geq 0\}$. Briefly, let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{X} \geq \mathbf{x}$ stand for $X_i \geq x_i$, $i = 1, 2$. The well-known reliability function is $R(\mathbf{x}) = P(\mathbf{X} \geq \mathbf{x})$. The partial conditional reliability function for the i^{th} component is

$$R^{(i)}(x; \mathbf{x}) = P(X_i - x_i > x | \mathbf{X} \geq \mathbf{x}), \quad \mathbf{x} > \mathbf{0},\ x > 0,\ i = 1, 2. \tag{1}$$

The BMRL function can be written as

$$\mathbf{m}(\mathbf{x}) = E(\mathbf{X} - \mathbf{x} | \mathbf{X} > \mathbf{x}) = (m_1(\mathbf{x}), m_2(\mathbf{x})). \tag{2}$$

It can be verified that $m_1(\mathbf{x})R(\mathbf{x}) = \int_{x_1}^{\infty} R(t, x_2)dt$, and with differentiation

$$-\frac{\partial \ln R(\mathbf{x})}{\partial x_1} = (1 + \frac{\partial m_1(\mathbf{x})}{x_1})\frac{1}{m_1(\mathbf{x})}. \tag{3}$$

Similar equation holds in x_2 direction, and in turn by the fundamental theorem of line integrals, the integral $\int_{\mathbf{a}}^{\mathbf{b}} - \ln R(\mathbf{x})d\mathbf{x}$ is independent of the path joining $\mathbf{a}$ to $\mathbf{b}$ and

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla(-\ln R(\mathbf{t})) \cdot d\mathbf{t} = \ln R(\mathbf{a}) - \ln R(\mathbf{b}),$$

in which $\nabla(-\ln R(\mathbf{t})) = (-\frac{\partial \ln R(\mathbf{x})}{\partial x_1}, -\frac{\partial \ln R(\mathbf{x})}{\partial x_2})$. More specifically,

$$\int_{\mathbf{0}}^{\mathbf{x}} \nabla(-\ln R(\mathbf{t})) \cdot d\mathbf{t} = \ln R(\mathbf{0}) - \ln R(\mathbf{x}) = -\ln R(\mathbf{x}).$$

Nair and Nair (1989) considered the particular paths $(0, 0)$ to $(x_1, 0)$ and $(x_1, 0)$ to (x_1, x_2) to evaluate the left hand side of this equation, and obtained

$$R(\mathbf{x}) = \frac{m_1(0, 0)m_2(x_1, 0)}{m_1(x_1, 0)m_2(x_1, x_2)} \exp\Big\{ - \int_0^{x_1} \frac{dt_1}{m_1(t_1, 0)} - \int_0^{x_2} \frac{dt_2}{m_2(x_1, t_2)} \Big\}.$$

3 The bivariate quantile residual life

The i^{th} partial α-QRL function can be defined by

$$q_\alpha^{(i)}(\mathbf{x}) = \inf\{t_i : R^{(i)}(t_i; \mathbf{x}) = \bar{\alpha}\}, \quad \mathbf{x} \in \mathbb{R}^{+2}, \tag{4}$$

Taking $i = 1$, it simplifies to

$$q_\alpha^{(1)}(\mathbf{x}) = R_1^{-1}(\bar{\alpha}R(x_1, x_2); x_2) - x_1, \quad \mathbf{x} \in \mathbb{R}^{+2},$$

in which $R_1^{-1}(p; x_2) = \inf\{x_1 :\ R(x_1, x_2) = p\}$. Similarly

$$q_\alpha^{(2)}(\mathbf{x}) = R_2^{-1}(\bar{\alpha}R(x_1, x_2); x_1) - x_2.$$

The α-BQRL function can be constructed by gathering these two partial functions in a vector as $\mathbf{q}_\alpha(\mathbf{x}) = (q_\alpha^{(1)}(\mathbf{x}), q_\alpha^{(2)}(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^{+2}$. Johnson and Kotz (1973) considered the bivariate failure rate function $\mathbf{r}(\mathbf{x}) = (r_1(\mathbf{x}), r_2(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^2$ in which $r_i(\mathbf{x}) = -\frac{\partial}{\partial x_i} \ln R(\mathbf{x})$ represents the i^{th} partial failure rate function. As univariate case,

$$\int_{x_1}^{x_1+q_\alpha^{(1)}(\mathbf{x})} r_1(t, x_2)dt = -\ln \bar{\alpha}, \tag{5}$$

and

$$\int_{x_2}^{x_2+q_\alpha^{(2)}(\mathbf{x})} r_2(x_1, t)dt = -\ln \bar{\alpha}, \tag{6}$$

and consequently we have

$$1 + \frac{\partial}{\partial x_1} q_\alpha^{(1)}(\mathbf{x}) = \frac{r_1(x_1, x_2)}{r_1(x_1 + q_\alpha^{(1)}(x_1, x_2), x_2)}, \tag{7}$$

and

$$1 + \frac{\partial}{\partial x_2} q_\alpha^{(2)}(\mathbf{x}) = \frac{r_2(x_1, x_2)}{r_2(x_1, x_2 + q_\alpha^{(2)}(x_1, x_2))}, \tag{8}$$

respectively. As a result, $\frac{\partial}{\partial x_i} q_\alpha^{(i)}(\mathbf{x}) \geq -1$ for $i = 1, 2$. If $r_i(\mathbf{x})$ be increasing (decreasing) in x_i, then $q_\alpha^{(i)}(\mathbf{x})$ decreases (increases) in x_i.

4 Characterization by bivariate quantile residual life

Here, we are interested to investigate how closely the α-BQRL determines the distribution function. This leads us to solve the system of functional equations

$$\begin{cases} R(\varphi_1(\mathbf{x}), x_2) = \bar{\alpha} R(\mathbf{x}), \\ R(x_1, \varphi_2(\mathbf{x})) = \bar{\alpha} R(\mathbf{x}), \end{cases} \tag{9}$$

where $\varphi_i(\mathbf{x}) = x_i + q_\alpha^{(i)}(\mathbf{x})$, $i = 1, 2$.

It can be shown that every solution R of (9) can be written as

$$R(\mathbf{x}) = R_0(\mathbf{x}) h^*(\frac{\ln R_0(\mathbf{x})}{\ln \bar{\alpha}} - 1), \quad \mathbf{x} \in \mathbb{R}^{+2}, \tag{10}$$

or equivalently

$$R(\mathbf{x}) = R_0(\mathbf{x}) K(-\ln R_0(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^{+2}, \tag{11}$$

in which R_0 is one special solution of (9) and K is a periodic function with period $-\ln \bar{\alpha}$. Clearly, $K(0) = 1$ and it must be restricted so that R be a reliability function.

Theorem 1. *Two BQRL functions $\mathbf{q}_\alpha(\mathbf{x})$ and $\mathbf{q}_\beta(\mathbf{x})$ which $\frac{\ln \bar{\alpha}}{\ln \bar{\beta}}$ is irrational uniquely determine the underlying distribution, namely F, in the class of continuous and strictly increasing bivariate distributions.*

Example 1. *It is simple to verify that the reliability function*

$$R_0(\mathbf{x}) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2\}, \quad \lambda_1, \lambda_2 > 0, \mathbf{x} \in \mathbb{R}^{+2}, \tag{12}$$

accommodate global constant BQRL function. Thus, by (11), the class of reliability functions

$$R(\mathbf{x}) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2\}(1 + \epsilon \sin(a\lambda_1 x_1 + a\lambda_2 x_2)), \quad \mathbf{x} \in \mathbb{R}^{+2}, \ |\epsilon| < \frac{1}{\sqrt{2}}, \ a > 0,$$

have global constant BQRL function for $\alpha = 1 - e^{-\frac{2k\pi}{a}}$, $k = 1, 2, \ldots$.

References

[1] Basu, A. P. (1971). Bivariate failure rate. *Journal of the American Statistical Association*, **66**, 103-104.

[2] Gupta, R. C. and Langford, E. S. (1984). On the determination of a distribution by its median residual life function: A functional equation. *Journal of Applied Probability,* **21**, 120-128.

[3] Johnson, N. L. and Kotz, S. (1973). A vector valued multivariate hazard rate. Bulletin of the International Statistical Institute, 45 (Book 1), 570-574.

[4] Lin, G. D. (2009). On the characterization of life distributions via percentile residual lifetimes. *Sankhya* Ser. A, **71**, Part 1, 64-72.

[5] Nair, K. R. and Nair, N. U. (1989). Bivariate mean residual life. *IEEE Transactions on Reliability*, **38**, 362-364.

[6] Roy, D. (1994). Classification of life distributions in multivariate models. *IEEE Transactions on Reliability*, **43(2)**, 224-229.

[7] Song, J. K. and Cho, G. Y. (1995). A note on percentile residual life. *Sankhya,* Ser. A, **57**, 333-335.

Distribution-Free Comparison of Mean Residual Life Functions of Two Populations

Sharafi, M. [1]

Department of Statistics, Razi University

Abstract

At any age the mean residual life function gives the expected remaining life at that age. This function can be useful in life-testing experiments in biological as well as industrial settings. In this paper, we first propose a nonparametric test to compare mean residual life functions based on two independent samples. Next, in order to assess the power properties of the proposal test statistic, we examine its empirical power properties, through a Monte Carlo simulation study under different lifetime distributions.

Keywords: Empirical distribution function, Mean residual life ordering, Power.

1 Introduction

In many fields such as reliability, survival analysis and actuarial studies, statistical inference based on the remaining lifetimes would be intuitively more appealing than the popular hazard function defined as the risk of immediate failure, whose interpretation could be sometimes difficult to be grasped. Common summary measures for the remaining lifetimes have been the mean and median residual lifetimes.

Definition 1. *Let X denote the lifetime of an item having a continuous distribution function F such that $F(0) = 0$ and let $\bar{F}(t) = 1 - F(t)$. The mean residual life (MRL) function is defined by*

$$m_F(t) = \int_t^\infty \bar{F}(u)du/\bar{F}(t), \quad if\ \bar{F}(t) > 0.$$

[1] m.sharafi@razi.ac.ir

For a detailed discussion and statistical applications of the MRL function, you can refer to [2]. In reliability engineering, it is interest to study of mean residual life (MRL) because the MRL function plays a key and important role in decision making, such as optimizing burn-in tests ([1]), planning accelerated life tests ([6], [4]), establishing warranty policies ([9]), and making maintenance decisions ([5], [7], [8]).

In this paper, we interest to compare the mean residual life functions from two populations or treatment groups.

Let X and Y be the lifetimes of two units with distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$, and mean residual life functions $m_F(t)$ and $m_G(t)$, respectively. We consider of testing the null hypothesis

$$H_0 : m_F(t) = m_G(t)$$

against

$$H_1 : m_F(t) \leq m_G(t) \quad (t > 0) \tag{1}$$

with strict inequality over a set of nonzero probability.
Therefore, we first propose a new distribution-free test for testing H_0 againt H_1. Next, we examine the performance of this test procedure under some statistical distributions through Monte Carlo simulations and compare it with a known test in the literature.

2 Main Result

In a particular life-testing experiment, suppose n units with independent lifetimes $X_1, \cdots, X_n$ from X and m units with independent lifetimes $Y_1, \cdots, Y_m$ from Y put on the test. On the basis these samples, we want to test H_0 versus H_1.
It can be show that H_1 holds if and only if, for all $t \geq 0$,

$$\delta(t) = \bar{F}(t) \int_t^\infty \bar{G}(u)du - \bar{G}(t) \int_t^\infty \bar{F}(u)du \geq 0, \tag{2}$$

Taking $t = 0$, in 2, we find that H_1 implies the different means.

with using 2, we define the following measure as

$$\Delta(F, G) = \int_0^\infty \delta(t)d(pF(t) + (1-p)G(t)). \tag{3}$$

It should be noted that under H_0, $\Delta(F, G) = 0$ while under H_1, $\Delta(F, G) > 0$.
For $p = \frac{1}{2}$,

$$\begin{aligned} \Delta(F, G) = \frac{1}{2} \int_0^\infty \int_t^\infty \bar{F}(u)du\bar{G}(t)dF(t) - \frac{1}{4} \int_0^\infty \bar{G}^2(u)\bar{F}(u)du + \frac{1}{4}\mu_F \\ - \left[\frac{1}{2} \int_0^\infty \int_t^\infty \bar{G}(u)du\bar{F}(t)dG(t) - \frac{1}{4} \int_0^\infty \bar{F}^2(u)\bar{G}(u)du + \frac{1}{4}\mu_G \right], \end{aligned} \tag{4}$$

where μ_F and μ_G are mean values of X and Y respectively.

Let us now assume that we have two independent samples of sizes n and m from the distributions F and G respectively. Furthermore, let F_n denote the empirical distribution function based on a random sample of size n from distribution F and similarly G_m denote the empirical distribution function based on a random sample of size m from distribution G. Then, under this setting, we propose the test statistic

$$T = \frac{1}{2m^2n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(Y_{j:m}-Y_{i:m})(m-i)(n-R_{Y_{i:m}}+i) - \frac{1}{4mn}\sum_{i=1}^{m-1}(Y_{i+1:m}-Y_{i:m})$$
$$\times(1-\frac{i}{m})^2(n-R_{Y_{i:m}}+i)+\frac{1}{4}\bar{Y} - \left[\frac{1}{2n^2m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(X_{j:n}-X_{i:n})(n-i)\right.$$
$$\left.\times(m-R_{X_{i:n}}+i) - \frac{1}{4mn}\sum_{i=1}^{n-1}(X_{i+1:n}-X_{i:n})(1-\frac{i}{n})^2(m-R_{X_{i:n}}+i)+\frac{1}{4}\bar{X}\right] \quad (5)$$

where the kth and lth order statistic of $Xi's$ and $Yi's$ by $X_{k:n}$ and $Y_{l:m}$, $k = 1\cdots,n$ and $l = 1,\cdots,m$. Also $R_{Y_{i:m}}$ and $R_{X_{i:n}}$ denote the rank of $Y_{i:m}$ and $X_{i:n}$ in the combined increasing arrangement of X and Yrespectively.

2.1 Empirical power study

We evaluate the performance of the proposed test and compare its power properties with kochar's test statistic. Kochar(1981) introduced the K statistic for comparing two hazard rate functions as:

$$K = \frac{1}{nm}\left(\sum_{j=1}^{n}a_jR_{X_{i:n}} - \sum_{j=1}^{n}ja_j\right) \quad (6)$$

where $a_j = \frac{1}{2}+\log\{1-\frac{j}{n+1}\}$.
Therefore, the empirical power values were obtained for the tests T and K through Monte Carlo simulations for the exponential, Weibull and Lomax distributions. We generated 10000 sets of data for equal sample sizes 10 and 30 in which the empirical power results thus determined are all presented in Tables 1. From Table, we see that the power of the introduced test increases with increasing sample sizes as well as increasing parameter. Furthermore, with using the power values, we find that of the proposed test is better than kochar's test.

References

[1] Block, H. W. and Savits,T. H. (1997) Burn-in,*Statistical Science*, **12**, 1-19.

[2] Guess, F. and Proschan, F. (1988). Mean residual life. In:Rao Krishnaiah, P.R., Rao, C.R. (Eds.), *Handbook of Statistics*, **7**,215-224.

[3] Kochar, S.C. (1981). A new distribution-free test for the equality of two failure rates, *Biometrika*, **68**, 423-426.

[4] Liao, C. M. and Tseng, S. T. (2006). Optimal design for step-stress accelerated degradation tests,*IEEE Trans. Reliability*, **55**, 59-66.

[5] Mi, J. (2002). On bound for some optimal policies in reliability, *Journal of Applied Probability*, **39**, 491-502.

Table 1: **Empirical near 5% critical values and power values for the proposed test and kochar's test.**

	$n_1 = n_2 = 10$					$n_1 = n_2 = 30$				
	stat.	$q_{0.95}$	θ_0	θ_1	θ_2	stat.	$q_{0.95}$	θ_0	θ_1	θ_2
	...	..	3	5	10	...	..	3	5	10
$\exp(\theta_0 = 3)$	T	0.357	0.051	0.248	0.614	T	0.707	0.051	0.592	1.00
	K	-0.255	0.055	0.254	0.753	K	-0.374	0.052	0.577	0.996
	..	..	5	10	20	..	..	5	10	20
$\exp(\theta_0 = 5)$	T	0.253	0.051	0.293	0.611	T	0.524	0.050	0.744	0.995
	K	-0.255	0.049	0.137	0.297	K	-0.376	0.052	0.454	0.861
	...	..	3	6	15	...	..	3	6	15
weibull(θ_0 = 3,0.7)	T	0.252	0.047	0.340	0.743	T	0.49	0.054	0.811	0.999
	K	-0.254	0.052	0.138	0.341	K	-0.374	0.049	0.430	0.896
	...	..	2	10	20	...	..	2	10	20
weibull(θ_0 = 4,2)	T	0.606	0.047	0.415	0.689	T	1.18	0.052	0.926	1.000
	K	-0.255	0.05	0.194	0.394	K	-0.373	0.049	0.598	0.937
	...	..	0.3	5	10	...	..	0.3	5	10
lumax(θ_0 = 0.3,3)	T	0.244	0.051	0.936	0.993	T	0.453	0.050	0.964	0.989
	K	-0.254	0.0496	0.486	0.514	K	-0.375	0.053	0.982	0.991
	...	..	3	10	15	...	..	3	10	15
lumax(θ_0 = 3,10)	T	0.367	0.051	0.577	0.741	T	0.642	0.052	0.805	0.914
	K	-0.254	0.050	0.253	0.354	K	-0.376	0.053	0.792	0.913

[6] Tang, L. C., Sun,Y. S., Goh, T. N. and Ong, H. L. (1996). Analysis of step-stress accelerated-life-test data: a new approach, *IEEE Trans. Reliability*, **45**, 69-74.

[7] Yeh R. H., and Chang, W. L. (2007). Optimal threshold value of failure-rate for leased products with preventive maintenance actions, *Mathematical and computer modeling*, **46**, 730-737.

[8] Yan, S., and Xie, M. (2010). Nonparametric Estimation of Decreasing Mean Residual Life With Type II Censored Data,*IEEE Trans. Reliability*, **59**, 38-44.

[9] Yun, W. Y., Murthy, D. N. P. and Jack, N. (2008). Warranty servicing with imperfect repair, *International Journal of Production Economics*, **111**, 159-169.

Recent Advances in Comparisons of Coherent Systems Based on Inactivity Times

Tavangar, M. [1]

Department of Statistics, University of Isfahan

Abstract

The purpose of the talk is to study the inactivity time of failed components of a coherent system consisting of n identical components with statistically independent lifetimes. Different aging and stochastic properties of this conditional random variable are obtained. Also we investigate stochastic properties of the inactivity time in the case where the component lifetimes are dependent random variables. Some results are extended to the case where the system has an arbitrary coherent structure with exchangeable components.

Keywords: Exchangeability, Joint reliability function, Signature, Likelihood ratio order.

1 Introduction

In the study of the reliability of engineering systems, the k-out-of-n structure plays a key role. A system with n components has a k-out-of-n structure if it operates as long as at least k of its components operate. The class of k-out-of-n systems is a special case of a class of systems which is known in the literature as coherent systems. A structure consisting of n components is known as a coherent system if the structure function of the system is monotone in its components, and each component of the system is relevant; see [2]. The concept of the signature of a coherent system, introduced by Samaniego [6], has become quite useful in studying the properties of coherent systems, and in comparing different systems. For a coherent system with lifetime T whose components' lifetimes $X_1, X_2, ..., X_n$ are statistically independent and identically distributed (i.i.d.) random variables with continuous distribution function F, the signature vector of the system is

[1] m.tavangar@stat.ui.ac.ir

defined as a probability vector $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ with $s_i = P\{T = X_{i:n}\}$, $i = 1, 2, ..., n$, where $X_{i:n}$ is the ith order statistic among $X_1, X_2, ..., X_n$; see [5], [6], [7].

Let X denote the lifetime of an alive unit having distribution F. Assuming that the unit has failed at or before time t, the inactivity time (IT) of X is defined as the conditional random variable $(t - X \mid X \leq t)$, which, in this context, represents the time that has elapsed since the failure of the unit. Among the researchers who have extended this concept to the coherent system, we can refer to [1], [4], [9].

On the basis of the structure of the coherent system, if the failure times of the components are not monitored continuously, then the exact failure times of some components of the system are unknown. Hence it might be important for reliability engineers and system designers to have some information about the time that has elapsed from a failure in the system. Suppose that an $(n-k+1)$-out-of-n system is equipped with a warning light that comes up at the time of the failure of the jth component, $j < k$. The system is still working then, but the operator may now consider some maintenance or replacement policies. In this paper, we first study the time that has elapsed from the ith failure in the system, $i = 1, 2, ..., j$, given that the component with lifetime $X_{j:n}$ has failed at or before time t, but the system is working at time t; that is, the random variable

$$(t - X_{i:n} \mid X_{j:n} \leq t < X_{k:n}), \text{ for } i = 1, 2, ..., j, \text{ and } j < k.$$

This random variable is called the conditional IT of the component with lifetime $X_{i:n}$. Now, assume that a *coherent* system (with lifetime T) is alive at time t, and at least j components have failed by time t. We then define the conditional IT of the failed component with lifetime $X_{i:n}$ as $(t - X_{i:n} \mid X_{j:n} \leq t < T)$. In what follows, we investigate several interesting properties of the IT of $X_{i:n}$ for both $(n-k+1)$-out-of-n and coherent systems.

We also investigate the properties of inactivity time of the components of a $(n-k+1)$-out-of-n system in the case where the components of the system are dependent. Let the vector $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ denote the lifetimes of the components and assume that $\boldsymbol{X}$ has an arbitrary joint distribution function $F(t_1, t_2, ..., t_n)$. Assume that the system has failed at or before time t. Following the notation in [9], we define the inactivity time of the component with lifetime $X_{r:n}$, $r = 1, 2, ..., k$, at the system level as $(t - X_{r:n} \mid X_{k:n} \leq t)$.

2 Main results

Consider two $(n-k+1)$-out-of-n systems $\mathcal{S}_1$, and $\mathcal{S}_2$ with i.i.d. components $X_1, X_2, ..., X_n$, and $Y_1, Y_2, ..., Y_n$, respectively. The following result shows that, when the components of two systems are ordered in terms of reversed hazard rates, then the corresponding systems are stochastically ordered in terms of their IT [11]. For definitions of different stochastic orders, see [8].

Theorem 1. *Let $X_1 \leq_{\text{rhr}} Y_1$. Then for any $t \geq 0$, and $1 \leq i \leq j < k \leq n$,*

$$(t - Y_{i:n} | Y_{j:n} \leq t < Y_{k:n}) \leq_{\text{st}} (t - X_{i:n} | X_{j:n} \leq t < X_{k:n}).$$

It can be shown that the condition about the *rhr*-order in Theorem 1 cannot be replaced by similar properties on *hr*-order.

Theorem 2. *Let $X_1 \leq_{\text{lr}} Y_1$. Then for any $t \geq 0$, $1 \leq i \leq j < k \leq n$, and $1 \leq i \leq p < q \leq m$,*

$$(t - Y_{i:n} \mid Y_{j:n} \leq t < Y_{k:n}) \leq_{\text{lr}} (t - X_{i:m} \mid X_{p:m} \leq t < X_{q:m}),$$

whenever $n \leq m$, $j \leq p$, *and* $k \leq q$.

Let T be the lifetime of a coherent system with n i.i.d. components and signature vector $\boldsymbol{s} = (s_1, s_2, ..., s_n)$, and let $X_1, X_2, ..., X_n$ be the lifetimes of the components with a common absolutely continuous distribution F. We now present a result regarding the likelihood ratio ordering of the IT $(t - X_{i:n} \mid X_{j:n} \leq t < T)$ with respect to j.

Theorem 3. *If the distribution function F is absolutely continuous, then for $1 \leq i \leq j < n$, we have*

$$(t - X_{i:n} \mid X_{j:n} \leq t < T) \leq_{\mathrm{lr}} (t - X_{i:n} \mid X_{j+1:n} \leq t < T).$$

In the next theorem, we examine the implication of likelihood ratio and hazard rate orderings of the signature vectors of two systems.

Theorem 4. *Let T_1 and T_2 be the lifetimes of two coherent systems with common i.i.d. components $X_1, X_2, ..., X_n$, and signature vectors $\boldsymbol{s}^{(1)}$ and $\boldsymbol{s}^{(2)}$, respectively. If $\boldsymbol{s}^{(1)} \leq_{\mathrm{lr}} (\leq_{\mathrm{hr}})\boldsymbol{s}^{(2)}$, then for any $t \geq 0$,*

$$(t - X_{i:n} \mid X_{j:n} \leq t < T_1) \leq_{\mathrm{lr}} (\leq_{\mathrm{hr}})(t - X_{i:n} \mid X_{j:n} \leq t < T_2).$$

The reversed hazard rate function is an important measure in the study of engineering systems. Let X be an absolutely continuous random variable with the distribution function $F(t)$, and the probability density function $f(t)$. The *reversed hazard rate* function of X is defined as $r(t) = f(t)/F(t)$, for all t such that $F(t) > 0$. We say that X has a decreasing reversed hazard rate (DRHR) distribution if $r(t)$ is a decreasing function; for more details, see [3], [8]. In [11], it is shown that, when the component lifetimes of the system are DRHR, then the IT $(t - X_{i:n} \mid X_{j:n} \leq t < X_{k:n})$ is stochastically increasing in t.

Now consider a $(n-k+1)$-out-of-n system consisting of n components and assume that the components of the system are dependent with lifetimes $X_1, X_2, \ldots, X_n$.

Theorem 5. *If the density function of the exchangeable random vector $(X_1, X_2, ..., X_n)$ satisfies the MTP$_2$ property, then*

$$(t - X_{r:n} \mid X_{k:n} \leq t) \leq_{\mathrm{st}} (t - X_{r:n} \mid X_{k+1:n} \leq t),$$

for any $t \geq 0$ and $1 \leq r \leq k < n$.

For definition of MTP$_2$ functions, we refer the reader to [8]. One can show that if the MTP$_2$ assumption in Theorem 5 is removed, then the conclusion of the theorem does not remain valid [10]. Tavangar and Asadi [10] derived some other results regarding the IT of a system with exchangeable components.

References

[1] Asadi, M. (2006). On the mean past lifetime of the components of a parallel system. *J. Statist. Plann. Inference*, **136**, 1197–1206.

[2] Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*, New York: Holt, Rinehart and Winston.

[3] Block, H.W., Savits, T.H. and Singh, H. (1998). The reversed hazard rate function. *Probab. Engrg. Inform. Sci.*, **12**(1), 69–90.

[4] Khaledi, B.E. and Shaked, M. (2007). Ordering conditional lifetimes of coherent systems. *J. Statist. Plann. Inference*, **137**, 1173–1184.

[5] Kochar, S.C., Mukerjee, H. and Samaniego, F.J. (1999). The 'signature' of a coherent system and its application to comparisons among systems. *Naval Res. Logist.*, **46**(5), 507–523.

[6] Samaniego, F.J. (1985). On closure of the IFR class under formation of coherent systems. *IEEE Trans. Rel.*, **R-34**(1), 69–72.

[7] Samaniego, F.J. (2007). *System Signatures and Their Applications in Engineering Reliability*, New York, Springer.

[8] Shaked, M. and Shanthikumar, J.G. (2007). *Stochastic Orders*, New York, Springer.

[9] Tavangar, M. and Asadi, M. (2010). A study on the mean past lifetime of the components of $(n-k+1)$-out-of-n system at the system level. *Metrika*, **72**(1), 59–73.

[10] Tavangar, M. and Asadi, M. (2015). The inactivity time of exchangeable components of k-out-of-n structures. *Sankhya B*, **77**(1), 141–164.

[11] Tavangar, M. (2015). Conditional inactivity time of components in a coherent operating system. To appear in *IEEE Trans. Rel.*

Some Results on Mean Vitality Function of Coherent Systems

Toomaj, A. [1] and Hashempour, M. [2]

[1] Department of Statistics, Gonbad Kavous University
[2] Department of Statistics, Ferdowsi University of Mashhad

Abstract

In this paper, we present some results on applications of mean vitality function to comparisons of coherent systems. We also obtain an upper bound for the mean vitality function of coherent system when the lifetimes of components are independent and identically distributed.

Keywords: Coherent System, IFRA, MVF, Stochastic Orders, System Signature.

1 Introduction

Let X be a random lifetime of a system or a component having the cumulative distribution function (cdf) F with a finite moment. The mean residual life (MRL) function is defined as

$$m(t) = E(X - t|X > t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)},$$

where $\bar{F}(t) = 1 - F(t)$ is the survival (reliability) function of F. If the cdf F has the probability density function (pdf) f, then

$$m(t) \quad = \quad v(t) - t, \qquad (1)$$

where $v(t) = E(X|X > t) = \int_t^\infty xf(x)dx/\bar{F}(t)$ is called *vitality function* (VF) or life expectancy; see, Kupka and Loo [5]. The functions VF and MRL play an important role in engineering reliability, biomedical sciences and survival analyzes; see e.g., Bairamov *et*

[1]ab.toomaj@gonbad.ac.ir
[2]ma.hashempour@stu.um.ac.ir

al. [2], Kotz and Shanbhag [4], Ruiz and Navarro [7] and the references therein. For a continuous random variable X with the pdf f, the Shannon [10] entropy of X is defined as $H(X) = -E[\log f(X)]$ where " log " stands for the natural logarithm. Recently, Rao *et al.* [6] introduced a new measure of information, called *cumulative residual entropy* (CRE) and is defined by

$$\mathcal{E}(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx. \tag{2}$$

In this paper, we obtain some results about the expectation of vitality function of coherent systems. A system is said to be coherent if every component of the system is relevant and the structure function of the system is monotone. Let T denote the lifetime of a coherent system consisting of n independent and identically distributed (i.i.d.) components with lifetimes $X_1, \cdots, X_n$ which follow the common cdf F. It follows that (see e.g., Samaniego [8])

$$\bar{F}_T(t) := P(T > t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \ t > 0, \tag{3}$$

where $\bar{F}_{i:n}(t)$ is the survival function of $X_{i:n}$. The vector of coefficients $\mathbf{s} = (s_1, \cdots, s_n)$ in (3) is called the *signature* of the system where $s_i = P(T = X_{i:n})$, for $1 \le i \le n$, is the probability that the i-th failure causes the system failure.

2 Main results

Here, we use the concept of *mean vitality function* (MVF) order to comparisons of coherent systems based on the signature of the system. The results is considered by Toomaj and Doostparast [11].

Definition 1. *Let X and Y be random variables with finite MVF's $E(v(X))$ and $E(v(Y))$, respectively. Then X is said to be smaller than Y in the MVF order, denoted by $X \le_{mvf} Y$, if $E(v(X)) \le E(v(Y))$.*

Since $\mathcal{E}(X) = E(m(X))$ (see, Asadi and Zohrevand [1]), therefore, from (1) the MVF of a random variable X with finite mean $\mu = E(X)$ is

$$E(v(X)) \quad = \quad E(m(X)) + E(X) = \mathcal{E}(X) + \mu. \tag{4}$$

It can be applied the concept of MVF to comparison of coherent systems. Therefore, we have the following corollary. Let T be the lifetime of the coherent system with signature $\mathbf{s} = (s_1, \cdots, s_n)$ consisting of n i.i.d. component lifetimes $X_1, \cdots, X_n$ coming from the cdf F. Then

$$E(v(X_{1:n})) \le E(v(T)) \le E(v(X_{n:n})). \tag{5}$$

Corollary 2 says that the MVF of coherent systems are between the MVF's of the series and parallel systems. Hence, Expression (5) motivates the comparison of coherent systems based on MVF measure.

Example 1. Let T_1 and T_2 be lifetimes of two coherent systems with signatures $\mathbf{s}_1 = (0, \frac{3}{7}, \frac{4}{7})$ and $\mathbf{s}_2 = (0, \frac{3}{8}, \frac{5}{8})$, respectively, having $n = 3$ i.i.d. component lifetimes coming from the standard exponential distribution. It is easy to verify that $E(v(T_1)) = 2.48$ and $E(v(T_2)) = 2.55$ and hence $T_1 \leq_{mvf} T_2$. $\square$

Now, we have the following proposition given by Toomaj and Doostparast [11]. To see the definition of usual stochastic order, we refer the reader to Shaked and Shanthikumar [9].

Let T_1 and T_2 be the lifetime of two coherent systems consisting of n i.i.d. component lifetimes from the cdfs F and G with signatures $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. If $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$ and $X \leq_{st} Y$, then $T_1 \leq_{mvf} T_2$.

Example 2. *Let $\boldsymbol{s}_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ and $\boldsymbol{s}_2 = (0, 0, \frac{1}{4}, \frac{3}{4})$ be signatures of two systems consisting $n = 4$ i.i.d. components with the common cdf F. Let T_1 and T_2 be the corresponding lifetimes of the systems. It is easy to verify that $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$. Then Proposition 2 implies that $T_1 \leq_{mvf} T_2$.*

In the sequel, we provide an upper bound for the MVF of a random variable by implementing some additional information. To see the definition of increasing failure rate average (IFRA), we refer the reader to Barlow and Proschan [3].

Let X be IFRA with the pdf f. Then, we have

$$\mathcal{E}(X) \leq \mu. \tag{6}$$

Proof. Since X is IFRA, it implies that

$$\left(-\frac{\log \bar{F}(t)}{t}\right)' \geq 0, \ t > 0.$$

Hence, for all $t > 0$ we have

$$-\bar{F}(t) \log \bar{F}(t) \leq t f(t), \ t > 0,$$

and the desired result follows. $\square$

If T denote the lifetime of a coherent system consisting of n i.i.d. components which are IFRA, then it is known that T is IFRA, see Barlow and Proschan [3]. Hence from Equation (4) and Lemma 2, we have the following corollary. If T denote the lifetime of a coherent system consisting of n i.i.d. components which are IFRA, then

$$E(v(T)) \leq 2\mu_T,$$

where $\mu_T = E(T) = \sum_{i=1}^{n} s_i \mu_{i:n}$ and $\mu_{i:n}$ for $i = 1, \cdots, n$ stands for the expected lifetimes of the order statistics.

References

[1] M. Asadi and Y. Zohrevand, On the dynamic cumulative residual entropy, *J. Stat. Plan. Inference* **137** (2007), 1931-1941.

[2] I. Bairamov, M. Ahsanullah and I. Akhundov, A residual life function of a system having parallel or series structures, *Journal of Statistical Theory and Applications.* vol. **1**, no. 2 (2002) 119-132.

[3] R. E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life testing*, (Silver springer, 1981).

[4] S. Kotz and D. Shanbhag, Some new approaches to probability distributions, *Adv. Appl. Probab.* vol **12** (1980) 903-921.

[5] J. Kupka and S. Loo, The hazard and vitality measures of aging, *J. Appl. Prob.* vol **26** (1989) 532-542.

[6] M. Rao, Y. Chen, B. Vemuri and W. Fei Cumulative residual entropy: a new measure of information, *IEEE Trans. Inf. Theory*, **50** (2004).

[7] J. M. Ruiz and J. Navarro, Characterization of distributions by relationships between the failure rate and the mean residual life, *IEEE Tran. Reliability.* **43** (1994) 640-644.

[8] F. J. Samaniego, *System Signatures and their Applications in Engineering Reliability*, (Springer Science+Business Media, LLC, New York, 2007).

[9] M. Shaked and J. G. Shanthikumar, *Stochastic Orders and their Applications*, (Academic Press, San Diego, 2006).

[10] C. E. Shannon, A mathematical theory of communication, *Bell System Tech. J.* **27** (1948) 379-423 and 623-656.

[11] Toomaj, A. and Doostparast, M. Comparisons of Mixed Systems Using Mean Vitality Function, *Journal of Statistical Theory and Applications*, **13** (2) (2013) 189-195.

Reliability Analysis of Multi-State k-out-of-n Systems with Components Having Random Weights

Zarezadeh, S. [1]

Department of Statistics, Shiraz University

Abstract

This paper is concerned with reliability modeling of multi-state k-out-of-n systems consisting of multi-state components. It is also assumed that each component of the system has an integer-valued random weight (capacity). A recursive algorithm is presented for reliability evaluation of this model. Some illustrative examples are also provided.

Keywords: Multi-state system, k-out-of-n system, Recursive algorithm.

1 Introduction

If a binary system with n components works if and only if at least k components work, the system is called k-out-of n:G system. In a binary k-out-of-n:F system, the system fails if and only if at least k components fail. There are many situations that each component of the system has different contribution to the system. Then, the system reliability is not defined only based on its structure and the total contribution of the components must be also considered. In the literature, these systems are well-known to weighted systems. Recently, the reliability of the weighted systems have extensively studied by the researchers, see [1]-[6]. Wu and Chen [6] explored the reliability of a k-out-of-n:G system whose components have unequal weights. This system works if and only if at least k components are working and the total weight of working components is at least k. Wu and Chen [6] and Higashiyama [3] presented algorithms for computing the reliability of such a system. Samaniego and Shaked [5] studied relations between the coherent systems and systems with weighted components. The reliability of multi-state k-out-of-n systems consisting of multi-state weighted components have been studied in [1], and [4]. In all

[1] s.zarezadeh@shirazu.ac.ir

of aforementioned research works, it is assumed that the weights of components are pre-specified constants. Eryilmaz [2] studied multi-state k-out-of-n systems with two-state components having random weights.

In this paper we consider a system with n components and assume that each component and the system may be in $M+1$ possible states: $0, 1, \ldots, M$. $J = M$ denotes the highest performance level of the system. $J = M-1,\ M-2,\ \ldots$ denote system states in the process of its gradual deterioration. State $J = 0$ corresponds to the total failure of the system. We propose a model to describe reliability of a multi-state k-out-of-n system with the above assumptions and the components having random weights. The proposed model is illustrated by some numerical examples.

2 Main results

Consider a system with n components. The system and each component of the system have $M+1$ possible states $(0, 1, 2, \ldots, M)$. When component i, $i = 1, \ldots, n$, is in state j $(0 \le j \le M)$, has a weight (utility value) of $W_{i,j}$. W_{ij} is a discrete random variable with the support $[a_{ij}, b_{ij}]$, $0 < a_{ij} < b_{ij} < \infty$. The system is defined to be in state j if the total weight of components which are in state j is greater than or equal to a pre-specified value c_j, $j = 0, 1, \ldots, M$. Define random variable X_{ij}, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, M$, as

$$X_{ij} = \begin{cases} 1, & \text{if component } i \text{ is in state } j; \\ 0, & \text{o.w.} \end{cases}$$

The components are independent and W_{ij}'s are assumed to be independent of X_{ij}'s. If ϕ denotes the structure function of the system, the system is in state l, $\{\phi(\mathbf{X}) = l\}$, if and only if

$$\{\sum_{i=1}^{n} X_{il} \ge k, \sum_{i=1}^{n} W_{il} X_{il} \ge c_l\}.$$

To evaluate the reliability of the system, a recursive algorithm is given. Let us first introduce the following notations.

- n the number of components
- M the best state of the system
- p_{ij} the probability that component i is in state j
- W_{ij} the weight of component i when it is in state j
- c_l the minimum total weight required to make certain that the system is in state l

It can be proved the following theorem.

Theorem 1. *If $R_l(c_l, k, n)$ denotes the reliability for the system to be in state l, then*

$$\begin{aligned} R_l(c_l, k, n) = & p_{nl} \sum_{w \ge a_{nl}} R_l(c_l - w, k-1, n-1) \mathbb{P}(W_{nl} = w) \\ & + (1 - p_{nl}) R_l(c_l, k, n-1) \end{aligned}$$

In the sequel, we give an example that the system reliability is calculated based on recursive algorithm presented in Theorem 1.

Example 1. *Consider a system with 5 components. Assume the system and each component may be in three states ($M = 2$). $J = 2$ denotes complete performance of the system (up state), $J = 1$ corresponds to partial performance, and $J = 0$ corresponds to complete failure (down state). Let p_{ij} be the ij-th array of matrix P and consider*

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.5 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

Suppose that random variable W_{ij}, $i = 1, 2, \ldots, 5$, $j = 0, 1, 2$, denotes the utility value of component i when it is in state j and have the following probability mass function:

$$\mathbb{P}(W_{ij} = w) = a_{ij}^{w}(1 - a_{ij})^{w-1}, \qquad w = 1, 2,$$

where a_{ij} is the ij-th array of matrix A considered as

$$A = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.5 & 0.8 & 0.4 \\ 0.25 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.3 & 0.2 \end{pmatrix}$$

Tables 1-3 show the reliability of the system in each state for $c = 2, 4, 6$. The computer program is developed in Matlab 8.0.

Table 1: Probability that the system is in down state

c_0	$R_0(c_0, 1, 5)$	$R_0(c_0, 2, 5)$	$R_0(c_0, 3, 5)$	$R_0(c_0, 4, 5)$	$R_0(c_0, 5, 5)$
2	0.8790	0.5550	0.2166	0.0454	0.0038
4	0.4084	0.2970	0.1320	0.0314	0.0031
6	0.1354	0.1468	0.0753	0.0193	0.0018

Table 2: Probability that the system has partial performance

c_1	$R_1(c_1, 1, 5)$	$R_1(c_1, 2, 5)$	$R_1(c_1, 3, 5)$	$R_1(c_1, 4, 5)$	$R_1(c_1, 5, 5)$
2	0.8628	0.5149	0.1845	0.0351	0.0027
4	0.3628	0.2688	0.1129	0.0232	0.0019
6	0.0875	0.0937	0.0549	0.0153	0.0013

Table 3: Probability that the system is in up state

c_2	$R_2(c_2, 1, 5)$	$R_2(c_2, 2, 5)$	$R_2(c_2, 3, 5)$	$R_2(c_2, 4, 5)$	$R_2(c_2, 5, 5)$
2	0.8824	0.5562	0.2140	0.0438	0.0036
4	0.4832	0.3511	0.1508	0.0333	0.0029
6	0.1100	0.1047	0.0518	0.0125	0.0012

As can be seen from Tables 1-3, $R_l(c_l, k, n)$ is decreasing with respect to each of the parameters c_l and k.

References

[1] Ding, Y., Zuo, M. J., Lisnianski, A., Li, W. (2010), A framework for reliability approximation of multi-state weighted k-out-of-n systems, *IEEE Transactions on Reliability*, **59**, 297-308.

[2] Eryilmaz, S. (2013), On reliability analysis of a k-out-of-n system with components having random weights, *Reliability Engineering & System Safety*, **109**, 41-44.

[3] Higashiyama Y. (2001), A factored reliability formula for weighted-k-out-of-n system, *Asia-Pacific Journal of Operational Research*, **18**, 61-66.

[4] Li, W., Zuo, M. J. (2008). Reliability evaluation of multi-state weighted k-out-of-n systems, *Reliability Engineering & System Safety*, **93**, 160-167.

[5] Samaniego, F. J., Shaked, M. (2008), Systems with weighted components. *Statistics and Probability Letters*, **78**, 815-23.

[6] Wu, J. S., Chen, R. J. (1994), An algorithm for computing the reliability of a weighted-k out-of-n system. *IEEE Transactions on Reliability*, **43**, 327-328.

The Generalized Joint Signature for Systems with Shared Components

Zarezadeh, S. [1] and Mohammadi, L. [2]

Department of Statistics, Shiraz University

Abstract

The concept of joint signatures which first defined by Navarro et al. [1] are useful tools for investigating the reliability of two systems with shared components. When several coherent systems share some components and the components have independent and identically distributed (i.i.d.) lifetimes, we obtain a pseudo-mixture representation for the joint distribution of the lifetimes of the systems based on a general notion of joint signatures. We present an R program to find the mentioned joint signature for any number of systems and components.

Keywords: Coherent system, Order statistic, Signature.

1 Introduction

This paper is a continuation of Navarro et al. [1] in which the joint behavior of several systems with at least one shared component is investigated. The joint distribution function of the system lifetimes is obtained generally in a theorem and then various illustrated examples are also provided.

An example of systems with shared components, which is used often, is in networked computing in which a server is used at the same time with several computers. A central server stores almost all of the files for the department's computers. If the central server breaks, some of the computers will not work at all and some will have limited capabilities. The performance of any given pair of PCs will depend on the performance of the shared components and that of its own individual components. Navarro et al. [1] found the joint distribution of the lifetimes of the computers when all component lifetimes are i.i.d.

[1] s.zarezadeh@shirazu.ac.ir
[2] mohammadileila@yahoo.com

random variables. They used a measure called signature which was defined previously in Samaniego [2], [3]. The signature of a coherent system with n i.i.d. components is defined as the probability vector $s = (s_1, s_2, ..., s_n)$, where s_i is the probability that the system fails when the i-th component fails. Hence, $s_i = P(T = X_{i:n})$, for $i \in \{1, 2, ..., n\}$, where T is the system's lifetime and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ are the order statistics corresponding to the n component lifetimes. A system's signature is useful and interesting specially because it is distribution-free. Using the signatues is an efficient method to find the precise features that influence the performance of a system's design.

Navarro et al. [1] proposed joint signatures by considering two systems sharing some components. We want to generalize this concept. Suppose we have a cluster of systems, some sharing components with some others. Then, we study how to define a very general notion of joint signatures. The following scenario can be imagined. Suppose we have three systems. Then, we can split the case into components that are shared by all three, components that are common only for 1 and 2, components that are common only for 1 and 3, and components that are common only for 2 and 3. By this setting, we obtain a pseudo-mixture representation for the joint distribution function of the lifetimes of three systems based on distributions of order statistics of component lifetimes and then develop a general notion of joint signatures. This will then generalize to more than three systems in the cluster.

In this article, we obtain the joint distribution of lifetimes of more than two cohorent systems with n i.i.d. components. The definition of signature in our representations is related to the one defined in Navarro et al. [1], but it is more general. We present R programs to obtain the multidimensional distribution of the lifetimes of at least three systems. Our programs give a signature matrix which provides, in fact, the coefficients of the distribution functions of the order statistics of the n iid component lifetimes in the representation of the multidimensional distribution function of the systems.

2 Main results

Suppose that $X_1, X_2, \ldots, X_n$ are non-negative independent random variables with common distribution function F. Consider three systems with lifetimes $T_1 = \phi_1(Y_1, Y_2, \ldots, Y_{n_1})$, $T_2 = \phi_2(Z_1, Z_2, \ldots, Z_{n_2})$, and $T_3 = \phi_3(W_1, W_2, \ldots, W_{n_3})$. Let $\{Y_1, Y_2, \ldots, Y_{n_1}\}$, $\{Z_1, Z_2, \ldots, Z_{n_2}\}$, and $\{W_1, W_2, \ldots, W_{n_3}\}$ be subsets of $\{X_1, X_2, \ldots, X_n\}$. If we denote the joint distribution function of $\mathbf{T} = (T_1, T_2, T_3)$ by $G(t_1, t_2, t_3) = \mathbb{P}(T_1 \le t_1, T_2 \le t_2, T_3 \le t_3)$, the following theorem is obtained.

Theorem 1. *Let $\{i_1, i_2, i_3\}$ be a permutation of $\{1, 2, 3\}$. Then the joint distribution function of $\mathbf{T}$, denoted by G, is written as*

$$G(t_{i_1}, t_{i_2}, t_{i_3}) = \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{i=1}^{n} s_{i,j,k}^{(i_1,i_2,i_3)} F_{i:n}(t_{i_1}) F_{j:n}(t_{i_2}) F_{k:n}(t_{i_3}) \qquad \text{for} \quad t_{i_1} \le t_{i_2} \le t_{i_3}. \tag{1}$$

Proof. The proof is removed because of the restriction on the number of the pages of the paper. □

If we have m coherent systems with respective lifetimes $T_1, T_2, \ldots, T_m$, then, for $t_{i_1} \le$

$t_{i_2} \leq \cdots \leq t_{i_m}$,

$$G(t_{i_1}, t_{i_2}, \ldots, t_{i_m}) = \sum_{i_1=1}^{n} \sum_{i_2=0}^{n-i_1} \cdots \sum_{i_m=0}^{n-\sum_{j=1}^{m-1} i_j} c_{i_1,i_2,\ldots,i_m} \prod_{j=1}^{m} F^{i_j}(t_{i_j}),$$

where $c_{i_1,i_2,\ldots,i_m}$ are integers which do not depend on the underlying distribution function F.

In the following remark, we present a matrix form for the joint distribution function of $\mathbf{T}$. We use the notation $B = \{b_i\}_n$ for a matrix B with n rows B_i, $i = 1, ..., n$. Define $\mathbf{a}'_{t_{i_1}} = (F_{1:n}(t_{i_1}), \ldots, F_{n:n}(t_{i_1}))$, and

$$\mathbf{a}'_{t_{i_j}} = (F_{0:n}(t_{i_j}), F_{1:n}(t_{i_j}), \ldots, F_{n:n}(t_{i_j})),\ j = 2, 3.$$

The joint distribution function of $\mathbf{T}$ in (1) can be rewritten as $G(t_1, t_2, t_3) = \mathbf{a}'_{t_{i_1}} W$, with $W = \{\mathbf{a}'_{t_{i_2}} S_l^{(i_1,i_2,i_3)} \mathbf{a}_{t_{i_3}}\}_n$ and $S_l^{(i_1,i_2,i_3)} = \{s_{l,j,k}^{(i_1,i_2,i_3)}\}_{(n+1)\times(n+1)}$, $l = 1, 2, \ldots, n$. Define $A = \{\overbrace{0, \ldots, 0}^{in \text{ times}}\ \mathbf{a}'_{t_2}\ \overbrace{0, \ldots, 0}^{n(n-i) \text{ times}}\}_{n+1}$ and $S^{(i_1,i_2,i_3)} = \{S_i^{(i_1,i_2,i_3)}\}_n$, then, $G(t_1, t_2, t_3) = \mathbf{a}'_{t_1} AS\mathbf{a}_{t_3}$.

Example 1. Let X_1, X_2, X_3 be i.i.d. random variables with common distribution function F. Consider three coherent systems with lifetimes $T_1 = \max\{X_1, X_2\}$, $T_2 = \max\{X_1, X_3\}$, and $T_3 = \max\{X_1, X_4\}$. Thus the systems have one shared component. Then the joint distribution function of T_1, T_2, and T_3, for $t_1 \leq t_2 \leq t_3$, is written as

$$G(t_1, t_2, t_3) = F_{2:2}(t_1)F_{1:1}(t_2)F_{1:1}(t_3).$$

The signature of order 4 of $X_{2:2}$ and $X_{1:1}$ are obtained, respectively, as $(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Hence, by some calculations, the joint signature $S_1^{(1,2,3)}$ is obtained as a 4×5 zero matrix and $S_2^{(1,2,3)}$, $S_3^{(1,2,3)}$ and $S_4^{(1,2,3)}$ are 4×5 matrices with the first row and first column all zero's and the rest a constant which are respectively $\frac{1}{96}$, $\frac{1}{48}$, and $\frac{1}{32}$.

Then, by letting

$$\mathbf{a}'_{t_1} = (F_{1:4}(t_1), F_{2:4}(t_1), F_{3:4}(t_1), F_{4:4}(t_1)),$$

and

$$\mathbf{a}'_{t_i} = (F_{0:4}(t_i), F_{1:4}(t_i), F_{2:4}(t_i), F_{3:4}(t_i), F_{4:4}(t_i)), \quad i = 2, 3,$$

one obtains $G(t_1, t_2, t_3) = \mathbf{a}'_{t_1} W$, where $W = \{\mathbf{a}'_{t_2} S_n^{(1,2,3)} \mathbf{a}_{t_3}\}_n$.

For any permutation (i_1, i_2, i_3) of $\{1, 2, 3\}$ such that $t_{i_1} \leq t_{i_2} \leq t_{i_3}$, the distribution function G is followed same as the case where $t_1 \leq t_2 \leq t_3$.

Example 2. *Consider three systems with lifetimes* $T_1 = \max\{X_1, X_2\}$, $T_2 = \max\{X_2, X_3\}$, *and* $T_3 = \max\{X_3, X_4\}$.

Using the same procedure as the previous example, we obtain $S_1^{(1,3,2)}$ *which is a* 5×5 *zero matrix. Also,* $S_2^{(1,3,2)}, S_3^{(1,3,2)}$ *and* $S_4^{(1,3,2)}$ *are* 5×5 *matrices with all elements zero except for the last three rows of the first column which are respectively* $(\frac{1}{36}, \frac{1}{18}, \frac{1}{12})$, $(\frac{1}{18}, \frac{1}{9}, \frac{1}{6})$, *and* $(\frac{1}{12}, \frac{1}{6}, \frac{1}{4})$.

We have written an R pacakge to find the joint distribution of $T_1, ..., T_m$, for general $m > 2$. Our package takes as input m matrices, each $n \times n$, containing the relation between each T_i and $(X_1, ..., X_n)$, $i = 1, ..., m$. Then by running our program, we receive the following outputs for each permutation of $1, ..., m$: The joint distribution function of the systems in terms of $F(t_i)$'s, $i = 1, ..., m$, and also in terms of $F_{j:n}(t_i)$'s, $i = 1, ..., m,$, $j = 1, ..., n$, a matrix of the coefficients of $F_{j:n}(t_i)$'s, $i = 1, ..., m,$, $j = 1, ..., n$, and the general joint signature matrix S, which is an $(m+1)(n+1) \times (n+1)$ matrix. Hence, for example 1, this matrix binds all the matrices $S_i^{(1,2,3)}$'s, $i = 1, ..., 4$ in rows and it is a 20×5 matrix.

References

[1] Navarro, J., Samaniego, F. J. and Balakrishnan, N. (2010), The joint signature of coherent systems with shared components, *Journal of Applied Probability*, **47**(1), 235-253.

[2] Samaniego, F. J. (1985), On closure of the IFR class under formation of coherent systems, *IEEE Transactions on Reliability*, **34**, 69-72.

[3] Samaniego, F. J. (2007), *System Signatures and Their Applications in Reliability Engineering*, New York, Berlin: Springer, 2007.